Oriented and Injective Oriented Colourings of Grid Graphs

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Abstract

An oriented colouring of a directed graphs is a vertex colouring in which no two adjacent vertices belong to the same colour class and all of the arcs between any two colour classes have the same direction. Injective oriented colourings are oriented colourings that satisfy the extra condition that no two in-neighbours of a vertex receive the same colour. The oriented chromatic number of an unoriented graph is the maximum oriented chromatic number over all possible orientations. Similarly, the injective oriented chromatic number of an unoriented graph is the maximum injective oriented chromatic number over all possible orientations. The main results obtained in this paper are bounds on the injective oriented chromatic number of two dimensional grid graphs.

1 Introduction

This paper discusses the oriented chromatic number and injective oriented chromatic number of two dimensional grid graphs, denoted $G(n,m)$. In [4] Fertin, Raspaud and Roychowdhury discuss bounds on the oriented chromatic number of grid graphs with two vertices in each row, $G(n,2)$, three vertices in each row, $G(n,3)$, and the general grid graph with n rows and m columns, $G(n,m)$. Szepietowski and Targan [14] use a computer to show that every orientation of $G(n,4)$ admits an oriented colouring with at most seven colours. These works are extended here by considering bounds of injective oriented chromatic numbers of grid graphs. We also obtain an upper bound of the oriented chromatic number of $G(n,4)$ whose proof does not rely on computers.

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The terminology and notation of Bondy and Murty [1] is followed unless otherwise stated. The term *graph* here refers to a simple graph, meaning a graph with no loops or multiple edges. An *orientation* of a graph or an *oriented graph*, $\vec{G}$, is obtained by assigning one of two possible directions to each edge of a graph G. In particular, $\vec{G}$ consists of a set of vertices $V(\vec{G}) = V(G)$, and a set of ordered pairs of vertices called *arcs*, $A(\vec{G})$, where $\overrightarrow{xy}$ denotes an arc from x to y and $xy \in E(G)$. A *directed path* on vertices $v_1, v_2, \ldots, v_n$, where $n \geq 2$, is an oriented path, $v_1v_2, \ldots, v_n$ such that $\overrightarrow{v_iv_{i+1}} \in A(\vec{G})$ for all $1 \leq i \leq n-1$ or $\overrightarrow{v_{i+1}v_i} \in A(\vec{G})$ for all $1 \leq i \leq n-1$. A $2-$*path* is a path of length two, $v_1v_2v_3$ where v_2 is the *middle vertex*. Let $G(n,m)$ be a grid graph with n rows and m columns and $\vec{G}(n,m)$ be an orientation of $G(n,m)$. We define a vertex of $G(n,m)$ by its coordinates (s,t) where $1 \leq s \leq n$ and $1 \leq t \leq m$; there is an edge between (i,j) and (p,q) if and only if $|p-i| = 1$ and $q = j$ or $|q-j| = 1$ and $p = i$.

Oriented colourings are a type of vertex colouring in which no two adjacent vertices belong to the same colour class and all of the arcs between any two colour classes have the same direction, i.e., for any oriented graph $\vec{G}$, if x, y, w, z are assigned colours $c(x), c(w), c(y), c(z)$, respectively such that $c(x) = c(w)$, $c(y) = c(z)$ and $\overrightarrow{xy} \in A(\vec{G})$, then $\overrightarrow{zw} \notin A(\vec{G})$. Injective oriented colourings are oriented colourings that satisfy the extra condition that no two in-neighbours of a vertex receive the same colour.

Let G be a graph and $\vec{G}$ an orientation of G. The *oriented chromatic number of* $\vec{G}$, denoted $\vec{\chi}(\vec{G})$, is the smallest number of colours required for an oriented colouring of $\vec{G}$. The *oriented chromatic number of* G, denoted $\vec{\chi}(G)$, is the maximum oriented chromatic number over all possible orientations of G. To determine upper bounds on $\vec{\chi}(G)$ it is enough to consider $\vec{\chi}(\vec{G})$ for an arbitrary orientation $\vec{G}$ of G. Similarly, the *injective oriented chromatic number of* $\vec{G}$, denoted $\vec{\chi}_i(\vec{G})$, is the smallest number of colours required for an injective oriented colouring of $\vec{G}$, and the *injective oriented chromatic number of* G, denoted $\vec{\chi}_i(\vec{G})$, is the maximum injective oriented chromatic number over all possible orientations of G. To determine upper bounds on $\vec{\chi}_i(G)$ it is enough to consider $\vec{\chi}_i(\vec{G})$ for an arbitrary orientation $\vec{G}$ of G. For convenience, an *injective colouring* refers to an injective oriented colouring.

Raspaud and Sopena [12] prove that the oriented chromatic number of any oriented planar graph is at most 80. Marshall [8] provides examples of planar graphs whose oriented chromatic numbers are at least 17. There is a large gap between these upper and lower bounds. Many authors have considered special classes of planar graphs [9, 2, 4, 11]. It has been shown that the oriented chromatic number of a planar graph with girth four is at most 47 [3]. Finding good upper bounds for oriented chromatic numbers of special classes of planar graphs has proven to be challenging [10]. Recently, there has been interest in injective oriented colourings [6, 7]. MacGillivray, Raspaud and Swarts [7] give an upper bound (which it tight) for the injective oriented chromatic number of oriented trees, in terms of the maximum in-degree of the vertices. They also describe graphs whose injective oriented chromatic numbers are equal to the number of vertices of the graph. Here we provide an upper bound, without the aid of a computer, for the oriented chromatic number of grid graphs with n rows and 4 columns. We also give upper bounds for the injective oriented chromatic numbers of grid graphs with specific dimensions. In particular, this paper deals with bounds for the injective oriented chromatic numbers of grid graphs with n rows and 2, 3, and $m \geq 4$ columns.

Our results are obtained by considering oriented and injective oriented colourings in terms of homomorphisms of directed graphs, or digraphs. The reader may wish to consult *Graphs and Homomorphisms* [5] for a great reference on homomorphisms. A homomorphism from a digraph $\vec{G}$ to a digraph $\vec{H}$ is a mapping from the vertex set of $\vec{G}$ to the vertex set of $\vec{H}$ that preserves arcs. An injective homomorphism is a homomorphism where no two in-neighbours of a vertex have the same image. An oriented colouring of a digraph $\vec{G}$ can be thought of as a homomorphism from an oriented graph $\vec{G}$ to an oriented graph $\vec{H}$. Here the vertices of the *target graph* $\vec{H}$ (the homomorphic image) are treated as colours, where we label the n vertices of $\vec{H}$ with $\{0, 1, 2, 3, \ldots n-1\}$. In particular, if a vertex of $\vec{G}$ is mapped to vertex i of the target graph it receives colour i. Saying that there exists an $\vec{H}$-*colouring* of a graph $\vec{G}$ means that there exists a homomorphism from $\vec{G}$ to $\vec{H}$. Similarly, an injective colouring of an oriented graph $\vec{G}$ can be thought of as an injective homomorphism from an oriented graph $\vec{G}$ to an oriented graph $\vec{H}$. We say there is an injective $\vec{H}$-*colouring* of an oriented graph $\vec{G}$ if there is an injective homomorphism from $\vec{G}$ to $\vec{H}$. The oriented chromatic number of an oriented graph $\vec{G}$ is the minimum number of vertices required for an oriented graph $\vec{H}$ so that there exists a homomorphism from $\vec{G}$ to $\vec{H}$. The injective oriented chromatic number of an oriented graph $\vec{G}$ is the minimum number of vertices required for an oriented graph $\vec{H}$ so that there exists an injective homomorphism from $\vec{G}$ to $\vec{H}$.

2 Preliminary Results

Oriented chromatic numbers of graphs have been extensively studied (see, for example, [2, 11, 13]). The oriented chromatic number of grid graphs has been considered in [4, 14]. Fertin, Raspaud and Roychowdhury prove the following.

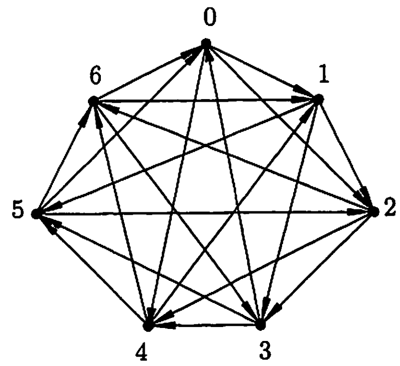

Figure 1: T_7

- For any $n \geq 4$, $\vec{\chi}(G(n,2)) = 6$.
- For any $n \geq 6$, $6 \leq \vec{\chi}(G(n,3)) \leq 7$.
- For any $n \geq 5$ and $m \geq 4$, $7 \leq \vec{\chi}(G(n,m)) \leq 11$.

Fertin, Raspaud, and Roychowdhury [4] prove that $\vec{\chi}(G(n,3)) \leq 7$ by providing a homomorphism from $\vec{G}(n,3)$ to T_7, the quadratic residue tournament on seven vertices, i.e., $V(T_7) = \{i \mid 0 \leq i \leq 6\}$ with $d^+(v) = d^-(v) = 3$ for all $v \in V(T_7)$ and $\overrightarrow{ij} \in A(T_7)$ if and only if $j - i \,(\text{mod } 7) \in \{1,2,4\}$ (see Figure 1). Szepietowski and Targan [14] provide an orientation of $G(7,3)$ whose oriented colouring requires seven colours, allowing one to conclude that $\vec{\chi}(G(n,3)) = 7$ for $n \geq 7$. Szepietowski and Targan [14] use a computer search to prove that every orientation of $G(n,4)$ can be mapped to T_7.

Here, we describe an alternate homomorphism to colour $\vec{G}(n,3)$ with T_7, even though the result is already established. It is included here because it was the motivation behind the techniques that are used to obtain the upper bounds for $\vec{\chi}(G(n,4)), \vec{\chi}_i(G(n,2))$ and $\vec{\chi}_i(G(n,3))$. Describing this alternative homomorphism provides insight into the colour method that is used for the main results of this paper.

Proposition 1. *Let u and v be distinct vertices of T_7. Then for each orientation of a 2-path, there exists at least one such 2-path between u and v.*

Proof. Since T_7 is vertex transitive, it is enough to observe that there is an entry in every row and column of Appendix A, Table 2. □

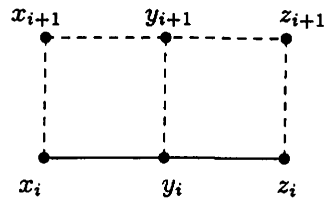

Figure 2: Colouring $G(n,3)$ with T_7

Lemma 2. *Let $\vec{H}$ be a digraph such that $d^+(v) \geq 1$ and $d^-(v) \geq 1$ for any $v \in V(\vec{H})$. Then there exists an $\vec{H}$-colouring of any oriented path.*

Proof. Let $\vec{P} = v_1v_2 \cdots v_s$ be an oriented path. We proceed by describing a homomorphism, c, from $\vec{P}$ to $\vec{H}$. Let $c(v_1) = i$ for any $i \in V(\vec{H})$. If $\overrightarrow{v_1v_2} \in A(\vec{P})$, then let $c(v_2)$ be any vertex in $N^+(c(v_1))$. This is always possible since $d^+(c(v_1)) \geq 1$. Similarly, if $\overrightarrow{v_2v_1} \in A(\vec{P})$ then let $c(v_2)$ be any vertex in $N^-(c(v_1))$. This is always possible since $d^-(c(v_2)) \geq 1$. By induction, the remaining vertices of $\vec{P}$ can be coloured with $\vec{H}$. □

Lemma 3. *For all $n \geq 1$, $\vec{\chi}(G(n,3)) \leq 7$.*

Proof. We construct a homomorphism from $\vec{G}(n,3)$ to T_7 by proceeding with induction on the number of rows of $\vec{G}(n,3)$. For convenience, let $x_i = (i,1)$, $y_i = (i,2)$, $z_i = (i,3)$ be the vertices in row i of $G(n,3)$, as illustrated in Figure 2.

When $n = 1$, $\vec{G}(1,3)$ is an oriented path. By Lemma 2 there is a T_7-colouring of $\vec{G}(1,3)$ since $d^-(v) = d^+(v) = 3$ for all $v \in V(T_7)$.

Assume that for some $i \geq 1$ there exists a homomorphism, c, from $\vec{G}(i,3)$ to T_7. We proceed by extending c to a T_7-colouring of $\vec{G}(i+1,3)$.

First colour y_{i+1}. If $\overrightarrow{y_iy_{i+1}} \in A(\vec{G}(i+1,3))$, then colour y_{i+1} arbitrarily with one of the colours in $N^+(c(y_i)) \setminus \{c(x_i), c(z_i)\}$. This is possible because $d^+(c(y_i)) = 3$. We do not want to colour y_{i+1} with $c(x_i)$ or $c(z_i)$ because $x_ix_{i+1}y_{i+1}$ or $z_iz_{i+1}y_{i+1}$ could be directed paths in
$G(i+1,3)$. Similarly, if $\overrightarrow{y_{i+1}y_i} \in A(G(i+1,3))$, then colour y_{i+1} with one of the colours in $N^-(c(y_i)) \setminus \{c(x_i), c(z_i)\}$. Next, by Proposition 1, there exists a path $c(x_i)\ p\ c(y_{i+1})$ in T_7 with the same orientation as $x_ix_{i+1}y_{i+1}$. Similarly, there exists a path $c(z_i)\ q\ c(y_{i+1})$ in T_7 with the same orientation as $z_iz_{i+1}y_{i+1}$. Set $c(z_{i+1}) = q$ and $c(x_{i+1}) = p$.

Since a T_7-colouring of $\vec{G}(i,3)$ can be extended to a T_7-colouring of $\vec{G}(i+1,3)$, it follows by induction that $\vec{G}(n,3)$ can be coloured by T_7 for all $n \geq 1$. Therefore $\vec{\chi}(G(n,3)) \leq 7$. □

3 Properties of T_9

The results of this section are used to obtain upper bounds on $\vec{\chi_i}(G(n,2))$ and $\vec{\chi}(G(n,4))$. Consider T_9, a vertex transitive tournament on nine vertices with $V(T_9) = \{i \mid 0 \leq i \leq 8\}$, $d^+(v) = d^-(v) = 4$ for all $v \in V(T_9)$, and $\overrightarrow{ij} \in A(T_9)$ if and only if $j - i \pmod 9 \in \{1,2,3,5\}$. This graph will be used as a target graph for Theorems 8 and 12. Appendix A, Tables 3 and 4, are supplied to aid the reader in verifying the following propositions.

Proposition 4. *Let u and v be distinct vertices of T_9. Then for each orientation of a 2-path, there exists at least one such 2-path between u and v.*

Proof. Since T_9 is vertex transitive, it is enough to observe that there is an entry in every row and column of Appendix A, Table 4. □

Proposition 5. *For distinct $u, v \in V(T_9)$, $|N^-(u) \cap N^-(v)| \leq 2$ and $|N^+(u) \cap N^+(v)| \leq 2$.*

Proof. Since T_9 is vertex transitive, it suffices to consider the case $u = 0$. From Appendix A, Table 4, Columns 1 and 3, we see that $|N^+(0) \cap N^+(v)| \leq 2$ and $|N^-(0) \cap N^-(v)| \leq 2$, for all $v \neq 0$ and the result follows. □

Proposition 6. *Let t, u, v and w be distinct vertices in T_9. For each orientation of a 2-path, there exists two such 2-paths between $\{t\}$ and $\{u, v, w\}$ whose middle vertices are distinct. In particular,*

$$|N^+(t) \cap (N^+(v) \cup N^+(u) \cup N^+(w))| \geq 2,$$
$$|N^-(t) \cap (N^+(v) \cup N^+(u) \cup N^+(w))| \geq 2,$$
$$|N^-(t) \cap (N^-(v) \cup N^-(u) \cup N^-(w))| \geq 2,$$
$$|N^+(t) \cap (N^-(v) \cup N^-(u) \cup N^-(w))| \geq 2.$$

Proof. The property can be observed by noting that the union of any three distinct rows in any column of Appendix A, Table 4 has at least two vertices. □

Proposition 7. *Let u, v and w be distinct vertices in T_9. Then there exist two directed 2-paths from $\{u\}$ to $\{v, w\}$ whose middle vertices are distinct, and two directed 2-paths from $\{v, w\}$ to $\{u\}$ whose middle vertices are distinct. In particular,*

$$|N^+(u) \cap (N^-(v) \cup N^-(w))| \geq 2 \text{ and } |N^-(u) \cap (N^+(v) \cup N^+(w))| \geq 2.$$

Proof. Since T_9 is vertex transitive, it suffices to consider $N^+(u) \cap (N^-(v) \cup N^-(w))$ for $u = 0$. If v or w is in $\{3,4,5,6,7,8\}$, then an inspection of Appendix A, Table 4, Column 2 reveals that $|N^+(0) \cap N^-(v)| \geq 2$ or $|N^+(0) \cap N^-(w)| \geq 2$. It follows that $|N^+(0) \cap (N^-(v) \cup N^-(w))| \geq 2$.

Otherwise, $\{v,w\} = \{1,2\}$ and $|N^+(0) \cap (N^-(1) \cup N^-(2))| = |\{1,5\}| = 2$. The result follows.

Now consider $N^-(u) \cap (N^+(v) \cup N^+(w))$ for $u = 0$. If v or w is in $\{1,2,3,4,5,6\}$, then an inspection of Appendix A, Table 4, Column 4 reveals that $|N^-(0) \cap N^+(v)| \geq 2$ or $|N^-(0) \cap N^+(w)| \geq 2$. It follows that $|N^-(0) \cap (N^+(v) \cup N^+(w))| \geq 2$. Otherwise, $\{v,w\} = \{7,8\}$ and $|N^+(0) \cap (N^-(7) \cup N^-(8))| = |\{4,8\}| = 2$. Again, the result follows.

□

4 Every orientation of $G(n,4)$ can be mapped to T_9

Fertin, Raspaud and Roychowdhury [4] provide an orientation of $G(5,4)$ whose oriented chromatic number is seven. Szepietowski and Targan [14] use a computer to show that every orientation of $G(n,4)$ can be mapped to T_7. Here we show, without the aid of a computer, that every orientation of $G(n,4)$ can be mapped to T_9.

Theorem 8. *For all $n \geq 1$, there exists a T_9-colouring of $\vec{G}(n,4)$.*

Proof. We proceed by induction on the number of rows of $\vec{G}(n,4)$. For convenience, let $w_i = (i,1), x_i = (i,2)$, $y_i = (i,3)$, $z_i = (i,4)$ refer to row i of $\vec{G}(n,4)$ (see Figure 3).

In the case $n = 1$, notice that $\vec{G}(1,4)$ is an oriented path. Since $d^+(u) = d^-(u) = 4$ for all $u \in V(T_9)$, Lemma 2 ensures there is a T_9-colouring of $\vec{G}(1,4)$.

Assume that for some $i \geq 1$ there exists a homomorphism, c, from $\vec{G}(i,4)$ to T_9. We proceed by extending c to a T_9-colouring of $\vec{G}(i+1,4)$.

Step 1. *Determine possible colours for y_{i+1}.*

If $\overrightarrow{y_iy_{i+1}} \in A(\vec{G}(i+1,4))$, let $S = N^+(c(y_i)) \setminus \{c(x_i), c(z_i)\}$ be the set of possible colours for y_{i+1}. Similarly, if $\overrightarrow{y_{i+1}y_i} \in A(\vec{G}(n,4))$, let $S = N^-(c(y_i)) \setminus \{c(x_i), c(z_i)\}$ be the set of possible colours for y_{i+1}. Since $d^+(v) = d^-(v) = 4$ for any $v \in T_9$, $|S| \geq 2$.

Step 2. *Colour y_{i+1} and x_{i+1}.*

If $|S| > 2$, Proposition 6 guarantees that there are paths $c(x_i)\ p_1\ a_1$ and $c(x_i)\ p_2\ a_2$ in T_9 having the same orientation as $x_ix_{i+1}y_{i+1}$, where $a_1, a_2 \in S$ and $p_1 \neq p_2$. At least one of p_1 and p_2 is not equal to $c(w_i)$. Choose $j \in \{1,2\}$ so that $p_j \neq c(w_i)$. Set $c(x_{i+1}) = p_j$ and $c(y_{i+1}) = a_j$.

If $|S| = 2$ and $x_ix_{i+1}y_{i+1}$ is a directed path, then Proposition 7 guarantees that there are paths $c(x_i)\ p_1\ a_1$ and $c(x_i)\ p_2\ a_2$ in T_9 having the same orientation as $x_ix_{i+1}y_{i+1}$, such that $a_1, a_2 \in S$ and $p_1 \neq p_2$. Choose $j \in \{1,2\}$ so that $p_j \neq c(w_i)$. Set $c(x_{i+1}) = p_j$ and $c(y_{i+1}) = a_j$.

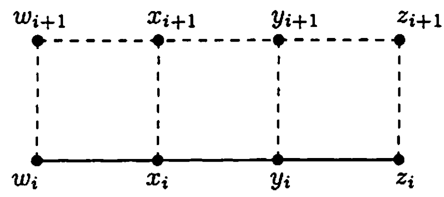

Figure 3: Colouring $G(n, 4)$ with T_9

Finally, if $|S| = 2$ and $x_i x_{i+1} y_{i+1}$ is not a directed path, then $c(x_i)$ is a possible colour for y_{i+1} so set $S := S \cup \{c(x_i)\}$. Now $|S| > 2$ and we proceed as before.

Step 3. *Colour w_{i+1} and z_{i+1}.*

By Proposition 4, there exists a path $c(y_{i+1})\ p\ c(z_i)$ in T_9 with the same orientation as $y_{i+1}z_{i+1}z_i$. Set $c(z_{i+1}) = p$. Also, there exists a path $c(x_{i+1})\ q\ c(w_i)$ in T_9 with the same orientation as $x_{i+1}w_{i+1}w_i$. Set $c(w_{i+1}) = q$.

Since a T_9-colouring of $\vec{G}(i, 4)$ can be extended to a T_9-colouring of $\vec{G}(i+1, 4)$, it follows by induction that there exists a T_9-colouring of $\vec{G}(n, 4)$ for all $n \geq 1$. □

5 General bounds on $\vec{\chi_i}(G(n, m))$

We now direct our attention to injective oriented colourings of two dimensional grid graphs. Since injective colourings are oriented colourings, similar techniques are used for describing homomorphisms, and additional care is taken to ensure that no two in-neighbours of a vertex have the same image.

We begin by considering the general case for n rows and m columns, $\vec{\chi_i}(G(n, m))$. Here we provide an injective homomorphism from $\vec{G}(n, m)$ to a graph with 23 vertices. Fertin, Raspaud and Roychowdhury [4] provide an orientation of $G(5, 4)$ whose oriented colouring requires seven colours, while Szepietowski and Targan [14] provide an orientation of $G(7, 3)$ whose oriented colouring requires seven colours. Hence $7 \leq \vec{\chi_i}(G(n, m))$ for $n \geq 5$ and $m \geq 4$, or $n \geq 7$ and $m \geq 3$, since an injective colouring is an oriented colouring. Combining the two results we conclude that $7 \leq \vec{\chi_i}(G(m, n)) \leq 23$ for $n \geq 5$ and $m \geq 4$, or $n \geq 7$ and $m \geq 3$.

Our target graph for injective colouring of the general two dimensional grid is T_{23}, a vertex transitive tournament on 23 vertices with $V(T_{23}) = \{i \mid 0 \leq i \leq 22\}$, $d^+(u) = d^-(u) = 11$ for all $u \in V(T_{23})$, and $\vec{ij} \in A(T_{23})$ if and only if $j - i \pmod{23} \in \{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}$. Note that T_{23}

is a quadratic residue tournament.

Lemma 9. *Let $\vec{H}$ be a digraph such that $d^+(v) \geq 2$ and $d^-(v) \geq 2$ for any $v \in V(H)$. Then there exists an injective $\vec{H}$-colouring of any oriented path.*

Proof. Let $\vec{P} = v_1 v_2 \cdots v_s$ be an oriented path. We proceed by describing a homomorphism, c, from $\vec{P}$ to $\vec{H}$. Begin by setting $c(v_1) = i$ for any $i \in V(\vec{H})$. If $\overrightarrow{v_1 v_2} \in A(\vec{P})$ then let $c(v_2)$ be any colour in $N^+(c(v_1))$; otherwise let $c(v_2)$ be any colour in $N^-(c(v_1))$. This is always possible since $d^+(c(v_1)) \geq 2$ and $d^-(c(v_1)) \geq 2$. If $\overrightarrow{v_2 v_3} \in A$ then let $c(v_3)$ be any colour in $N^+(c(v_2))$; otherwise let $c(v_3)$ be any colour in $N^-(c(v_2)) \setminus \{c(v_1)\}$. This is always possible because $d^+(c(v_2)) \geq 2$ and $d^-(c(v_2)) \geq 2$. By induction we can colour the remaining vertices of $\vec{P}$.

□

Proposition 10. *Let u and v be distinct vertices of T_{23}. Then for each orientation of a 2-path, there exist at least five such 2-paths between u and v whose middle vertices are distinct.*

Proof. Since T_{23} is vertex transitive, it is enough to consider $u = 0$ and observe that there are at least five entries in every row and column of Appendix A, Table 8. □

For T_{19}, a vertex transitive tournament on 19 vertices with $V(T_{19}) = \{i \mid 0 \leq i \leq 18\}$, $d^+(u) = d^-(u) = 9$ for all $u \in V(T_{19})$, and $\overrightarrow{ij} \in A(T_{19})$ if and only if $j - i \pmod{19} \in \{1, 4, 5, 6, 9, 11, 16, 17\}$, we have $|N^+(0) \cap N^+(1)| = 4$. Therefore, T_{23} is the smallest quadratic residue tournament that exhibits the property in Proposition 10.

Theorem 11. *For all $n, m \geq 1$, $\vec{\chi_i}(G(n,m)) \leq 23$.*

Proof. We proceed by induction on the number of rows, n of $\vec{G}(n,m)$ for any given number of columns, m. For convenience, let $x_{i,j} = (i,j)$ refer to the vertex in row i and column j.

In the case $n = 1$, $\vec{G}(1,m)$ is an oriented path. Lemma 9 ensures that there is an injective T_{23}-colouring of $\vec{G}(1,m)$, since $d^+(v) = d^-(v) = 11$ for all $v \in V(T_{23})$.

For $n = 2$ (see Figure 4), assume that the first row of $\vec{G}(2,m)$ has been coloured with T_{23}, i.e., there is a homomorphism, c, from $\vec{G}(1,m)$ to T_{23}.
Step 1. *Colour $x_{2,1}$.* If $\overrightarrow{x_{2,1}x_{1,1}} \in A(\vec{G}(2,m))$, then colour $x_{2,1}$ with any colour in $N^-(c(x_{1,1})) \setminus \{c(x_{1,2})\}$. Similarly, if $\overrightarrow{x_{1,1}x_{2,1}} \in A(\vec{G}(2,m))$, colour $x_{2,1}$ with any colour in $N^+(c(x_{1,1})) \setminus \{c(x_{1,2})\}$. Note that there are ten or more colour choices for $x_{2,1}$ since $d^+(u) = d^-(u) = 11$ for all $u \in V(T_{23})$.

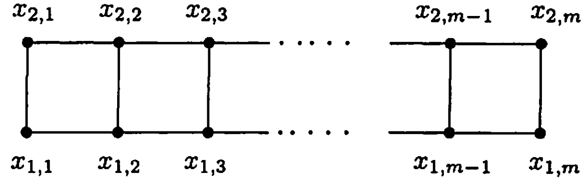

Figure 4: Injective T_{23} colouring of $\vec{G}(2,m)$.

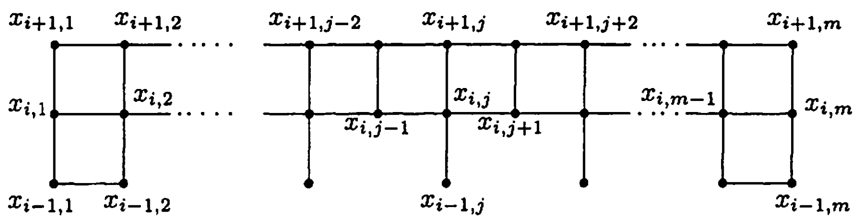

Figure 5: Injective T_{23} colouring of $\vec{G}(1+1,m)$.

Step 2. *Colour* $x_{2,2}$. Proposition 10 guarantees that there is a path $c(x_{2,1})\ p\ c(x_{1,2})$ in T_{23} with the same orientation as $x_{2,1}\ x_{2,2}\ x_{1,2}$ so that $p \notin \{c(x_{1,3}), c(x_{1,1})\}$. Set $c(x_{2,2}) = p$.
Step 3. *Colour* $x_{2,j}$ *for* $2 < j \le m-1$. Proposition 10 guarantees that there is a path $c(x_{2,j-1})\ p\ c(x_{1,j})$ in T_{23} with the same orientation as $x_{2,j-1}\ x_{2,j}\ x_{1,j}$ such that $p \notin \{c(x_{1,j-1}), c(x_{1,j+1}), c(x_{2,j-2})\}$. Set $c(x_{2,j}) = p$.
Step 4. *Finish by colouring* $x_{2,m}$. Proposition 10 guarantees that there is a path $c(x_{2,m-1})\ p\ c(x_{1,m})$ in T_{23} with the same orientation as $x_{2,m-1}\ x_{2,m}\ x_{1,m}$ so that $p \notin \{c(x_{1,m-1}), c(x_{2,m-2})\}$. Set $c(x_{2,m}) = p$.

Hence we have an injective T_{23}-colouring of $\vec{G}(2,m)$ and we conclude that $\vec{\chi_i}(G(2,m)) \le 23$.

Suppose $i \ge 2$ and there exists an injective T_{23}-colouring of $\vec{G}(i,m)$, i.e., there exists a homomorphism, c, from $\vec{G}(i,m)$ to T_{23}. To colour $\vec{G}(i+1,m)$, proceed as follows (see Figure 5).
Step1. *Colour* $x_{i+1,1}$. If $\overrightarrow{x_{i+1,1}x_{i,1}} \in A(G(i+1,m))$, then colour $x_{i+1,1}$ with any colour in $N^-(c(x_{i,1}))\backslash\{c(x_{i,2}), c(x_{i-1,1})\}$. Similarly, if $\overrightarrow{x_{i,1}x_{i+1,1}} \in A(G(i+1,m))$, colour $x_{i+1,1}$ with any colour in $N^+(c(x_{i,1}) \setminus \{c(x_{i,2}))\}$.
Step 2. *Colour* $x_{i+1,2}$. Proposition 10 guarantees that there is a path

$c(x_{i+1,1})\ p\ c(x_{i,2})$ in T_{23} with the same orientation as $x_{i+1,1}\ x_{i+1,2}\ x_{i,2}$ such that $p \notin \{c(x_{i,1}), c(x_{i,3}), c(x_{i-1,2})\}$. Set $c(x_{i+1,2}) = p$.

Step 3. *Colour $x_{i+1,j}$ for $2 < j \leq m-1$.* Proposition 10 guarantees that there is a path $c(x_{i+1,j-1})\ p\ c(x_{i,j})$ in T_{23} with the same orientation as $x_{i+1,j-1}\ x_{i+1,j}\ x_{i,j}$ such that $p \notin \{c(x_{i+1,j-2}), c(x_{i,j-1}), c(x_{i-1,j}), c(x_{i,j+1})\}$. Set $c(x_{i+1,j}) = p$.

Step 4. *Finish by colouring $x_{i+1,m}$.* Proposition 10 guarantees there is a path $c(x_{i+1,m-1})\ p\ c(x_{i,m})$ in T_{23} with the same orientation as $x_{i+1,m-1}\ x_{i+1,m}\ x_{i,m}$ such that $p \notin \{c(x_{i+1,m-2}), c(x_{i,m-1}), c(x_{i-1,m})\}$. Set $c(x_{i+1,m}) = p$.

We have extended an injective T_{23}-colouring of $\vec{G}(i,m)$ to an injective T_{23}-colouring of $\vec{G}(i+1,m)$. Therefore by induction, there exists an injective T_{23}-colouring of $G(n,m)$ for all $n \geq 1$, $m \geq 1$. We conclude that $\vec{\chi_i}(G(n,m)) \leq 23$. □

6 Bounds on $\vec{\chi_i}(G(n,2))$

We now restrict the size of $G(n,m)$ to $m = 2$ to improve the upper bound on $\vec{\chi_i}(G(n,2))$. Here we provide an injective T_9-colouring of $\vec{G}(n,2)$ so that $\vec{\chi_i}(G(n,2)) \leq 9$ for $n \geq 1$. An orientation of $G(2,4)$ provided by Fertin, Raspaud and Roychowdhury [4] shows that $\vec{\chi}(G(4,2)) = 6$. Since an injective colouring is an oriented colouring, $\vec{\chi_i}(G(n,2)) \geq 6$ for all $n \geq 4$. Combining the two results we have $6 \leq \vec{\chi_i}(G(n,2)) \leq 9$ for $n \geq 4$.

Theorem 12. *For all $n \geq 1$, $\vec{\chi_i}(G(n,2)) \leq 9$.*

Proof. We construct an injective T_9-colouring of $\vec{G}(n,2)$ by proceeding with induction on the number of rows of $\vec{G}(n,2)$. For convenience, let $x_i = (i,1)$, $y_i = (i,2)$, refer to row i of $\vec{G}(n,2)$.

In the case $n = 1$, $\vec{G}(1,2)$ has just one edge. Choose any edge in T_9 and colour the vertices of $\vec{G}(1,2)$ accordingly.

For $n = 2$, $\vec{G}(2,2)$ forms an oriented 4-cycle, $C = stuvs$. To prove that there is an injective T_9-colouring of C, we consider three cases: C is oriented so there are two vertices with indegree two, one vertex with indegree two, or no vertices with indegree two.

If C is oriented so that there are two vertices with indegree two, then these vertices are not adjacent, and so without loss of generality assume $d^-(v) = d^-(t) = 2$ (see Figure 6). Then $c(s) = 0$, $c(t) = 1$, $c(u) = 5$ and $c(v) = 1$ is an injective T_9-colouring of C. If C has one vertex with indegree two, then without loss of generality, assume that v is the vertex of indegree two, as depicted in Figure 7. There are two non-isomorphic orientations to consider. For Figure 7(a), $c(s) = 8$, $c(t) = 7$, $c(u) = 4$, and $c(v) = 0$ is an

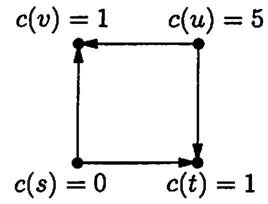

Figure 6: $\vec{G}(2,2)$ with two vertices of indegree two.

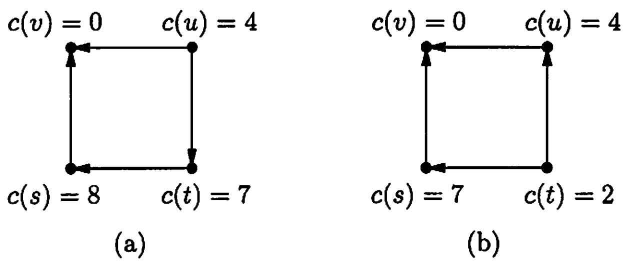

Figure 7: $\vec{G}(2,2)$ with one vertex of indegree two.

injective T_9-colouring of C. In Figure 7(b), $c(s) = 7$, $c(t) = 2$, $c(u) = 4$, and $c(v) = 0$ is an injective T_9-colouring of C. Finally, if C has no vertices of indegree two, then $G(2,2)$ is a directed cycle, as illustrated in Figure 8, and $c(s) = 4$, $c(t) = 2$, $c(u) = 1$ and $c(v) = 0$ is an injective T_9-colouring of C. In all three cases there is an injective T_9-colouring of C, and hence there is an injective T_9-colouring of $\vec{G}(2,2)$.

Suppose $i \geq 2$ and that there exists an injective homomorphism, c, from $\vec{G}(i,2)$ to T_9. Assume that x_i, x_{i-1}, y_i and y_{i-1} have been injectively coloured with T_9. We now proceed by induction on the number of rows and extend c to an injective T_9-colouring of $\vec{G}(i+1,2)$. We consider two cases based on the orientations of x_ix_{i+1} and y_iy_{i+1}.

Case 1: Suppose that at least one of $\overrightarrow{y_iy_{i+1}}$ or $\overrightarrow{x_ix_{i+1}}$ is in $A(\vec{G}(i+1,2))$. Without loss of generality assume that $\overrightarrow{y_iy_{i+1}} \in A(\vec{G}(i+1,2))$. Notice that injective colouring properties are not violated if y_{i+1} and y_{i-1} receive the same colour. This is an important insight when colouring y_{i+1}.

Consider the orientation of x_ix_{i+1}.

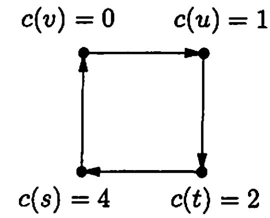

Figure 8: $\vec{G}(2,2)$ with no vertices of indegree two

Case 1(a): If $\overrightarrow{x_i x_{i+1}} \in A(\vec{G}(i+1,2))$ then $S = N^+(c(x_i)) \setminus \{c(y_i)\}$ is the set of possible colours for x_{i+1}. Since $|S| \geq 3$, Proposition 6 guarantees that there are paths $c(y_i)\ p_1\ a_1$ and $c(y_i)\ p_2\ a_2$ in T_9 with the same orientation as $y_i y_{i+1} x_{i+1}$, such that $a_1, a_2 \in S$ and $p_1 \neq p_2$. Choose $j \in \{1,2\}$ so that $p_j \neq c(x_i)$ (notice that y_{i+1} may be coloured with $c(y_{i-1})$ as mentioned previously). Set $c(y_{i+1}) = p_j$ and $c(x_{i+1}) = a_j$.
Case 1(b): If $\overrightarrow{x_{i+1} x_i} \in A(\vec{G}(i+1,2))$ then $S = N^-(x_i) \setminus \{c(x_{i-1}), c(y_i)\}$ is the set of possible colours for x_{i+1}. Since $\overrightarrow{x_{i+1} x_i}, \overrightarrow{y_i y_{i+1}} \in A(\vec{G}(i+1,2))$, y_{i+1} may be assigned $c(x_i)$ in an injective colouring of $\vec{G}(i+1,2)$. Recall that y_{i+1} may also be assigned $c(y_{i-1})$. Colour x_{i+1} with any colour in S. By Proposition 4, there is a path $c(x_{i+1})\ p\ c(y_i)$ in T_9 with the same orientation as $x_{i+1} y_{i+1} y_i$. Set $c(y_{i+1}) = p$.
Case 2: We may now assume that $\overrightarrow{y_{i+1} y_i}, \overrightarrow{x_{i+1} x_i} \in A(\vec{G}(i+1,2))$. Without loss of generality, assume that $\overrightarrow{x_i y_i} \in A(\vec{G}(i+1,2))$. When extending the colouring c to x_{i+1}, we are guaranteed to have $c(x_{i+1}) \neq c(y_i)$ since $x_{i+1} x_i y_i$ is a directed path from x_{i+1} to y_i.
Step 1. *Determine T, the set of possible colours for y_{i+1}.*
Let $T = N^-(c(y_i)) \setminus \{c(x_i), c(y_{i-1})\}$ be the set of possible colours for y_{i+1}. Since $d^-(c(y_i)) = 4$, $|T| \geq 2$.
Step 2. *Colour x_{i+1} and y_{i+1}.*
Consider the orientation of $x_{i+1} y_{i+1}$.
Case 2(a): If $\overrightarrow{y_{i+1} x_{i+1}} \in A(\vec{G}(i+1,2))$, as depicted in Figure 9(a), then $x_i x_{i+1} y_{i+1}$ is a directed path from y_{i+1} to x_i. Proposition 7 guarantees there are two paths, $c(x_i)\ p_1\ b_1$ and $c(x_i)\ p_2\ b_2$, in T_9, with the same orientation as $x_i x_{i+1} y_{i+1}$ such that $b_1, b_2 \in T$ and $p_1 \neq p_2$. Choose $j \in \{1,2\}$ so that $p_j \neq c(x_{i-1})$. Set $c(x_{i+1}) = p_j$ and $c(y_{i+1}) = b_j$.
Case 2(b): Assume that $\overrightarrow{x_{i+1} y_{i+1}} \in A(\vec{G}(i+1,2))$, as illustrated in Figure 9(b). If $|T| \geq 3$, then Proposition 6 guarantees there are two paths, $c(x_i)\ p_1\ b_1$ and $c(x_i)\ p_2\ b_2$ in T_9 with the same orientation as $x_i x_{i+1} y_{i+1}$, such that $b_1, b_2 \in T$ and $p_1 \neq p_2$. Choose $j \in \{1,2\}$ so that $p_j \neq c(x_{i-1})$.

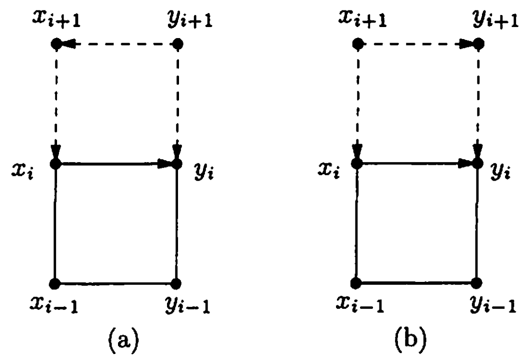

Figure 9: Injective T_9-colouring of $\vec{G}(n,2)$.

Set $c(x_{i+1}) = p_j$ and $c(y_{i+1}) = b_j$.

We may now assume that $|T| = 2$, and let $T = \{b_1, b_2\}$. By Proposition 4, there exist paths $c(x_i)p_1b_1$ and $c(x_i)p_2b_2$ in T_9 with the same orientation as $x_ix_{i+1}y_{i+1}$. If possible, choose $j \in \{1,2\}$ so that $p_j \neq c(x_{i-1})$ and set $c(x_{i+1}) = p_j$, and $c(y_{i+1}) = b_j$. Otherwise, it must be the case that $p_1 = p_2 = c(x_{i-1})$. Since T_9 is vertex transitive, we may assume, without loss of generality, that $c(x_i) = 0$. By examination of Appendix A, Table 4, Column 3, we see that it must be the case that $p_1 = p_2 = c(x_{i-1}) = 4$ and $\{b_1, b_2\} = \{5, 6\}$. Thus we must have $\{0,5,6\} \subset N^-(c(y_i))$. A review of Appendix A, Table 3 reveals that there is no such vertex in T_9, a contradiction. Therefore, it is the case that there exists a $j \in \{1,2\}$ so that $p_j \neq c(x_{i-1})$.

Since an injective T_9-colouring of $\vec{G}(i,2)$ can be extended to an injective T_9-colouring of $\vec{G}(i+1,2)$, it follows by induction that there is an injective T_9-colouring of $\vec{G}(n,2)$. Therefore, for all $n \geq 1$, $\vec{\chi_i}(G(n,2)) \leq 9$.

□

7 Properties of T_{11}

The results of this section are used to prove an upper bound on $\vec{\chi_i}(G(n,3))$. Consider T_{11}, a vertex transitive tournament on eleven vertices with $V(T_{11}) = \{i \mid 0 \leq i \leq 10\}$, $d^+(v) = d^-(v) = 5$ for all $v \in V(T_{11})$ and arcs $\vec{ij} \in A(T_{11})$ if and only if $j - i \pmod{11} \in \{1,3,4,5,9\}$. The graph T_{11} is a quadratic residue tournament and will be used as a target graph in Theorem 18. Appendix A, Tables 5 and 6, are supplied to aid the reader in verifying the following propositions.

Proposition 13. *Let u and v be distinct vertices of T_{11}. Then for each orientation of a 2-path there exist at least two such 2-paths between u to v whose middle vertices are distinct.*

Proof. Since T_{11} is vertex transitive, it is enough to consider the case $u = 0$ and observe that there are at least two entries in every row and column of Appendix A, Table 6. □

Since T_{11} is vertex transitive, the proofs for the next two propositions are immediately obvious by considering the case $t = 0$ and observing the entries in Appendix A, Table 6.

Proposition 14. *For distinct $t, u, v \in V(T_{11})$,*

$$\begin{aligned} N^+(t) \cap N^+(u) &\neq N^+(t) \cap N^+(v) \\ N^-(t) \cap N^-(u) &\neq N^-(t) \cap N^-(v) \\ N^+(t) \cap N^-(u) &\neq N^+(t) \cap N^-(v) \\ N^-(t) \cap N^+(u) &\neq N^-(t) \cap N^+(v) \end{aligned}$$

Proposition 15. *For distinct $t, v \in V(T_{11})$,*

$$\begin{aligned} |N^+(t) \cap N^+(v)| = 2, \\ |N^-(t) \cap N^-(v)| = 2, \\ 2 \leq |N^+(t) \cap N^-(v)| \leq 3, \\ 2 \leq |N^-(t) \cap N^+(v)| \leq 3. \end{aligned}$$

Proposition 16. *Let t, u, v be distinct vertices in T_{11}. For each orientation of a 2-path there are at least three such 2-paths from $\{t\}$ to $\{u, v\}$ whose middle vertices are distinct. In particular,*

$$\begin{aligned} |N^+(t) \cap (N^+(v) \cup N^+(u))| \geq 3, \\ |N^-(t) \cap (N^+(v) \cup N^+(u))| \geq 3, \\ |N^-(t) \cap (N^-(v) \cup N^-(u))| \geq 3, \\ |N^+(t) \cap (N^-(v) \cup N^-(u))| \geq 3. \end{aligned}$$

Proof. We consider the case when $t = 0$ and only consider one of the inequalities since T_{11} is vertex transitive and the proofs of the other inequalities are analogous. Note that

$$N^+(t) \cap (N^+(v) \cup N^+(u)) = (N^+(t) \cap N^+(v)) \cup (N^+(t) \cap N^+(u)).$$

Proposition 15 lets us conclude that $|N^+(t) \cap N^+(v))| \geq 2$ and $|N^+(t) \cap N^+(u))| \geq 2$. Proposition 14 tells us that $N^+(t) \cap N^+(v) \neq N^+(t) \cap N^+(u)$

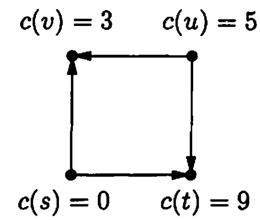

Figure 10: Injective T_{11}-colouring with two vertices of indegree two.

and thus there is at least one element in $N^+(t) \cap N^+(u)$ that is not in $N^+(t) \cap N^+(u)$. Combining the two facts gives

$$|(N^+(t) \cap N^+(v)) \cup (N^+(t) \cap N^+(u))| \geq 3.$$

□

8 Bounds on $\vec{\chi_i}(G(n,3))$

By proceeding with induction on the number of rows of $\vec{G}(n,3)$, an injective T_{11}-colouring is provided showing that $\vec{\chi_i}(G(n,3)) \leq 11$ for all $n \geq 1$. Combining these results with an orientation of $G(7,3)$ found in [14] whose oriented colouring requires seven colours, gives $7 \leq \vec{\chi_i}(G(n,3)) \leq 11$ for $n \geq 7$, since an injective colouring is an oriented colouring.

We first consider injective T_{11}-colourings of 4-cycles, as these colourings will be used to colour $\vec{G}(2,3)$.

Lemma 17. *There exists an injective T_{11} – colouring of any oriented 4-cycle.*

Proof. There are three cases to consider for an injective T_{11}-colouring of an oriented 4-cycle: a 4-cycle with two vertices of indegree two, one vertex of indegree two, or no vertices of indegree two.

Let $C = stuvs$ be a 4-cycle. If C is oriented so that there are two vertices with indegree two, then these vertices are not adjacent, and so without loss of generality assume $d^-(v) = d^-(t) = 2$, as illustrated in Figure 10. Then $c(s) = 0$, $c(t) = 9$, $c(u) = 5$ and $c(v) = 3$ is an injective T_{11}-colouring of C.

If C is oriented so that there is one vertex with indegree two, then without loss of generality assume that v is the vertex of indegree two. There are two non-isomorphic orientations to consider as depicted in Figure 11. For Figure 11(a), $c(s) = 0$, $c(t) = 2$, $c(u) = 1$, and $c(v) = 4$ is an injective

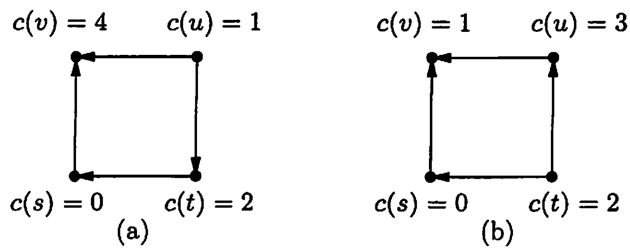

Figure 11: Injective T_{11}-colouring with one vertex of indegree two.

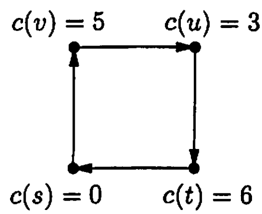

Figure 12: Injective T_{11}-colouring with no vertices of indegree two.

T_{11}-colouring, while for Figure 11(b), $c(s) = 0$, $c(t) = 2$, $c(u) = 3$, and $c(v) = 1$ is an injective T_{11}-colouring of C.

Finally, if C is oriented so that there are no vertices of indegree two, then C is a directed cycle as shown in Figure 12, and $c(s) = 0$, $c(t) = 6$, $c(u) = 3$ and $c(v) = 5$ is an injective T_{11}-colouring of C.

□

Theorem 18. *For all $n \geq 1$, $\vec{\chi_i}(G(n,3)) \leq 11$.*

Proof. For convenience, let $x_i = (i,1)$, $y_i = (i,2)$, $z_i = (i,3)$ refer to row i of $G(n,3)$.

When $n = 1$, $x_1y_1z_1$ is an oriented path. Since $d^+(u) = d^-(u) = 5$ for all $u \in V(T_{11})$, by Lemma 9 there is an injective T_{11}-colouring of $\vec{G}(1,3)$.

For $n = 2$, begin with an injective T_{11}-colouring of the 4-cycle $y_1y_2z_2z_1y_1$. Next, find the colour choices of x_2. If $\overrightarrow{y_2x_2} \in A(\vec{G}(2,3))$, let $S = N^+(c(y_2)) \setminus \{c(y_1)\}$ be the colour choices for x_2. Similarly, if $\overrightarrow{x_2y_2} \in A(\vec{G}(2,3))$, let $S = N^-(c(y_2)) \setminus \{c(z_2), c(y_1)\}$. Since $d^-(v) = d^+(v) = 5$ for all $v \in V(T_{11})$, $|S| \geq 3$. Next, we colour x_1 and x_2. Proposition 16 guarantees that are paths $a_1\ p_1\ c(y_1)$, $a_2\ p_2\ c(y_1)$ and $a_3\ p_3\ c(y_1)$ in T_{11} with the same orientation as $x_2x_1y_1$ such that $a_1, a_2, a_3 \in S$ and p_1, p_2, p_3 are distinct. Choose

$j \in \{1,2,3\}$ so that $p_j \notin \{c(z_1), c(y_2)\}$. Setting $c(x_2) = a_j$ and $c(x_1) = p_j$ results in an injective T_{11}-colouring of $\vec{G}(2,3)$.

Assume for some $i \geq 2$ that there exists an injective T_{11}-colouring, c, of $\vec{G}(i,3)$. We extend c to an injective T_{11}-colouring of $\vec{G}(i+1,3)$.

Step 1. *Determine, S, the set of possible colours for y_{i+1}.*

If $\overrightarrow{y_i y_{i+1}} \in A(G(i+1,3))$, let $S = N^+(c(y_i)) \setminus \{c(x_i), c(z_i)\}$. Note that $|S| \geq 3$, since $d^+(u) = 5$ for all $u \in V(T_{11})$. Similarly, if $\overrightarrow{y_{i+1} y_i} \in A(G(i+1,3))$, let $S = N^-(c(y_i)) \setminus \{c(x_i), c(z_i), c(y_{i-1})\}$. Note that $|S| \geq 2$, since $d^-(u) = 5$ for all $u \in V(T_{11})$. Write $S = \{a_j \mid 1 \leq j \leq |S|\}$.

Step 2. *Restrict S so that x_{i+1} can be coloured.*

For an oriented colouring of row $i+1$ to be injective, we must take care when colouring x_{i+1} for certain orientations. In particular, an oriented colouring of $\vec{G}(i+1,3)$ may have $c(x_{i+1}) = c(x_{i-1})$ when $x_{i+1}, x_{i-1} \in N^-(x_i)$, or $c(y_i) = c(x_{i+1})$ when $x_{i+1}, y_i \in N^-(y_{i+1})$ or $x_{i+1}, y_i \in N^-(x_i)$. To ensure that the colouring is injective, let

$$P_j := \begin{cases} N^+(c(x_i)) \cap N^+(a_j) & \text{if } \overrightarrow{x_i x_{i+1}}, \overrightarrow{y_{i+1} x_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^+(c(x_i)) \cap N^-(a_j)) \setminus \{c(y_i)\} & \text{if } \overrightarrow{x_i x_{i+1}}, \overrightarrow{x_{i+1} y_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^-(c(x_i)) \cap N^-(a_j)) \setminus \{c(y_i), c(x_{i-1})\} & \text{if } \overrightarrow{x_{i+1} x_i}, \overrightarrow{x_{i+1} y_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^-(c(x_i)) \cap N^+(a_j)) \setminus \{c(y_i), c(x_{i-1})\} & \text{if } \overrightarrow{x_{i+1} x_i}, \overrightarrow{y_{i+1} x_{i+1}} \in A(\vec{G}(i+1,3)), \end{cases}$$

for each j, $1 \leq j \leq |S|$. Since the entries in each column of Table 6 are pairwise distinct, there is at most one j for which $P_j = \emptyset$. If such a j exists, then without loss of generality $j = |S|$, and set $S_x := S \setminus \{a_{|S|}\}$. If no such j exists, then set $S_x := S$.

Step 3. *Restrict S_x so that z_{i+1} can be coloured.*

For an oriented colouring of row $i+1$ to be injective, we must take care when colouring z_{i+1} for certain orientations. In particular, an oriented colouring of $\vec{G}(i+1,3)$ may have $c(z_{i+1}) = c(z_{i-1})$ when $z_{i+1}, z_{i-1} \in N^-(z_i)$, or $c(y_i) = c(z_{i+1})$ when $z_{i+1}, y_i \in N^-(y_{i+1})$ or $z_{i+1}, y_i \in N^-(z_i)$.

To ensure that the colouring is injective, let

$$Q_j := \begin{cases} N^+(c(z_i)) \cap N^+(a_j) & \text{if } \overrightarrow{z_i z_{i+1}}, \overrightarrow{y_{i+1} z_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^+(c(z_i)) \cap N^-(a_j)) \setminus \{c(y_i)\} & \text{if } \overrightarrow{z_i z_{i+1}}, \overrightarrow{z_{i+1} y_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^-(c(z_i)) \cap N^-(a_j)) \setminus \{c(y_i), c(z_{i-1})\} & \text{if } \overrightarrow{z_{i+1} z_i}, \overrightarrow{z_{i+1} y_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^-(c(z_i)) \cap N^+(a_j)) \setminus \{c(y_i), c(z_{i-1})\} & \text{if } \overrightarrow{z_{i+1} z_i}, \overrightarrow{y_{i+1} z_{i+1}} \in A(\vec{G}(i+1,3)), \end{cases}$$

for each for each j, $1 \leq j \leq |S_x|$. Since the entries in each column of Table 6 are pairwise distinct, there is at most one j for which $Q_j = \emptyset$. If such a j exists, then without loss of generality $j = |S_x|$, and we set $S_{xz} := S_x \setminus \{a_{|S_x|}\}$. If no such j exists, then set $S_{xz} := S_x$.

Claim 19. $|S_{xz}| \geq 1$

Proof. Consider two cases depending on the orientation of $y_i y_{i+1}$. If $\overrightarrow{y_i y_{i+1}} \in A(G(i+1,3))$ then $|S| \geq 3$ and the result is immediate because $|S_{xz}| \geq |S| - 2$. Otherwise, $\overrightarrow{y_{i+1} y_i} \in A(G(i+1,3))$. Again, if $|S| \geq 3$ the result is immediate. Assume that $S = \{a_1, a_2\}$, which implies $\overrightarrow{x_i y_i} \in A(G(i+1,3))$. If $P_j \neq \emptyset$ for $j = 1, 2$ then $|S_{xz}| \geq 2 - 1$ and the result follows. Without loss of generality, assume that $P_1 = \emptyset$. Since $P_1 = \emptyset$, we have

$$c(x_{i-1}) \in \begin{cases} N^-(c(x_i)) \cap N^-(a_1) & \text{if } \overrightarrow{x_{i+1} x_i}, \overrightarrow{x_{i+1} y_{i+1}} \in A(\vec{G}(i+1,3)), \\ N^-(c(x_i)) \cap N^+(a_1) & \text{if } \overrightarrow{x_{i+1} x_i}, \overrightarrow{y_{i+1} x_{i+1}} \in A(\vec{G}(i+1,3)). \end{cases}$$

In either case, $x_{i+1} x_i y_i$ is a directed 2-path and

$$c(y_i) \notin \begin{cases} N^-(c(x_i)) \cap N^-(a_1) & \text{if } \overrightarrow{x_{i+1} x_i}, \overrightarrow{x_{i+1} y_{i+1}} \in A(\vec{G}(i+1,3)), \\ N^-(c(x_i)) \cap N^+(a_1) & \text{if } \overrightarrow{x_{i+1} x_i}, \overrightarrow{y_{i+1} x_{i+1}} \in A(\vec{G}(i+1,3)). \end{cases}$$

This contradicts the assumption that $P_1 = \emptyset$, and thus we conclude that $|S_{xz}| \geq 1$. □

Step 4. *Colour row* $(i+1)$.

If $\overrightarrow{y_{i+1} x_{i+1}} \in A(\vec{G}(i+1,3))$ or $\overrightarrow{y_{i+1} z_{i+1}} \in A(\vec{G}(i+1,3))$ then by setting $c(y_{i+1}) = a_1$, assigning x_{i+1} a colour from P_1 and z_{i+1} a colour from Q_1, we obtain an injective colouring of $\vec{G}(i+1,3)$. This is possible since $|S_{xz}| \geq 1$.

Otherwise, $\overrightarrow{x_{i+1} y_{i+1}}, \overrightarrow{z_{i+1} y_{i+1}} \in A(\vec{G}(i+1,3)$. First suppose that $|P_j| \geq 2$ or $|Q_j| \geq 2$ for some j. Without loss of generality, assume that $|P_j| \geq 2$ for some j. Set $c(y_{i+1}) = a_j$, assign z_{i+1} a colour from Q_j and x_{i+1} a colour from $P_j \setminus \{c(z_{i+1})\}$. The result is an injective colouring of $\vec{G}(i+1,3)$.

If $|P_j| = |Q_j| = 1$ for all j, $1 \leq j \leq |S_{xz}|$ then, if possible, choose j so that $P_j \neq Q_j$. Set $c(y_{i+1}) = a_j$, assign z_{i+1} the colour from Q_j and x_{i+1} the colour from P_j. This results in an injective T_{11}-colouring.

The only thing left to consider is when $|P_j| = |Q_j| = 1$ and $P_j = Q_j = \{r_j\}$ for all j, $1 \leq j \leq |S_{xz}|$. Recall $\overrightarrow{x_{i+1}y_{i+1}} \in A(\vec{G}(i+1,3)$,

$$P_j := \begin{cases} (N^+(c(x_i)) \cap N^-(a_j)) \setminus \{c(y_i)\} & \text{if } \overrightarrow{x_ix_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^-(c(x_i)) \cap N^-(a_j)) & \text{if } \overrightarrow{x_{i+1}x_i} \in A(\vec{G}(i+1,3)). \\ \quad \setminus\{c(y_i), c(x_{i-1})\} & \end{cases}$$

Similarly, $\overrightarrow{z_{i+1}y_{i+1}} \in A(\vec{G}(i+1,3)$ so

$$Q_j := \begin{cases} (N^+(c(z_i)) \cap N^-(a_j)) \setminus \{c(y_i)\} & \text{if } \overrightarrow{z_iz_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^-(c(z_i)) \cap N^-(a_j)) & \text{if } \overrightarrow{z_{i+1}z_i} \in A(\vec{G}(i+1,3)). \\ \quad \setminus\{c(y_i), c(z_{i-1})\} & \end{cases}$$

We look at two cases based on the orientation of y_iy_{i+1}.

Case 1: $\overrightarrow{y_{i+1}y_i} \in A(\vec{G}(i+1,3))$.

As $x_{i+1}y_{i+1}y_i$ is a directed 2-path from x_{i+1} to y_i, it is easy to observe that

$$c(y_i) \notin \begin{cases} N^-(c(x_i)) \cap N^-(a_j) & \text{if } \overrightarrow{x_{i+1}x_i} \in A(\vec{G}(i+1,3)), \\ N^+(c(x_i)) \cap N^-(a_j) & \text{if } \overrightarrow{x_ix_{i+1}} \in A(\vec{G}(i+1,3)), \end{cases}$$

for all j, $1 \leq j \leq |S|$. Using the fact that $|P_j| = 1$, we conclude that $\overrightarrow{x_{i+1}x_i}, \overrightarrow{x_{i-1}x_i} \in A(\vec{G}(i+1,3))$. In addition, $c(y_i) \notin N^-(c(x_i)) \cap N^-(a_j)$ for all j, $1 \leq j \leq |S|$, implies that $P_j \neq \emptyset$ for all $j \leq |S|$ and hence $S_x = S$. Similarly, as $z_{i+1}y_{i+1}y_i$ is a directed 2-path from z_{i+1} to y_i, it is easy to observe that

$$c(y_i) \notin \begin{cases} N^-(c(z_i)) \cap N^-(a_j) & \text{if } \overrightarrow{z_{i+1}z_i} \in A(\vec{G}(i+1,3)), \\ N^+(c(z_i)) \cap N^-(a_j) & \text{if } \overrightarrow{z_iz_{i+1}} \in A(\vec{G}(i+1,3)), \end{cases}$$

for all j, $1 \leq j \leq |S|$. Therefore, $|Q_j| = 1$ implies that $\overrightarrow{z_{i+1}z_i}, \overrightarrow{z_{i-1}z_i} \in A(\vec{G}(i+1,3))$. Note that $c(y_i) \notin N^-(c(z_i)) \cap N^-(a_j)$ implies that $Q_j \neq \emptyset$ for any j, $1 \leq j \leq |S_x|$. We conclude that $S_{xz} = S_x = S$ and hence $|S_{xz}| \geq 2$.

Suppose that $c(x_{i-1}) \neq c(z_{i-1})$. Repeatedly using the fact that for all distinct pairs $\{t, u\} \subseteq V(T_{11})$, $|N^-(t) \cap N^-(u)| = 2$ (Proposition 15) we have:

$$\begin{aligned}
N^-(a_1) &= \{c(x_{i-1}), c(z_{i-1}), r_1, v_1, v_2\} \\
N^-(a_2) &= \{c(x_{i-1}), c(z_{i-1}), r_2, v_3, v_4\} \\
N^-(c(x_i)) &= \{c(x_{i-1}), r_1, r_2, v_5, v_6\} \\
N^-(c(z_i)) &= \{c(z_{i-1}), r_1, r_2, v_7, v_8\}
\end{aligned}$$

If $r_1, r_2, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, c(x_{i-1}), c(z_{i-1})$ are distinct vertices of T_{11}, then $|V(T_{11})| \geq 12$, clearly a contradiction.

Since all the pairs in Appendix A, Table 6, column 4 are distinct we have that $r_1, r_2, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are distinct. Therefore we are left with $c(x_{i-1}) = c(z_{i-1})$. Since $\{c(x_{i-1}), r_1, r_2\} \subseteq N^-(c(x_i)) \cap N^-(c(z_i))$ and $|N^-(c(x_i)) \cap N^-(c(z_i))| = 2$ we conclude that $c(x_i) = c(z_i)$. The only way for $c(x_i) = c(z_i)$ is to have $\overrightarrow{y_i x_i}, \overrightarrow{y_i z_i} \in A(\vec{G}(i+1, 3))$, since $\vec{G}(i, 3)$ has been injectively coloured. Moreover, all edges are oriented inwards to x_i, so x_i is not an in-neighbour of any vertex. We recolour x_i respecting only the homomorphism, since this will necessarily produce an injective colouring. By Proposition 13, there are paths $c(x_{i-1})\ p_1\ c(y_i)$ and $c(x_{i-1})\ p_2\ c(y_i)$ in T_{11} having the same orientation as $x_{i-1}x_iy_i$ such that $p_1 \neq p_2$. Recolour x_i with p_1 or p_2 so that it is different from $c(z_i)$. Returning to Step 1 and following the prescribed algorithm will successfully produce an injective colouring; if it did not, then the new colours $r_1, r_2, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, c(x_{i-1}), c(z_{i-1})$ would all be distinct, which contradicts the fact that $|V(T_{11})| = 11$.

Case 2: $\overrightarrow{y_i y_{i+1}} \in A(\vec{G}(i+1, 3))$. Notice that y_{i+1} is not an in-neighbour of any vertex.

(a) First, consider the case when there is no directed 2-path between x_i and y_{i+1} or between z_i and y_{i+1}. Without loss of generality, assume that there is no directed 2-path between x_i and y_{i+1}. Then $\overrightarrow{x_{i+1} x_i}, \overrightarrow{y_i x_i} \in A(\vec{G}(i+1, 3))$, as depicted in Figure 13.

Begin by colouring y_{i+1} with $c(x_i)$. Let

$$Q := \begin{cases} (N^-(c(x_i)) \cap N^+(c(z_i))) \setminus \{c(y_i)\} & \text{if } \overrightarrow{z_i z_{i+1}} \in A(\vec{G}(i+1, 3)), \\ (N^-(c(x_i)) \cap N^-(c(z_i))) \\ \quad \setminus \{c(y_i), c(z_{i-1})\} & \text{if } \overrightarrow{z_{i+1} z_i} \in A(\vec{G}(i+1, 3)). \end{cases}$$

If $Q \neq \emptyset$, then colour z_{i+1} with any colour in Q and x_{i+1} with any colour in $N^-(c(x_i)) \setminus \{c(x_{i-1}), c(y_i), c(z_{i+1})\}$. This is an injective T_{11}-colouring of $\vec{G}(i+1, 3)$.

If $Q = \emptyset$, then $N^-(c(x_i)) \cap N^-(c(z_i)) = \{c(y_i), c(z_{i-1})\}$ which implies that $\overrightarrow{z_{i+1} z_i}, \overrightarrow{y_i z_i}, \overrightarrow{z_{i-1} z_i} \in A(\vec{G}(i+1, 3))$. Notice that there is no directed

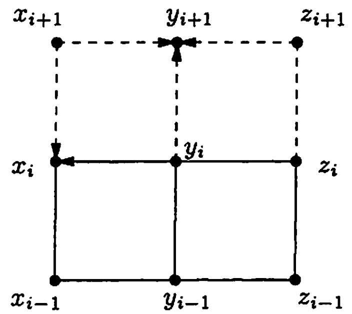

Figure 13: there is no directed 2-path between x_i and y_{i+1}.

2-path between y_{i+1} and z_i. Change the colour of y_{i+1} to $c(z_i)$ and let

$$P := \quad (N^-(c(x_i)) \cap N^-(c(z_i))) \setminus \{c(y_i), c(x_{i-1})\}.$$

We show that $P \neq \emptyset$. Assume to the contrary that $P = \emptyset$ and $Q = \emptyset$. Since $Q = \emptyset$, $N^-(c(x_i)) \cap N^-(c(z_i)) = \{c(y_i), c(z_{i-1})\}$. Similarly, since $P = \emptyset$, $N^-(c(x_i)) \cap N^-(c(z_i)) = \{c(y_i), c(x_{i-1})\}$. These imply that $\{c(y_i), c(x_{i-1})\} = \{c(y_i), c(z_{i-1})\}$ and so $c(x_{i-1}) = c(z_{i-1})$. Since $P_1 = Q_1 = \{r_1\}$, $r_1 \in N^-(c(x_i)) \cap N^-(c(z_i)) = \{c(y_i), c(x_{i-1})\}$. However, by the definition of P_1, $r_1 \notin \{c(y_i), c(x_{i-1})\}$. We conclude that either $P \neq \emptyset$ or $Q \neq \emptyset$.

Colour x_{i+1} with any colour in P and z_{i+1} with any colour in $N^-(c(z_i)) \setminus \{c(z_{i-1}), c(y_i), c(x_{i+1})\}$. This is an injective T_{11}-colouring of $G(i+1, 3)$.

(b) Finally, assume that there is a directed 2-path between x_i and y_{i+1} and a directed 2-path between z_i and y_{i+1}. Since $N^+(y_{i+1}) = \emptyset$, such paths go from x_i to y_{i+1} and from z_i to y_{i+1}. We claim that a directed 2-path from x_i and y_{i+1} uses $\overrightarrow{x_iy_i} \in A(\vec{G}(i+1, 3)$, and that a directed 2-path from z_i to y_{i+1} uses $\overrightarrow{z_iy_i} \in A(\vec{G}(i+1, 3)$. In addition, we show that $|S_{xz}| = 5$ which leads us to conclude $c(x_i) = c(z_i)$. This contradicts the induction assumption that row i is injectively coloured.

If $\overrightarrow{x_ix_{i+1}} \in A(\vec{G}(i+1, 3)$, then using the fact $|P_j| = 1$, $c(y_i) \in N^+(c(x_i)) \cap N^-(a_j)$ and so $\overrightarrow{x_iy_i} \in A(\vec{G}(i+1, 3))$. It follows that $S_x = S$.

If $\overrightarrow{x_{i+1}x_i} \in A(\vec{G}(i+1, 3)$, then $\overrightarrow{x_iy_i} \in A(\vec{G}(i+1, 3)$ because there has to be a directed 2-path from x_i to y_{i+1}. This implies that $c(y_i) \notin N^-(c(x_i)) \cap N^-(a_j)$ for all j, $1 \leq j \leq |S|$, since $x_{i+1}x_iy_i$ is a directed path. Therefore, $S_x = S$.

An analogous argument shows that $\overrightarrow{z_iy_i} \in A(\vec{G}(i+1, 3))$ and $S_{xz} = S_x$. We conclude that $S_{xz} = S_x = S$ and $|S_{xz}| = 5$.

Since $\overrightarrow{x_iy_i}, \overrightarrow{z_iy_i} \in A(\vec{G}(i+1,3))$, it follows that

$$P_j := \begin{cases} (N^+(c(x_i)) \cap N^-(a_j)) \setminus \{c(y_i)\} & \text{if } \overrightarrow{x_ix_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^-(c(x_i)) \cap N^-(a_j)) \setminus \{c(x_{i-1})\} & \text{if } \overrightarrow{x_{i+1}x_i} \in A(\vec{G}(i+1,3)), \end{cases}$$

$$Q_j := \begin{cases} (N^+(c(z_i)) \cap N^-(a_j)) \setminus \{c(y_i)\} & \text{if } \overrightarrow{z_iz_{i+1}} \in A(\vec{G}(i+1,3)), \\ (N^-(c(z_i)) \cap N^-(a_j)) \setminus \{c(z_{i-1})\} & \text{if } \overrightarrow{z_{i+1}z_i} \in A(\vec{G}(i+1,3)). \end{cases}$$

Recall that $P_j = Q_j = \{r_j\}$ for $1 \le j \le |S_{xz}| = 5$. We will show by contradiction that $P_j \neq Q_j$ for some $1 \le j \le |S_{xz}| = 5$. Assume to the contrary that $P_j = Q_j = \{r_j\}$ for all1 $\le j \le |S_{xz}| = 5$ and begin by showing that we cannot have all r_j's distinct. If they were all distinct then

$$\{r_1, r_2, r_3, r_4, r_5\} = \begin{cases} N^+(c(x_i)) & \text{if } \overrightarrow{x_ix_{i+1}} \in A(\vec{G}(i+1,3)), \\ N^-(c(x_i)) & \text{if } \overrightarrow{x_{i+1}x_i} \in A(\vec{G}(i+1,3)), \end{cases}$$

and

$$\{r_1, r_2, r_3, r_4, r_5\} = \begin{cases} N^+(c(z_i)) & \text{if } \overrightarrow{z_iz_{i+1}} \in A(\vec{G}(i+1,3)), \\ N^-(c(z_i)) & \text{if } \overrightarrow{z_{i+1}z_i} \in A(\vec{G}(i+1,3)). \end{cases}$$

By observing Appendix A, Table 5 this can only occur if $c(x_i) = c(z_i)$. This contradicts our assumption that row i was injectively coloured. Therefore, not all the r_j's are distinct and we may assume that $r_s = r_t$ for some $1 \le s < t \le 5$. If $\overrightarrow{x_ix_{i+1}} \in A(\vec{G}(i+1,3))$, then $N^+(c(x_i)) \cap N^-(a_s) = N^+(c(x_i)) \cap N^-(a_t) = \{c(y_i), r_s\}$, a contradiction of Proposition 14. If $\overrightarrow{x_{i+1}x_i} \in A(\vec{G}(i+1,3))$ then $N^-(c(x_i)) \cap N^-(a_s) = N^-(c(x_i)) \cap N^-(a_t) = \{r_s, c(x_{i-1})\}$, also a contradiction of Proposition 14. Therefore, it must be the case that for some j, $1 \le j \le S_{xz}$, we don't have $P_j = Q_j = \{r_j\}$ and hence row $i+1$ would have been injectively coloured.

Therefore, an injective T_{11} colouring of $\vec{G}(i,3)$ can be extended to an injective T_{11} colouring of $\vec{G}(i+1,3)$. We conclude that for all $n \ge 1$, $\vec{\chi}_i(G(n,3)) \le 11$.

□

9 Conclusion/Summary

We've shown that $\vec{\chi}_i(G(n,m)) \le 23$, something that is not discussed in any other literature, so that we can conclude that $7 \le \vec{\chi}_i(G(n,m)) \le 23$ for $n \ge 5$ and $m \ge 4$, or $n \ge 7$ and $m \ge 3$. That bound is improved by considering specific dimensions of grid graphs. In particular, we show that $6 \le \vec{\chi}_i(G(n,2)) \le 9$ for $n \ge 4$, and $7 \le \vec{\chi}_i(G(n,3)) \le 11$ for $n \ge 7$. Providing orientations of grid graphs to improve the lower bounds of the injective oriented chromatic number of grid graphs remains open.

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A Neighbourhoods of T_7, T_9, T_{11} and T_{23}.

u	$N^+(u)$	$N^-(u)$
0	{1,2,4}	{3,5,6}
1	{2,3,5}	{0,4,6}
2	{3,4,6}	{0,1,5}
3	{0,4,5,}	{1,2,6}
4	{1,5,6}	{0,2,3}
5	{0,2,6}	{1,3,4}
6	{0,1,3}	{2,4,5}

Table 1: In-neighbours and out-neighbours in T_7.

Since all the target graphs are vertex transitive, we consider only the vertex 0 when verifying properties.

	Column 1	Column 2	Column 3	Column 4
Vertex v	$N^+(0) \cap N^+(v)$	$N^+(0) \cap N^-(v)$	$N^-(0) \cap N^-(v)$	$N^-(0) \cap N^+(v)$
1	{2}	{4}	{6}	{3,5}
2	{4}	{1}	{5}	{3,6}
3	{4}	{1,2}	{6}	{5}
4	{1}	{2}	{3}	{5,6}
5	{2}	{1,4}	{3}	{6}
6	{1}	{2,4}	{5}	{3}

Table 2: Neighbourhood intersections with vertex 0 in T_7.

u	$N^+(u)$	$N^-(u)$
0	{1,2,3,5}	{4,6,7,8}
1	{2,3,4,6}	{0,5,7,8}
2	{3,4,5,7}	{0,1,6,8}
3	{4,5,6,8}	{0,1,2,7}
4	{0,5,6,7}	{1,2,3,8}
5	{1,6,7,8}	{0,2,3,4}
6	{0,2,7,8}	{1,3,4,5}
7	{0,1,3,8}	{2,4,5,6}
8	{0,1,2,4}	{3,5,6,7}

Table 3: In-neighbours and out-neighbours in T_9.

	Column 1	Column 2	Column 3	Column 4
Vertex v	$N^+(0) \cap N^+(v)$	$N^+(0) \cap N^-(v)$	$N^-(0) \cap N^-(v)$	$N^-(0) \cap N^+(v)$
1	{2,3}	{5}	{7,8}	{4,6}
2	{3,5}	{1}	{6,8}	{4,7}
3	{5}	{1,2}	{7}	{4,6,8}
4	{5}	{1,2,3}	{8}	{6,7}
5	{1}	{2,3}	{4}	{6,7,8}
6	{2}	{1,3,5}	{4}	{7,8}
7	{1,3}	{2,5}	{4,6}	{8}
8	{1,2}	{3,5}	{6,7}	{4}

Table 4: Neighbourhood intersections with vertex 0 in T_9.

u	$N^+(u)$	$N^-(u)$
0	{1,3,4,5,9}	{2,6,7,8,10}
1	{2,4,5,6,10}	{0,3,7,8,9}
2	{0,3,5,6,7}	{1,4,8,9,10}
3	{1,4,6,7,8}	{0,2,5,9,10}
4	{2,5,7,8,9}	{0,1,3,6,10}
5	{3,6,8,9,10}	{0,1,2,4,7}
6	{0,4,7,9,10}	{1,2,3,5,8}
7	{0,1,5,8,10}	{2,3,4,6,9}
8	{0,1,2,6,9}	{3,4,5,7,10}
9	{1,2,3,7,10}	{0,4,5,6,8}
10	{0,2,3,4,8}	{1,5,6,7,9}

Table 5: In-neighbours and out-neighbours in T_{11}.

	Column 1	Column 2	Column 3	Column 4
Vertex v	$N^+(0) \cap N^+(v)$	$N^+(0) \cap N^-(v)$	$N^-(0) \cap N^+(v)$	$N^-(0) \cap N^-(v)$
1	{4,5}	{3,9}	{2,6,10}	{7,8}
2	{3,5}	{1,4,9}	{6,7}	{8,10}
3	{1,4}	{5,9}	{6,7,8}	{2,10}
4	{5,9}	{1,3}	{2,7,8}	{6,10}
5	{3,9}	{1,4}	{6,8,10}	{2,7}
6	{4,9}	{1,3,5}	{7,10}	{2,8}
7	{1,5}	{3,4,9}	{8,10}	{2,6}
8	{1,9}	{3,4,5}	{2,6}	{7,10}
9	{1,3}	{4,5}	{2,7,10}	{6,8}
10	{3,4}	{1,5,9}	{2,8}	{6,7}

Table 6: Neighbourhood intersections with vertex 0 in T_{11}.

u	$N^+(u)$	$N^-(u)$
0	{1,2,3,4,6,8,9,12,13,16,18}	{5,7,10,11,14,15,17,19,20,21,22}
1	{2,3,4,5,7,9,10,13,14,17,19}	{0,6,8,11,12,15,16,18,20,21,22}
2	{3,4,5,6,8,10,11,14,15,18,20}	{0,1,7,9,12,13,16,17,19,21,22}
3	{4,5,6,7,9,11,12,15,16,19,21}	{0,1,2,8,10,13,14,17,18,20,22}
4	{5,6,7,8,10,12,13,16,17,20,22}	{0,1,2,3,9,11,14,15,18,19,21}
5	{0,6,7,8,9,11,13,14,17,18,21}	{1,2,3,4,10,12,15,16,19,20,22,}
6	{1,7,8,9,10,12,14,15,18,19,22}	{0,2,3,4,5,11,13,16,17,20,21}
7	{0,2,8,9,10,11,13,15,16,19,20}	{1,3,4,5,6,12,14,17,18,21,22}
8	{1,3,9,10,11,12,14,16,17,20,21}	{0,2,4,5,6,7,13,15,18,19,22}
9	{2,4,10,11,12,13,15,17,18,21,22}	{0,1,3,5,6,7,8,14,16,19,20}
10	{0,3,5,11,12,13,14,16,18,19,22}	{1,2,4,6,7,8,9,15,17,20,21}
11	{0,1,4,6,12,13,14,15,17,19,20}	{2,3,5,7,8,9,10,16,18,21,22}
12	{1,2,5,7,13,14,15,16,18,20,21}	{0,3,4,6,8,9,10,11,17,19,22}
13	{2,3,6,8,14,15,16,17,19,21,22}	{0,1,4,5,7,9,10,11,12,18,20}
14	{0,3,4,7,9,15,16,17,18,20,22}	{1,2,5,6,8,10,11,12,13,19,21}
15	{0,1,4,5,8,10,16,17,18,19,21}	{2,3,6,7,9,11,12,13,14,20,22}
16	{1,2,5,6,9,11,17,18,19,20,22}	{0,3,4,7,8,10,12,13,14,15,21}
17	{0,2,3,6,7,10,12,18,19,20,21}	{1,4,5,8,9,11,13,14,15,16,22}
18	{1,3,4,7,8,11,13,19,20,21,22}	{0,2,5,6,9,10,12,14,15,16,17}
19	{0,2,4,5,8,9,12,14,20,21,22}	{1,3,6,7,10,11,13,15,16,17,18}
20	{0,1,3,5,6,9,10,13,15,21,22}	{2,4,7,8,11,12,14,16,17,18,19}
21	{0,1,2,4,6,7,10,11,14,16,22}	{3,5,8,9,12,13,15,17,18,19,20}
22	{0,1,2,3,5,7,8,11,12,15,17}	{4,6,9,10,13,14,16,18,19,20,21}

Table 7: In-neighbours and out-neighbours in T_{23}.

	Column 1	Column 2	Column 3	Column 4
Vertex v	$N^+(0) \cap N^+(v)$	$N^+(0) \cap N^-(v)$	$N^-(0) \cap N^+(v)$	$N^-(0) \cap N^-(v)$
1	{2,3,4,9,13}	{6,8,12,16,18}	{5,7,10,14,17,19}	{11,15,20,21,22}
2	{3,4,6,8,18}	{1,9,12,13,16}	{5,10,11,14,15,20}	{7,17,19,21,22}
3	{4,6,9,12,16}	{1,2,8,13,18}	{5,7,11,15,19,21}	{10,14,17,20,22}
4	{6,8,12,13,16}	{1,2,3,9,18}	{5,7,10,17,20,22}	{11,14,15,19,21}
5	{6,8,9,13,18}	{1,2,3,4,12,16}	{7,11,14,17,21}	{10,15,19,20,22}
6	{1,8,9,12,18}	{2,3,4,13,16}	{7,10,14,15,19,22}	{5,11,17,20,21}
7	{2,8,9,13,16}	{1,3,4,6,12,18}	{10,11,15,19,20}	{5,14,17,21,22}
8	{1,3,9,12,16}	{2,4,6,13,18}	{10,11,14,17,20,21}	{5,7,15,19,22}
9	{2,4,12,13,18}	{1,3,6,8,16}	{10,11,15,17,21,22}	{5,7,14,19,20}
10	{3,12,13,16,18}	{1,2,4,6,8,9}	{5,11,14,19,22}	{7,15,17,20,21}
11	{1,4,6,12,13}	{2,3,8,9,16,18}	{14,15,17,19,20}	{5,7,10,21,22}
12	{1,2,13,16,18}	{3,4,6,8,9}	{5,7,14,15,20,21}	{10,11,17,19,22}
13	{2,3,6,8,16}	{1,4,9,12,18}	{14,15,17,19,21,22}	{5,7,10,11,20}
14	{3,4,9,16,18}	{1,2,6,8,12,13}	{7,15,17,20,22}	{5,10,11,19,21}
15	{1,4,8,16,18}	{2,3,6,9,12,13}	{5,10,17,19,21}	{7,11,14,20,22}
16	{1,2,6,9,18}	{3,4,8,12,13}	{5,11,17,19,20,22}	{7,10,14,15,21}
17	{2,3,6,12,18}	{1,4,8,9,13,16}	{7,10,19,20,21}	{5,11,14,15,22}
18	{1,3,4,8,13}	{2,6,9,12,16}	{7,11,19,20,21,22}	{5,10,14,15,17}
19	{2,4,8,9,12}	{1,3,6,13,16,18}	{5,14,20,21,22}	{7,10,11,15,17}
20	{1,3,6,9,13}	{2,4,8,12,16,18}	{5,10,15,21,22}	{7,11,14,17,19}
21	{1,2,4,6,16}	{3,8,9,12,13,18}	{7,10,11,14,22}	{5,15,17,19,20}
22	{1,2,3,8,12}	{4,6,9,13,16,18}	{5,7,11,15,17}	{10,14,19,20,21}

Table 8: Neighbourhood intersections with vertex 0 in T_{23}.