Independence number and connectivity for fractional ID-k-factor-critical graphs *†

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Abstract

Let G be a graph, and k a positive integer. A graph G is fractional independent-set-deletable k-factor-critical (in short, fractional ID-k-factor-critical) if $G - I$ has a fractional k-factor for every independent set I of G. In this paper, it is proved that if $\kappa(G) \geq \max\{\frac{k^2+6k+1}{2}, \frac{(k^2+6k+1)\alpha(G)}{4k}\}$, then G is fractional ID-k-factor-critical.

Keywords: graph, independence number, connectivity, fractional k-factor, fractional ID-k-factor-critical graph.

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1 Introduction

In this paper we consider only finite undirected graphs which have neither multiple edges nor loops. Let G be a graph. We denote by $V(G)$ and $E(G)$

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its vertex set and edge set, respectively. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of vertices adjacent to v in G and $d_G(v) = |N_G(v)|$ be the degree of v in G. We write $N_G[v]$ for $N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S,T)$ the number of edges joining S and T, and set $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$. We use $\alpha(G)$ and $\kappa(G)$ to denote the independence number and the connectivity of G, respectively.

Let $k \geq 1$ be an integer. A spanning subgraph F of G such that $d_F(x) = k$ for each $x \in V(G)$ is called a k-factor of G. If $k = 1$, then we say that a 1-factor is a perfect matching. A graph G is factor-critical [1] if $G - v$ has a perfect matching for every $v \in V(G)$. In [2], the concept of the factor-critical graph was generalized to the ID-factor-critical graph. A graph G is independent-set-deletable factor-critical (shortly, ID-factor-critical) if for every independent set I of G which has the same parity with $|V(G)|$, $G - I$ has a perfect matching. Apparently, every ID-factor-critical graph with odd vertices is factor-critical.

Let $h : E(G) \to [0,1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for each $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. A fractional 1-factor is also called a fractional perfect matching. A graph G is fractional independent-set-deletable k-factor-critical (in short, fractional ID-k-factor-critical) [3] if $G - I$ has a fractional k-factor for every independent set I of G. If $k = 1$, then a fractional ID-k-factor-critical graph is called a fractional ID-factor-critical graph.

Many authors have investigated graph factors [4–9]. Chang, Liu and Zhu [3] showed a minimum degree condition for a graph to be fractional ID-k-factor-critical. Zhou, Xu and Sun [10] obtained an independence number and minimum degree condition for graphs to be fractional ID-k-factor-critical graphs. The following results on fractional ID-k-factor-critical graphs are known.

Theorem 1 (*Chang, Liu and Zhu* [3]). *Let k be a positive integer and G be a graph of order n with $n \geq 6k - 8$. If $\delta(G) \geq \frac{2n}{3}$, then G is fractional ID-k-factor-critical.*

Theorem 2 (*Zhou, Xu and Sun* [10]). *Let G be a graph, and let k be an integer with $k \geq 1$. If*

$$\alpha(G) \leq \frac{4k(\delta(G) - k + 1)}{k^2 + 6k + 1},$$

then G is fractional ID-k-factor-critical.

In this paper, we proceed to study fractional ID-k-factor-critical graphs, and use independence number $\alpha(G)$ and connectivity $\kappa(G)$ to obtain a new sufficient condition for a graph to be fractional ID-k-factor-critical. The main result is the following theorem.

Theorem 3 *Let G be a graph, and let k be a positive integer. If*

$$\kappa(G) \geq \max\{\frac{k^2+6k+1}{2}, \frac{(k^2+6k+1)\alpha(G)}{4k}\},$$

then G is fractional ID-k-factor-critical.

Unfortunately, the authors do not know whether the independence number and connectivity condition in Theorem 3 is best possible or not. Thus, we pose the following conjecture.

Conjecture 1 *Let G be a graph, and let k be a positive integer. If*

$$\kappa(G) \geq \max\{\frac{k^2+6k+1}{2}, \frac{(k^2+6k+1)\alpha(G)}{4k}\} - 1,$$

then G is fractional ID-k-factor-critical.

If G is a complete graph, then G is fractional ID-factor-critical. If G is a non-complete graph, then $\alpha(G) \geq 2$. Combining these with $k = 1$ in Theorem 3, we obtain the following corollary.

Corollary 1 *Let G be a graph. If*

$$\kappa(G) \geq 2\alpha(G),$$

then G is fractional ID-factor-critical.

2 Proof of Theorem 3

Liu and Zhang [11] showed a necessary and sufficient condition for a graph to have a fractional k-factor, which is very useful in the proof of Theorem 3.

Lemma 2.1 (*Liu and Zhang* [11]). *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$.

Proof of Theorem 3. Let I be an independent set of G and $H = G-I$. In order to prove Theorem 3, we need only to prove that H has a fractional k-factor. By contradiction, we suppose that H has no fractional k-factor. Then from Lemma 2.1, there exists some subset $S \subseteq V(H)$ such that

$$\delta_H(S,T) = k|S| + d_{H-S}(T) - k|T| \leq -1, \tag{1}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq k\}$. Obviously, $T \neq \emptyset$ by (1).

Now we take $x_1 \in T$ such that x_1 is the vertex with the least degree in $H[T]$. Let $N_1 = N_H[x_1] \bigcap T$ and $T_1 = T$. For $i \geq 2$, if $T - \bigcup_{1 \leq j < i} N_j \neq \emptyset$, let $T_i = T - \bigcup_{1 \leq j < i} N_j$. Then take $x_i \in T_i$ such that x_i is the vertex with the least degree in $H[T_i]$, and $N_i = N_H[x_i] \bigcap T_i$. We continue these procedures until we reach the situation in which $T_i = \emptyset$ for some i, say for $i = r+1$. Then from the above definition we know that $\{x_1, x_2, \cdots, x_r\}$ is an independent set of H. Since $T \neq \emptyset$, we have $r \geq 1$.

Let $|N_i| = n_i$. From the definition of N_i, we can obtain the following properties ((2) and (3) are trivial; (4) follows because our choice of x_i implies that all vertices in N_i have degree at least $n_i - 1$ in T_i).

$$\alpha(H[T]) \geq r, \tag{2}$$

$$|T| = \sum_{1 \leq i \leq r} n_i, \tag{3}$$

$$\sum_{1 \leq i \leq r} \Big(\sum_{x \in N_i} d_{T_i}(x)\Big) \geq \sum_{1 \leq i \leq r} (n_i^2 - n_i). \tag{4}$$

Let $U = V(H) - (S \bigcup T)$ and $\kappa(H - S) = t$. We now prove the following claims.

Claim 1. $\kappa(G) \geq \frac{(2k-1)(2k+3)}{4k} + \alpha(G)$.

Proof. If $\alpha(G) \geq 2k$, then by the assumption of Theorem 3 we have

$$\begin{aligned}
\kappa(G) &\geq \max\{\frac{k^2+6k+1}{2}, \frac{(k^2+6k+1)\alpha(G)}{4k}\} \\
&\geq \frac{(k^2+6k+1)\alpha(G)}{4k} = \frac{(k^2+2k+1)\alpha(G)}{4k} + \alpha(G) \\
&\geq \frac{k^2+2k+1}{2} + \alpha(G) \geq \frac{(2k-1)(2k+3)}{4k} + \alpha(G).
\end{aligned}$$

If $\alpha(G) < 2k$, then by the assumption of Theorem 3 we obtain

$$\kappa(G) \geq \max\{\frac{k^2+6k+1}{2}, \frac{(k^2+6k+1)\alpha(G)}{4k}\}$$

$$\geq \frac{k^2+6k+1}{2} = \frac{k^2+2k+1}{2} + 2k$$
$$> \frac{k^2+2k+1}{2} + \alpha(G) \geq \frac{(2k-1)(2k+3)}{4k} + \alpha(G).$$

This completes the proof of Claim 1.

Claim 2.
$$\sum_{x \in T} d_{H-S}(x) \geq \sum_{1 \leq i \leq r} (n_i^2 - n_i) + \frac{rt}{2}.$$

Proof. On the left-hand side of (4), an edge joining a vertex x in N_i and a vertex y in N_j $(i < j)$ is counted only once, that is to say, it is counted in the term $d_{T_i}(x)$ but not in the term $d_{T_j}(y)$. From this observation, we have

$$\sum_{x \in T} d_{H-S}(x) \geq \sum_{1 \leq i \leq r} (n_i^2 - n_i) + \sum_{1 \leq i < j \leq r} e_H(N_i, N_j) + e_H(T, U). \quad (5)$$

Since $\kappa(H-S) = t$, we obtain

$$e_H(N_i, \bigcup_{j \neq i} N_j) + e_H(N_i, U) \geq t \quad (6)$$

for each N_i $(1 \leq i \leq r)$. (We do not get (6) in the case where $r = 1$ and $U = \emptyset$. But in that case, we have $\delta_H(S,T) = k|S| + d_{H-S}(T) - k|T| = k|S| + n_1^2 - n_1 - kn_1 \leq -1$ by our choice of x_1, which implies that $|V(H)| = |S| + n_1 \leq \frac{-n_1^2+n_1+kn_1-1}{k} + n_1 = \frac{-n_1^2+(2k+1)n_1-1}{k} \leq \frac{(2k-1)(2k+3)}{4k}$. Note that $H = G - I$ and $|I| \leq \alpha(G)$. Then using Claim 1 we obtain $|V(G)| = |V(H)| + |I| \leq \frac{(2k-1)(2k+3)}{4k} + \alpha(G) \leq \kappa(G) < |V(G)|$, which is a contradiction.) Summing up these inequalities for all i $(1 \leq i \leq r)$, we have

$$\sum_{1 \leq i \leq r} (e_H(N_i, \bigcup_{j \neq i} N_j) + e_H(N_i, U)) = 2 \sum_{1 \leq i < j \leq r} e_H(N_i, N_j) + e_H(T, U) \geq rt. \quad (7)$$

From (7), we obtain

$$\sum_{1 \leq i < j \leq r} e_H(N_i, N_j) + e_H(T, U) \geq \frac{rt}{2}. \quad (8)$$

Using (5) and (8), we get

$$\sum_{x \in T} d_{H-S}(x) \geq \sum_{1 \leq i \leq r} (n_i^2 - n_i) + \frac{rt}{2}.$$

This completes the proof of Claim 2.

According to (1), (3) and Claim 2, we have

$$\begin{aligned} 0 &> -1 \geq \delta_H(S,T) = k|S| + d_{H-S}(T) - k|T| \\ &\geq k|S| + \sum_{1\leq i\leq r} (n_i^2 - n_i) + \frac{rt}{2} - k\sum_{1\leq i\leq r} n_i \\ &= k|S| + \sum_{1\leq i\leq r} (n_i^2 - (k+1)n_i) + \frac{rt}{2}, \end{aligned}$$

that is,

$$0 > k|S| + \sum_{1\leq i\leq r} (n_i^2 - (k+1)n_i) + \frac{rt}{2}. \qquad (9)$$

Let $f(n_i) = n_i^2-(k+1)n_i$. By differentiation we know that the minimum value of $f(n_i)$ is $-\frac{(k+1)^2}{4}$, that is, $n_i^2 - (k+1)n_i \geq -\frac{(k+1)^2}{4}$. Combining this with (9), we have

$$0 > k|S| - \sum_{1\leq i\leq r} \frac{(k+1)^2}{4} + \frac{rt}{2} = k|S| - \frac{(k+1)^2 r}{4} + \frac{rt}{2}. \qquad (10)$$

Note that $H = G - I$. By (2) we obtain

$$\alpha(G) \geq \alpha(H) \geq \alpha(H[T]) \geq r. \qquad (11)$$

Obviously, the following inequalities hold.

$$\kappa(G) \leq |I|+\kappa(G-I) = |I|+\kappa(H) \leq |I|+|S|+\kappa(H-S) = |I|+|S|+t. \qquad (12)$$

From (10) and $|S| \geq 0$, we obtain

$$-\frac{(k+1)^2}{4} + \frac{t}{2} < 0. \qquad (13)$$

In view of (10)–(13), $|I| \leq \alpha(G)$ and the assumption of Theorem 3, we have

$$\begin{aligned} 0 &> k|S| - \frac{(k+1)^2 r}{4} + \frac{rt}{2} \\ &\geq k(\kappa(G) - |I| - t) - \frac{(k+1)^2}{4}\cdot\alpha(G) + \frac{t}{2}\cdot\alpha(G) \\ &\geq k(\kappa(G) - \alpha(G) - t) - \frac{(k+1)^2}{4}\cdot\alpha(G) + \frac{t}{2}\cdot\alpha(G) \end{aligned}$$

$$
\begin{aligned}
&\geq k(\kappa(G) - \frac{4k\kappa(G)}{k^2+6k+1} - t) - \frac{(k+1)^2}{4} \cdot \frac{4k\kappa(G)}{k^2+6k+1} \\
&\quad + \frac{t}{2} \cdot \frac{4k\kappa(G)}{k^2+6k+1} \\
&= kt(\frac{2\kappa(G)}{k^2+6k+1} - 1) + k\kappa(G)(1 - \frac{4k}{k^2+6k+1} - \frac{(k+1)^2}{k^2+6k+1}) \\
&= kt(\frac{2\kappa(G)}{k^2+6k+1} - 1) \\
&\geq kt(\frac{2 \cdot \frac{k^2+6k+1}{2}}{k^2+6k+1} - 1) = 0,
\end{aligned}
$$

which is a contradiction.

From the argument above, we deduce the contradictions. Hence, H has a fractional k-factor, that is, G is fractional ID-k-factor-critical. This completes the proof of Theorem 3.

References

[1] L. Lovasz, M. D. Plummer, Matching Theory, Elsevier Science Publishers, B. V. North Holland, 1986.

[2] J. Yuan, Independent-set-deletable factor-critical power graphs, Acta Mathematica Scientia Ser. B 26(4)(2006), 577–584.

[3] R. Chang, G. Liu, Y. Zhu, Degree conditions of fractional ID-k-factor-critical graphs, Bulletin of the Malaysian Mathematical Sciences Society 33(3)(2010), 355–360.

[4] J. Ekstein, P. Holub, T. Kaiser, L. Xiong, S. Zhang, Star subdivisions and connected even factors in the square of a graph, Discrete Mathematics, in press.

[5] D. Bauer, H. J. Broersma, J. van den Heuvel, N. Kahl, E. Schmeichel, Degree Sequences and the Existence of k-Factors, Graphs and Combinatorics, in press.

[6] T. Nishimura, Independence number, connectivity, and r-factors, Journal of Graph Theory 13(1989), 63–69.

[7] M. Kouider, Z. Lonc, Stability number and $[a, b]$-factors in graphs, Journal of Graph Theory 46(2004), 254–264.

[8] S. Zhou, A sufficient condition for graphs to be fractional (k, m)-deleted graphs, Applied Mathematics Letters 24(9)(2011), 1533–1538.

[9] G. Liu, Q. Yu, L. Zhang, Maximum fractional factors in graphs, Applied Mathematics Letters 20(2007), 1237–1243.

[10] S. Zhou, L. Xu, Z. Sun, Independence number and minimum degree for fractional ID-k-factor-critical graphs, Aequationes Mathematicae 84(1-2)(2012), 71C-76.

[11] G. Liu, L. Zhang, Fractional (g, f)-factors of graphs, Acta Mathematica Scientia 21B(4)(2001), 541–545.