Labeling Hamiltonian Cycles of the Johnson Graph

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Abstract

A non-empty r-element subset A of an n-element set X_n and a partition π of X_n are said to be *orthogonal* if every class of π meets A in exactly one element. A *partition type* is determined by the number of classes of each distinct size of the partition. The Johnson graph $J(n, r)$ is the graph whose vertices are the r-element subsets of X_n, with two sets being adjacent if they intersect in $r - 1$ elements. A partition of a given type τ is said to be a τ-label for an edge AB in $J(n, r)$ if the sets A and B are orthogonal to the partition. A cycle $\mathcal{H}$ in the graph $J(n, r)$ is said to be *τ-labeled* if for every edge of $\mathcal{H}$ there exists a τ-label, and the τ-labels associated with distinct edges are distinct. Labeled Hamiltonian cycles are used to produce minimal generating sets for transformation semigroups. We identify a large class of partition types τ with a non-zero gap for which every Hamiltonian cycle in the graph $J(n, r)$ can be τ-labeled, showing, for example, that this class includes all the partition types with at least one class of size larger than 3 or at least three classes of size 3.

Key words: partition, graph, Hamiltonian cycle, labels.

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1 Introduction

This paper presents combinatorial results concerning interconnections between the graph of all fixed size subsets of a given finite set and partitions of a subset of this set. Let X_n be a non-empty set $\{1, 2, \ldots, n\}$, and let r be a positive integer less than n. The Johnson graph $J(n,r)$ is the graph whose vertices are the r-element subsets of X_n, with two sets being adjacent if they intersect in $r-1$ elements.

For a non-negative integer $g < n$ a partition of an $(n-g)$-element subset of X_n is a *partition of X_n with gap g*. A subset A of X_n and a partition π of X_n are said to be *orthogonal* if every class of π meets A in exactly one element. A partition π of X_n with gap g has type $\tau = d_1{}^{\mu_1} d_2{}^{\mu_2} \ldots d_k{}^{\mu_k}$ if π has μ_i classes of size d_i, where $d_1 > d_2 \ldots > d_k$ for $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} d_i\mu_i = n - g$. The number $r = \sum_{i=1}^{k} \mu_i$ of classes of τ is its *weight*.

A partition γ of a given type τ is said to be a *τ-label* for an edge AB in $J(n,r)$ if the sets A and B both are orthogonal to γ. A cycle $\mathcal{H}$ in the graph $J(n,r)$ is said to be τ-labeled if for every edge of $\mathcal{H}$ there exists a τ-label, and the τ-labels associated with distinct edges of $\mathcal{H}$ are distinct. A *Hamiltonian* path or cycle contains every vertex of the graph exactly once. Hamiltonian cycles of $J(n,r)$ are among the earliest examples of *Gray codes* [9] that were used to minimize errors in certain computer operations.

The τ-labeled Hamiltonian cycles in $J(n,r)$ are used to construct minimal generating sets for semigroups of total transformations of X_n if τ has zero gap [5], and for semigroups of partial transformations of X_n if τ has a non-zero gap [2]. To this end, existence of τ-labeled Hamiltonian cycles in $J(n,r)$ was established in [4] for $\tau = 2^r$ with a non-zero gap and for $\tau = 2^{r-t}1^t$ with zero gap, where $r, t \geq 1$. The authors of [4] conjectured that for any partition type τ of weight r defined on X_n for which there exists at least as many partitions as there are r-element subsets of X_n, there exists an alternating list all the r-element subsets and distinct partitions of type τ such that an adjacent set and a partition are orthogonal to each other. Subsequently, it was shown in [6] that for every partition type τ with zero gap having at least two non-singleton classes there exists a τ-labeled Hamiltonian cycle in $J(n,r)$. For partition types τ with zero gap and a unique non-singleton class, the existence of orthogonally τ-labeled Hamiltonian cycles is equivalent to the difficult and generally yet unsolved Middle Levels problem [10], [6].

Constructions of orthogonally labeled Hamiltonian cycles can be difficult. The authors of [5] identified large classes of partition types τ with zero gap that can be used to orthogonally label every Hamiltonian cycle in $J(n,r)$ for appropriate n and r (see Theorem 4.1 and Remark 5.2 below). They also presented the following problem for partition types with zero gap.

Problem 1 *Characterize all the partition types τ of weight r defined on X_n that can be used to τ-label every Hamiltonian cycle in $J(n, r)$.*

This paper presents a significant contribution to the solution of the problem above. We expand the set of techniques used in [5] and solve several problems posed there. Moreover, we expand the scope of the problem, focusing on the partition types with non-zero gaps.

A partition type τ of weight r defined on X_n is said to be *flexible* if every Hamiltonian cycle of the graph $J(n, r)$ can be τ-labeled. Not every partition type τ for which there exists a τ-labeled Hamiltonian cycle in $J(n, r)$ is flexible. For example, it was shown in [7] that there exists 2^4-labeled Hamiltonian cycle in $J(8, 4)$, and there exists another Hamiltonian cycle in $J(8, 4)$ which can not be 2^4-labeled. A partition type is *non-trivial* if it has at least one non-singleton class. Our objective is to prove the following results.

Theorem 1.1 *Let $n \geq 3$ be an integer, and let τ be a non-trivial partition type defined on X_n with a non-zero gap g and weight $r \geq 1$.*

1. *If τ has at least one class of size greater than two then τ is flexible unless $g = 1$ and τ is one of the following:*
   - *(a)* $3\,2^2$;
   - *(b)* $3^2\,1^t$ *with* $t \geq 0$.
2. *If the classes of τ are of sizes at most two, then τ is flexible unless it is one of the following:*
   - *(a)* $2^2\,1^t$ *with* $t \geq 0$ *and* $g \geq 1$;
   - *(b)* $2^3\,1^t$ *with* $t \geq 0$ *and* $g \geq 1$;
   - *(c)* 2^4 *with* $1 \leq g \leq 4$;
   - *(d)* $2^4\,1$ *with* $g = 1, 2$ *or* $2^4\,1^2$ *with* $g = 1$;
   - *(e)* 2^5 *with* $1 \leq g \leq 3$;
   - *(f)* 2^6 *with* $g = 1$.

For each partition type τ as in Theorem 1.1(1a),(1b),(2a)-(2f) there exists at least one τ-labeled Hamiltonian cycle (Theorem 1.4 in [2] and Theorem

3.1 in [4]). However, the question whether every Hamiltonian cycle of an appropriate size can be τ-labeled for these partition types τ is still an open problem.

It was shown in [7] that partition type $2^s 1^t$ with zero gap is flexible for $s \geq 9$ and $t \geq 0$. We expand this result in the first and second parts of the theorem below, addressing an open Problem 2(6) posed in [5] of determining if the partition type $2^h 1^t$ with $h = 2, \ldots, 8$, $t \geq 0$ and zero gap are flexible. In the third part we address another open problem Problem 2(3) posed in [5] to determine if the partition type $3^2\,2\,1^t$ with $t \geq 0$ and zero gap is flexible.

Theorem 1.2 *1. Let $r \geq 4$ and $t \geq 1$. The partition type $2^r\,1^t$ with zero gap is flexible unless it is one of the following:*

(a) $2^4\,1^t$ with $t = 1, 2, 3, 4$;

(b) $2^5\,1^t$ with $t = 1, 2$;

(c) $2^6\,1$.

2. $2^r\,1^t$ with zero gap is flexible when $r \geq 4$, $t \geq 1$ and $2r + t \geq 14$.

3. The partition type $3^2\,2\,1^t$ with zero gap is flexible if $t \geq 2$.

The remainder of the paper is focused on proving the above results. A proof of Theorem 1.1 is presented in Section 7 of the paper, while a proof of Theorem 1.2 is the content of Section 6.

2 Number of partitions

Let $\mathcal{N}(\tau, g)$ denote the number of partitions of type τ of X_n with gap g. The partition type τ of weight r is said to be *exceptional* [4] if $\mathcal{N}(\tau, g) < \binom{n}{r}$, otherwise a partition type is said to be *non-exceptional.* Clearly if there exists a τ-labeled Hamiltonian cycle in $J(n, r)$ the partition type τ has to be non-exceptional. The number of distinct partitions of type $\tau = {d_1}^{\mu_1}{d_2}^{\mu_2}\ldots{d_k}^{\mu_k}$ with zero gap is given by the following well-known formula (see, for example, [1]):

$$\mathcal{N}(\tau, 0) = \frac{n!}{\prod_{i=1}^{k}(d_i!)^{\mu_i}\mu_i!}$$

Given a partition of type τ of X_n with a gap g, let $\hat{\tau}$ denote the corresponding partition type of X_{n-g} with zero gap. It is easy to see that the number of partitions of type τ with a non-zero gap g is

$$\mathcal{N}(\tau,g)=\binom{n}{g}\mathcal{N}(\hat{\tau},0).$$

For example, if $\tau = 3^2 1$ is a partition type with gap 2 defined on X_9, $\hat{\tau} = 3^2 1$ is a partition type with zero gap defined on X_7. The partition $\gamma = 134|578|9$ of X_9 is of type τ; γ has classes$\{1,3,4\},\{5,7,8\},\{9\}$ and the gap set $\{2,6\}$. The partition $\delta = 134|567|2$ of X_7 is of type $\hat{\tau}$; δ has zero gap and classes $\{1,3,4\},\{5,6,7\},\{2\}$. We have that $\mathcal{N}(\hat{\tau},0) = \frac{7!}{(3!)^2 2!}$ and $\mathcal{N}(\tau,2)=\binom{9}{2}\mathcal{N}(\hat{\tau})$.

It was shown in [4], there are "very few" exceptional partitions with zero gap. We show that there are no exceptional partitions with non-zero gap.

Lemma 2.1 *1. A partition type τ of weight $r \geq 2$ with zero gap is exceptional if and only if and only if τ is of the form 2^2, 2^3, 3^2 or $d\,1^{r-1}$ with $r < d$.*

2. Every non-trivial partition type with non-zero gap is non-exceptional.

Proof. Since the first part of the lemma was proved in [4], we assume that τ is a partition type of X_n with a non-zero gap g. If $\tau = d$ for some $d \geq 2$, then $\mathcal{N}(\tau,g) = \binom{n}{d} \geq n$. Hence assume that the weight r of τ is at least 2. Suppose first that $\hat{\tau}$ is non-exceptional. In this case

$$\mathcal{N}(\tau,g)=\binom{n}{g}\mathcal{N}(\hat{\tau},0)\geq\binom{n}{g}\binom{n-g}{r}=\binom{n}{r}\binom{n-r}{g}\geq\binom{n}{r},$$

since $n > r+g$.

Now suppose that $\hat{\tau}$ is exceptional. If $r \leq g \leq n-3$ then $\mathcal{N}(\tau,g) = \binom{n}{g}\mathcal{N}(\hat{\tau},0) \geq \binom{n}{r}$. Therefore, referring to the first part of the present lemma, we may assume that $g < r$. If $\hat{\tau} = 2^2$ or 3^2, then $g = 1$ and the inequality $\mathcal{N}(d^2,1) \geq \binom{2d+1}{2}$ is easily verified for $d = 2,3$ by direct computation. Similarly if $\hat{\tau} = 2^3$, then $g \leq 2$ and $\mathcal{N}(2^3,g) = 30\binom{g+6}{g} \geq \binom{g+6}{3}$ by direct computation. Finally, if $\hat{\tau} = d\,1^{r-1}$ with $r < d$ then $n = d+g+r-1$ and

$$\mathcal{N}(\tau,g)=\binom{n}{g}\frac{(d+r-1)!}{d!(r-1)!}=\binom{n}{r}\frac{(g+d-1)!r}{g!d!}\geq\binom{n}{r},$$

for all $d \geq 2$ and $g,r \geq 1$. □

3 Counting techniques

An important technique for identification of flexible partition types is based on the use of the *Edge Inequality*, introduced in [5] for partition types with zero gap. We generalize the technique to identify large classes of flexible partition types τ with gap $g \geq 0$. For fixed n and r let $\tau = d_1^{\mu_1} d_2^{\mu_2} \dots d_k^{\mu_k}$ be a partition type on X_n with a gap g. Recall that a partition γ of type τ is a *τ-label* for an edge AB in the graph $J(n,r)$ if the sets A and B are orthogonal to γ. The total number of available τ-labels for the edge AB is denoted by $M(\tau, g)$. The number $M(\tau, 0)$ has been calculated in [5] as

$$M(\tau, 0) = \frac{(n-r)!(r-1)!}{\prod_{i=1}^{k}((d_i - 1)!)^{\mu_i}\mu_i!}.$$

It is not difficult to see that if τ is a partition type on X_n with any gap g and $\hat{\tau}$ is the corresponding partition type on X_{n-g} with zero gap, then the number of τ-labels for the edge AB in $J(n,r)$ is equal to

$$M(\tau, g) = \binom{n-r-1}{g} M(\hat{\tau}, 0) = \binom{n-r-1}{g} \frac{(n-r-g)!(r-1)!}{\prod_{i=1}^{k}((d_i - 1)!)^{\mu_i}\mu_i!} \quad (3.1)$$

The term $\binom{n-r-1}{g}$ reflects an observation that the union of the classes of a τ-label γ for the edge AB has to contain $A \cup B$, so the complement of the union of classes of γ is a g-element subset of $X_n \setminus (A \cup B)$. The number $M(\tau, g)$ of τ-labels that may be used to label the edge AB is independent of the choice of A and B.

Now suppose that $\mathcal{H}$ is a Hamiltonian cycle in $J(n,r)$. Assume that we start at a fixed edge of $\mathcal{H}$ and transverse the edges of $\mathcal{H}$ sequentially in a fixed direction assigning distinct τ-labels to each edge. Suppose γ is a possible τ-label for an arbitrary fixed edge AB in $\mathcal{H}$, and let $C(n,r)$ be the set of all the "competitor" edges in $\mathcal{H}$ that could also be labeled by the same γ. If the number of available partition labels $M(\tau, g)$ is at least as large as $|C(n,r)|$ then we can simply choose a distinct τ-label for each edge of $\mathcal{H}$.

A bound for $|C(n,r)|$ may be calculated as follows. Since γ is a τ-label for the edge AB, the symmetric difference $\{a, b\}$ of the sets A and B has to be a subset of one of the classes of γ. Thus if an edge $DE \in C(n,r)$ then the set D is orthogonal to γ, so D contains at most one of the elements of $\{a, b\}$. Thus the total number $|C(n,r)|$ of "competitor" edges DE is at most

$$m(n,r) = \binom{2}{1}\binom{n-2}{r-1} + \binom{2}{0}\binom{n-2}{r} = \frac{(n-2)!(n+r-1)}{r!(n-r-1)!}.$$

If every class of τ consists of at most two elements, then $\{a, b\}$ is a class of γ, and so D contains exactly one element of the set $\{a, b\}$. In this case $|\mathcal{C}(n,r)|$ does not exceed

$$m2(n,r) = \binom{2}{1}\binom{n-2}{r-1} < m(n,r).$$

The above bounds on the size of $\mathcal{C}(n,r)$ do not depend on the choices of the Hamiltonian cycle $\mathcal{H}$, the edge AB or the partition γ. We summarize this discussion in the following result.

Proposition 3.1 *For fixed n and r let τ be a non-trivial partition type defined on X_n with a gap g.*

1. *If τ satisfies the inequality*

$$M(\tau, g) \geq |\mathcal{C}(n,r)| \tag{3.2}$$

then τ is flexible.

2. *For any τ, $|\mathcal{C}(n,r)| \leq m(n,r)$.*

3. *If all classes of τ contain at most two elements then $|\mathcal{C}(n,r)| \leq m2(n,r)$.*

Note that Inequality (3.2) holds if

$$M(\tau, g) \geq m(n,r). \tag{3.3}$$

Inequality (3.3) was first introduced in [5] as the *Edge Inequality.* In view of this definition we refer to the Inequality (3.2) as the *Generalized Edge Inequality.*

4 Partitions with no singleton classes

The Edge Inequality was used in [5] to describe large classes of flexible partition types with zero gap. In many cases we will be able to use the fact that if τ is a partition type defined on X_n with gap g and $\hat{\tau}$ is the corresponding partition defined on X_{n-g} that satisfies the Edge Inequality then τ satisfies the Edge Inequality as well.

Theorem 4.1 [5] *Let τ be a non-exceptional partition type with zero gap, no singleton classes and weight $r \geq 2$. Then τ satisfies the Edge Inequality, unless τ belongs to the set $\mathcal{E}$ below:*

$$\mathcal{E} = \{d\,3(d \geq 4), d\,2^s(d \geq 3, s \geq 1), 3^2\,2, 3^2, 4^2, 5^2, 3^3, 2^h(h \leq 8)\}.$$

The objective of this section is to prove the following result describing the flexible partition types with no singletons and a non-zero gap g:

Theorem 4.2 *Let τ be a non-trivial partition type defined on X_n with a non-zero gap g, no singleton classes and weight $r \geq 2$.*

1. *If $n \geq 9$ and τ has at least one class of size greater than two, then τ satisfies the Generalized Edge Inequality $M(\tau, g) \geq |C(n,r)|$, and τ is flexible.*
2. *If $n \geq 3$ and $\tau = 2^r$ then τ satisfies the Generalized Edge Inequality $M(2^r, g) \geq m2(2r + g, r)$, and τ is flexible unless it is one of the following:*
   (a) *2^2 or 2^3; or*
   (b) *2^4 and $g \leq 4$; or*
   (c) *2^5 and $g \leq 3$; or*
   (d) *2^6 and $g = 1$.*
3. *If $3 \leq n \leq 8$ and τ has at least one class of size greater than two, then τ satisfies the Generalized Edge Inequality $M(\tau, g) \geq |C(n,r)|$ and τ is flexible unless it is is one of the following:*
   (a) *$3\,2$ with $g = 1$ or 2; or*
   (b) *$4\,2$ or $5\,2$ and $g = 1$; or*
   (c) *$3\,2^2$ with $g = 1$; or*
   (d) *3^2 with $g = 1$.*

While partition types τ in (3a) and (3b) above do not satisfy the Generalized Edge Inequality $M(\tau, g) \geq |C(n,r)|$, we will show that they are flexible using constructive methods. We prove the above theorem in a series of results beginning with a general case below.

Proposition 4.3 *Suppose that τ is a partition type with a non-zero gap g and weight $r \geq 2$ defined on X_n. If τ has no singleton classes, has at least one class with at least three elements, and if the partition type $\hat{\tau}$ of X_{n-g} satisfies the Edge Inequality, then τ satisfies the Edge Inequality also, and τ is flexible.*

Proof. Since $\hat{\tau}$ satisfies the Edge Inequality, in view of Equation (3.1), we have that

$$\begin{aligned} M(\tau, g) &= \binom{n-r-1}{g} M(\hat{\tau}, 0) \geq \binom{n-r-1}{g} m(n-g, r) \\ &= \frac{(n-r-1)!(n-g-2)!(n-g+r-1)}{g!r!((n-g-r-1)!)^2}. \end{aligned}$$

To prove that τ satisfies the Edge Inequality, we only need to show that the last quotient is at least as large as $m(n,r)$. Define a function $f(n,r,g)$ such that the last quotient is its numerator and $m(n,r)$ is its denominator; we will show that $f(n,g,r) \geq 1$. The function easily simplifies to

$$f(n,g,r) = \frac{((n-r-1)!)^2(n-g-2)!(n-g+r-1)}{g!((n-r-g-1)!)^2(n-2)!(n+r-1)}.$$

We show that $f(n,g,r)$ in an increasing function in n. Since τ has no singleton classes, and it has at least one class with at least three elements, we have that $n \geq 2r+g+1$. Consider

$$\frac{f(n+1,g,r)}{f(n,g,r)} = \frac{(n-g+r)(n+r-1)}{(n+r)(n-g+r-1)} \cdot \frac{(n-r)^2(n-g-1)}{(n-g-r)^2(n-1)}.$$

Direct computations show that the numerator of the first quotient is larger than its denominator by g. The difference between the numerator and the denominator in the second quotient can be written as $g(((n-r)(n-1) - (n-r)^2) + ((n-r)(n-1) - g(n-1))) \geq 0$ since $n-r \geq r+g+1 \geq g$. So each quotient is at least 1 and consequently, $f(n+1,g,r) \geq f(n,g,r)$. Thus to show that $f(n,r,g) \geq 1$, we only need to show that

$$f(2r+g+1,g,r) = \frac{3((r+g)!)^2(2r-1)!r}{g!(r!)^2(2r+g-1)!(3r+g)} \geq 1.$$

To this end, to see that $f(2r+g+1,g,r)$ increases as g increases, note that

$$\frac{f(2r+(g+1)+1,g+1,r)}{f(2r+g+1,g,r)} = \frac{(r+g+1)^2(3r+g)}{(g+1)(3r+g+1)(2r+g)}.$$

The last quotient is at least 1 since direct computations show that the difference between its numerator and denominator is positive. If $g=1$ then $n = 2r+2$, and $f(2r+2,1,r) = \frac{3(r+1)^2}{2(3r+1)} \geq 3/2 > 1$ for all $r \geq 1$. Thus $f(n,g,r) \geq 1$ for $n = 2r+g+1$. □

It was shown in [5] that the partition type 2^r with zero gap satisfies the Edge Inequality for $r \geq 9$. However, if the gap g is non-zero, the partition type 2^r may not satisfy the Edge Inequality even for large values of r. Indeed, assume that $g \geq 1$, $r \geq 2$ and take $n = 2r + g$, then

$$M(2^r, g) = (g + r - 1)!/g! = (g + 1)(g + 2) \dots (g + r - 1),$$

and

$$m(g+2r, r) = \frac{(g + 2r - 2)!(g + 3r - 1)}{r!(g + r - 1)!} = \frac{1}{r!}(g+r)(g+r+1) \dots (g+2r-2)(g+3r-1).$$

Observe that $M(2^r, g)$ is a polynomial in g of degree $r-1$, while $m(g+2r, r)$ is a polynomial in g of degree r, so for any fixed r there exists g_0 large enough so that $M(2^r, g) < m(g + 2r, r)$ for all $g \geq g_0$. However, we use Proposition 3.1(3) to identify a large number of flexible partition types $\tau = 2^r$ with non-zero gap g.

Proposition 4.4 *1. Let $r \geq 2$. The partition type $\tau = 2^r$ with non-zero gap g satisfies the Generalized Edge Inequality $M(2^r, g) \geq m(2r + g, r)$ unless one of the following holds*

(a) $r = 2$ or 3 with $g \geq 1$; or

(b) $r = 4$ and $1 \leq g \leq 4$; or

(c) $r = 5$ and $1 \leq g \leq 3$; or

(d) $r = 6$ and $g = 1$.

2. If $n \geq 14$ and $r \geq 4$ then $\tau = 2^r$ with gap $g \geq 1$ satisfies the Generalized Edge Inequality $M(2^r, g) \geq m(2r + g, r)$.

Proof. The second part of the proposition follows directly from the first part. To prove the first part of the proposition, in view of Proposition 3.1(3), we will show that the inequality $M(2^r, g) \geq m2(n, r)$ holds unless r and g satisfy the conditions (1a)-(1d) above. Observe that $n = 2r + g$, $M(2^r, g) = (r + g - 1)!/g!$, $m2(2r + g, r) = 2\binom{2r+g-2}{r-1}$ and define the function

$$f(g, r) = \frac{M(2^r, g)}{m2(2r + g, r)} = \frac{((r + g - 1)!)^2(r - 1)!}{2g!(2r + g - 2)!}.$$

To show that $f(g, r) \geq 1$ for all r and g other than described in (1a)-(1d), observe that $f(g, r)$ is increasing in g as

$$\frac{f(g + 1, r)}{f(g, r)} = \frac{(r + g)^2}{(g + 1)(2r + g - 1)} = 1 + \frac{(r - 1)^2}{(g + 1)(2r + g - 1)}.$$

Moreover $f(g,r)$ is increasing in r for $r \geq 4$ as

$$\frac{f(g,r+1)}{f(g,r)} = \frac{r(r+g)^2}{(2r+g)(2r+g-1)} = \frac{r}{(1+\frac{r}{r+g})(1+\frac{r-1}{r+g})} \geq \frac{r}{4} \geq 1.$$

Also direct computations show that $f(5,4) = 56/55$, $f(4,5) = 56/33$, $f(2,6) = 35/22$ and $f(1,7) = 210/143$, while $f(4,4)$, $f(3,5)$ and $f(1,6) < 1$. Finally, for any $g \geq 1$, we have $f(g,3) = \frac{(g+1)(g+2)}{(g+3)(g+4)} < 1$ and $f(g,2) = \frac{g+1}{2(g+2)} < 1$. □

Next we take up partition types τ with a non-zero gap and no singleton classes such that $\hat{\tau}$ is one the partition types in the set $\mathcal{E}$ of Theorem 4.1, and $\hat{\tau}$ contains at least one class with more than 2 elements. We will show that many such partition types satisfy the Edge Inequality.

Proposition 4.5 *For $d \geq 4$, the partition type $d\,3$ with non-zero gap g satisfies the Edge Inequality.*

Proof. Observe that $n = d+g+3$, $r = 2$, and $M(d\,3,g) = \binom{d+g}{g}\frac{(d+1)!}{2(d-1)!}$, while $m(n,r) = m(d+g+3,2) = (d+g+1)(d+g+4)/2$. We need to show that $M(d\,3,g) - m(d+4,2) \geq 0$ for $d \geq 4$ and $g \geq 1$.

For $d \geq 4$ and $g = 1$, the difference $M(d\,3,g) - m(d+4,2) = d(d+1)^2/2 - (d+2)(d+5)/2$ is positive for $d = 4$ and the difference is increasing (the derivative is positive) for $d \geq 4$.

For $d \geq 4$ and $g \geq 2$, $\binom{d+g}{g} \geq \binom{d+g}{2}$, and $d(d+1)/2 \geq 10$ so that $M(d\,3,g) \geq 10\binom{d+g}{2}$. It follows that the difference $M(d\,3,g) - m(d+g+3,2) \geq 10((d+g)(d+g-1)/2) - (d+g+1)(d+g+4)/2$. Since $2(d+g-1) \geq (d+g+4)$, it follows that $M(d\,3,g) \geq m(d+g+3,2)$. □

While for some values of d and r, the partition type $d\,2^{r-1}$ with zero gap satisfies the Edge Inequality, generally this is not the case. The following result was proved in [5].

Lemma 4.6 *1. For any fixed $r \geq 3$ there exists an integer $d_0 \geq 2$ such that for all $d \geq d_0$, the partition type $d\,2^{r-1}$ with zero gap does not satisfy the Edge Inequality.*

2. For any fixed $d \geq 2$, there exists an integer $r_0 \geq 3$ such that for all $r \geq r_0$, the partition type $d\,2^{r-1}$ with zero gap satisfies the Edge Inequality.

In contrast, if the partition type $d\,2^{r-1}$ has positive gap, then, with a very few exceptions, it satisfies the Edge Inequality.

Proposition 4.7 *Let $r \geq 2$ and $d \geq 3$.*

1. *The partition type $d\,2^{r-1}$ with non-zero gap g satisfies the Edge Inequality unless it is of the following:*
   (a) $3\,2^2$ *with* $g = 1$;
   (b) $3\,2$ *with* $g = 1$ *or* 2;
   (c) $4\,2$ *or* $5\,2$ *with* $g = 1$.

2. *If $n \geq 9$ then the partition type $d\,2^{r-1}$ with non-zero gap g satisfies the Edge Inequality.*

Proof. Observe that for the partition type $d\,2^{r-1}$ with a gap $g \geq 1$, we have that $n = d + g + 2r - 2$, and

$$M(d\,2^{r-1}, g) = \frac{(d+g+r-3)!(d+r-2)}{g!(d-1)!},$$

$$m(d+g+2r-2, r) = \frac{(d+g+2r-4)!(d+g+3r-3)}{r!(d+g+r-3)!}.$$

We show that unless d, g, r as in either of the conditions (1a)-(1c) above, the value of the function $f(d,g,r)$, defined as $f(d,g,r) = M(d\,2^{r-1}, g)/m(n,r)$, is at least 1. First we show that $f(d,g,r)$ is an increasing function of d by considering

$$\frac{f(d+1,g,r)}{f(d,g,r)} = \frac{(d+r-1)(d+g+3r-3)}{(d+g+3r-2)(d+r-2)} \cdot \frac{(d+g+r-2)^2}{d(d+g+2r-3)}.$$

Each quotient above is at least 1 as the difference between its numerator and denominator is non-negative, indeed $(d+r-1)(d+g+3r-3) - (d+g+3r-2)(d+r-2) = g + 2r - 1 \geq 0$ and $(d+g+r-2)^2 - d(d+g+2r-3) = ((r+g)-2)^2 + d(g-1) \geq 0$ for $g \geq 1$ and $r \geq 2$. Therefore $f(d,g,r) \geq f(3,g,r)$ for all $d \geq 4, g \geq 1, r \geq 2$.

Now $f(3,g,r) = \frac{((g+r)!)^2(r+1)!}{2g!(g+2r-1)!(g+3r)}$ is an increasing function of g, indeed

$$\frac{f(3,g+1,r)}{f(3,g,r)} = \frac{(g+r+1)^2(g+3r)}{(g+1)(g+2r)(g+3r+1)} \geq 1,$$

since $(g+r+1)^2(g+3r) - (g+1)(g+2r)(g+3r+1) = 3r^3 + r^2g + rg + r \geq 0$. Therefore $f(3,g,r) \geq f(3,1,r)$ for all $g \geq 1, r \geq 2$.

Also $f(3,1,r) = \frac{((r+1)!)^3}{2(2r)!(3r+1)}$ is an increasing function of r when $r \geq 4$, as

$$\frac{f(3,1,r+1)}{f(3,1,r)} = \frac{r+2}{r+1} \cdot \frac{3r+1}{2r+1} \cdot \frac{(r+2)^2}{6r+8} \geq 1.$$

Thus, for $d \geq 3, g \geq 1$ and $r \geq 4$, we have that $f(d,g,r) \geq f(3,1,4) = 150/91 > 1$, so $d\,2^{r-1}$ satisfies the Edge Inequality if $r \geq 4$.

Assume $r = 3$, then for $d \geq 3$ and $g \geq 2$ we have that $f(d,g,3) \geq f(3,2,3) = 120/77 > 1$, and for $d \geq 4$, $g = 1, r = 3$, we have that $f(d,1,3) \geq f(4,1,3) = 100/77 > 1$. Finally, $f(3,1,3) = 24/25 < 1$, so $d\,2^2$ satisfies the Edge Inequality unless $d = 3$ and $g = 1$.

If $r = 2$ we have that $f(d,g,2) \geq f(3,3,2) = 10/9 > 1$ for $d \geq 3$ and $g \geq 3$. Also $f(d,g,2) \geq f(4,2,2) = 40/27 > 1$ for $g \geq 2$ and $d \geq 4$, while $f(3,2,2) = 9/10 < 1$. Finally if $r = 2$, $g = 1$ and $d \geq 6$ we have that $f(d,1,2) \geq f(6,1,2) = 36/35$, while $f(5,1,2) = 25/27, f(4,1,2) = 4/5, f(3,1,2) = 9/14$. Thus $d\,2$ with a gap g satisfies the Edge Inequality unless either $d = 3$ and $g = 1,2$ or $d = 4,5$ and $g = 1$. □

To complete a proof of Theorem 4.2 we consider the following partition types in the set $\mathcal{E}$ (Theorem 4.1).

Proposition 4.8 *Partition types $3^2\,2$, 3^3, 4^2, 5^2 with non-zero gap g, and 3^2 with gap $g \geq 2$ satisfy the Edge Inequality.*

Proof. For a partition type $3^2\,2$ we have that $n = g+8$, $r = 3$, and $M(3^2\,2,g) = 5(g+1)(g+2)(g+3)(g+4)/4$, $m(g+8,3) = (g+5)(g+6)(g+10)/6$ are increasing functions of $g \geq 1$. Moreover $M(3^2\,2,g)$ increases faster than $m(g+8,3)$ for $g \geq 1$, and the functions have respective values of 150 and 77 and when $g = 1$. Thus $M(3^2\,2,g) \geq m(g+8,3)$ for $g \geq 1$, so $3^2\,2$ satisfies the Edge Inequality for $g \geq 1$.

For a partition type 3^3 we have $n = g+9$, $r = 3$, $M(3^3,g) = (g+1)(g+2)(g+3)(g+4)(g+5)/4$, $m(g+9,3) = (g+6)(g+7)(g+11)/6$ are increasing functions of $g \geq 1$. Again, $M(3^3,g)$ increases faster than $m(g+9,3)$, and the functions take on values 180 and 112 respectively when $g = 1$. Thus $M(3^3,g) \geq m(g+9,3)$ for $g \geq 1$, and 3^3 satisfies the Edge Inequality.

For a partition type d^2 we have that $n = 2d+g$, $r = 2$, $M(d^2,g) = \frac{(2d+g-3)!(d-1)}{g!((d-1)!)^2}$, $m(2d+g,2) = (2d+g-2)(2d+g+1)/2$. If $d = 3$ we have $M(3^2,g)/m(g+6,2) = (g+1)(g+2)(g+3)/((g+4)(g+7)) \geq 1$ for $g \geq 2$. If $d = 4$ we have $M(4^2,g)/m(g+8,2) = (g+1)(g+2)(g+3)(g+4)(g+5)/(6(g+6)(g+9)) \geq 1$ for $g \geq 1$. If $d = 5$ we have $M(5^2,g)/m(g+10,2) = (g+1)(g+2)(g+3)(g+4)(g+5)(g+6)(g+7)/(72(g+8)(g+11) \geq 1$ for $g \geq 1$. □

5 Partition types with singleton classes

The question whether a partition type with singleton classes is flexible is often resolved based on the information about its non-singleton classes. To this end we will use the following notation throughout this section. For a partition type σ of weight $s \geq 1$ with a gap q defined on X_k and a positive integer t, the partition type $\sigma \oplus 1^t$ defined X_{q+t} of weight $s+t$ and with the same gap q is obtained from σ by adjoining t singleton classes.

Let τ be a partition type with $t \geq 1$ singleton classes, weight $r \geq 2$ and a gap $g \geq 0$ defined on X_n. We write $\tau = \bar{\tau} \oplus 1^t$ where $\bar{\tau}$ is a partition type with no singletons, weight $r-t$ and gap g, defined on a set of $n-t$ elements. It was shown in Lemma 2.11 of [5] that if a partition type $\bar{\tau}$ of weight at least two, with zero gap and no singletons satisfies the Edge Inequality then $\tau = \bar{\tau} \oplus 1^t$ can be used to label any Hamiltonian cycle in $J(n,r)$, that is $\bar{\tau} \oplus 1^t$ is flexible. The proof readily extends to include partition types τ with a non-zero gap g that satisfy the Generalized Edge Inequality $M(\bar{\tau}, g) \geq |\mathcal{C}(n,r)|$. As the proof is short, we present it here for completeness.

Lemma 5.1 *Let $\tau = \bar{\tau} \oplus 1^t$ be a partition type with $t \geq 1$ singleton classes, weight $r \geq 2$ and a gap $g \geq 0$ defined on X_n. If $\bar{\tau}$ contains no singleton classes and satisfies the Generalized Edge Inequality $M(\bar{\tau}, g) \geq |\mathcal{C}(n,r)|$, then τ is flexible.*

Proof. Let $\mathcal{H}$ be a Hamiltonian cycle in $J(n,r)$ and let AB be an arbitrary but fixed edge in $\mathcal{H}$. Fix a t-element subset T of $A \cap B$. The number of τ-labels of AB with singleton classes in the set T equals the number of $\bar{\tau}$-labels of the edge $(A \setminus T)(B \setminus T)$ in the graph $J(n-t, r-t)$, which in turn equals to $M(\bar{\tau}, g)$. Let γ be a τ-label for AB such that T is the set of the singleton classes of γ, and let $\bar{\gamma}$ be the partition of type $\bar{\tau}$ of $X_n \setminus T$ obtained from γ by removing its singleton classes. If DE is a "competitor" edge that can be labeled by γ, then $T \subseteq D \cap E$ and the edge $(D \setminus T)(E \setminus T)$ is a "competitor" edge of $(A \setminus T)(B \setminus T)$ that can be labeled by the partition $\bar{\gamma}$, so $(D \setminus T)(E \setminus T) \in \mathcal{C}(n-t, r-t)$. The result follows by the assumption that $\bar{\tau}$ satisfies the Generalized Edge Inequality $M(\bar{\tau}, g) \geq |\mathcal{C}(n,r)|$. □

Remark 5.2 *It follows directly from the above result that if $\bar{\tau}$ is a non-exceptional partition type with zero gap that does not belong to set $\mathcal{E}$ in Theorem 4.1 then $\bar{\tau} \oplus 1^t$ is flexible for $t \geq 1$.*

Recall that a partition type $\tau = 2^s$ with $s \geq 7$ and gap $g \geq 1$ satisfies the Generalized Edge Inequality $M(2^s, g) \geq m2(2s+g, s)$ (Proposition 4.4), and so for any $t \geq 1$, the partition type $2^s 1^t$ is flexible. However, the partition type $\tau = 2^6$ with gap $g = 1$ does not satisfy the Generalized Edge Inequality $M(2^6, 1) \geq m2(13, 6)$, and generally the partition type $2^6 1^t$ with $g \geq 1$ does not satisfy the Generalized Edge Inequality $M(2^6 1^t, g) \geq m2(t+g+12, t+6)$. For example if $g = 1$ then $M(2^6 1^t, 1) = 6(t+5)!/t!$ and $m2(t+13, t+6) = 2\binom{t+11}{t+5}$, and $\frac{M(2^6 1^t,1)}{m2(t+13,t+6)} = \frac{3 \cdot 6!(t+1)...(t+5)}{(t+6)...(t+11)}$ takes on values less than 1 for large enough values of t since the numerator is a polynomial in t of degree 5, while the denominator is a polynomial in t of degree 6.

The counting technique developed below allows us to identify flexible partition types $\tau = \bar{\tau} \oplus 1^t$ with t singleton classes where $\bar{\tau}$ does not satisfy the Generalized Edge Inequality $M(\bar{\tau}, g) \geq |C(n,r)|$. Such partition types with no singletons are described in Theorem 4.1 (if the gap is zero) and Theorem 4.2 (if the gap is non-zero).

Lemma 5.3 *Let $\tau = \bar{\tau} \oplus 1^t$ be a a partition type with $t \geq 1$ singleton classes and weight $r \geq 2$, defined on X_n, and let $\mathcal{H}$ be a Hamiltonian cycle in $J(n,r)$ to be τ-labeled. Then the number of "competitor" edges $|C(n,r)|$ in $\mathcal{H}$ does not exceed*

$$b(n,r,t) = \sum_{k=t}^{r-1} \left(2\binom{r-1}{k}\binom{n-r-1}{r-k-1} + \binom{r-1}{k}\binom{n-r-1}{r-k} \right) \tag{5.1}$$

If $\bar{\tau} = 2^{r-t}$ with gap $g \geq 1$, then $|C(n,r)|$ does not exceed

$$b2(n,r,t) = \sum_{k=t}^{r-1} 2\binom{r-1}{k}\binom{n-r-1}{r-k-1} \tag{5.2}$$

with $n = 2r - t + g$.

Proof. Take an edge AB in $\mathcal{H}$, and suppose that γ is a τ-label for AB. If T is the set of singleton classes of γ, then for any "competitor" edge DE that could also be labeled by γ we have that $T \subseteq D$ and so $|D \cap (A \cap B)| = k$ for some integer k with $t \leq k \leq |A \cap B| = r - 1$. Hence using an argument similar to that of Section 3, if γ has at least one class of size at least three, an upper bound for the number of edges DE that could have been labeled by γ previously can be estimated as the number of choices for D with $|D \cap (A \cap B)| = k \geq t$ and $|D \cap \{a,b\}| \leq 2$, where $\{a,b\}$ is the symmetric difference of A and B. This number is equal to $b(n,r,t)$ in Equation (5.1).

If every non-singleton class of γ is of size two, this bound, similarly as in Proposition 3.1(3), is reduced to $b2(n,r,t)$, as in this case $|D \cap \{a,b\}| = 1$ since $\{a,b\}$ has to be a class of γ. □

Note that using the well-known Vandermonde's identity $\sum_{j=0}^{x}\binom{u}{j}\binom{v}{x-j} = \binom{u+v}{x}$ we can compare the values of $b(n,r,t)$ and $m(n,r)$ as follows. Since $\binom{r-1}{k}\binom{n-r-1}{r-k} = 0$ when $k = r$, we have that

$$\begin{aligned} m(n,r) &= 2\binom{n-2}{r-1} + \binom{n-2}{r} = \sum_{k=0}^{r-1} 2\binom{r-1}{k}\binom{n-r-1}{r-1-k} + \sum_{k=0}^{r}\binom{r-1}{k}\binom{n-r-1}{r-k} \\ &= b(n,r,t) + \sum_{k=0}^{t-1}\left(2\binom{r-1}{k}\binom{n-r-1}{r-1-k} + \binom{r-1}{k}\binom{n-r-1}{r-k}\right) \end{aligned}$$

Similarly, if $\tau = 2^{r-t}\,1^t$, we have that

$$m2(n,r) = b2(n,r,t) + \sum_{k=0}^{t-1} 2\binom{r-1}{k}\binom{n-r-1}{r-1-k}.$$

So using $b(n,r,t)$ or $b2(n,r,t)$ for establishing finer bounds on the number of "competitor" edges may yield fruitful results only for partition types with singleton classes. Moreover, for $\tau = 2^r$ with gap $g \geq 1$ we have that $n = 2r - t + g$, and

$$m2(n,r) - b2(n,r,t) = \sum_{k=0}^{t-1} 2\binom{r-1}{k}\binom{r-t+g-1}{r-k-1} = \sum_{k=t-g}^{t-1} 2\binom{r-1}{k}\binom{r-t+g-1}{r-k-1},$$

since $\binom{r-t+g-1}{r-k-1} = 0$ if $k < t - g$. Therefore we will use criterion in (5.2) only for partition types with a non-zero gap. We use the above developed criterion to prove the next result.

Lemma 5.4 *Let $t \geq 1$ and $r - t \geq 4$.*

1. *The partition type $2^{r-t}\,1^t$ with a non-zero gap g is flexible unless it is one of the following:*
   (a) *$2^4\,1$ with $g = 1$ or 2,*
   (b) *$2^4\,1^2$ with $g = 1$.*
2. *$2^{r-t}\,1^t$ with a non-zero gap g is flexible when $n = 2r - t + g \geq 12$.*

Proof. Let $\tau = 2^{r-t}\,1^t$ with gap $g \geq 1$ and write $\tau = \bar{\tau} \oplus 1^t$, where $\bar{\tau} = 2^{r-t}$ with a gap g. If $\bar{\tau}$ satisfies the Generalized Edge Inequality $M(2^{r-t}, g) \geq m2(2r-2t+g, r-t)$ then by Lemma 5.1 the partition type $2^{r-t}\,1^t$ is flexible. Thus in view of Theorem 4.2 we consider the following cases: $\bar{\tau} = 2^4$ with $1 \leq g \leq 4$, $\bar{\tau} = 2^5$ with $1 \leq g \leq 3$, and $\bar{\tau} = 2^6$ with $g = 1$.

First we show that for a fixed $g \geq 1$, if $2^{r-t}\,1^t$ with gap g satisfies the Generalized Edge Inequality $M(2^{r-t}\,1^t, g) \geq b2(n,r,t)$ then $2^{r-t}\,1^t$ with gap $g+1$ satisfies the Generalized Edge Inequality $M(2^{r-t}\,1^t, g+1) \geq b2(n+1,r,t)$ also. Indeed,

$$M(2^{r-t}\,1^t, g+1) = \binom{r-t+g}{g+1}\frac{(r-1)!}{t!} = \frac{r-t+g}{g+1}M(2^{r-t}\,1^t, g).$$

Also, since $\binom{n-r}{r-k-1} = \binom{n-r-1}{r-k-1}\frac{n-r}{n-2r+k+1}$ and $\frac{n-r}{n-2r+k+1} \leq \frac{n-r}{n-2r+t+1} = \frac{r-t+g}{g+1}$ for all $k = t, \ldots, r+t-1$ and $n = 2r-t+g$, we have that

$$b2(n+1,r,t) = \sum_{k=t}^{r-1} 2\binom{r-1}{k}\binom{n-r}{r-k+1} \leq \frac{r-t+g}{g+1}b2(n,r,t).$$

Therefore $M(2^{r-t}\,1^t, g+1)/b2(n+1,r,t) \geq M(2^{r-t}\,1^t, g)/b2(n,r,t) \geq 1$ as required.

We proceed to calculate $M(2^{r-t}\,1^t, g) - b2(n,r,t)$ to show that unless r, g and t are such that $2^{r-t}\,1^t$ with gap g satisfy conditions (1a)-(1b) above, the difference is positive, and so the partition type $2^r\,1^t$ is flexible.

If $\bar{\tau} = 2^4$ with $g = 1$, direct computation shows that

$$\begin{aligned} M(2^4\,1^t, 1) - b2(t+9, t+4, t) &= 4(t+3)!/t! - \sum_{k=t}^{t+3} 2\binom{t+3}{k}\binom{4}{t+3-k} \\ &= \frac{1}{3}(8t^3 + 30t^2 - 26t - 138) > 0 \end{aligned}$$

for all $t \geq 3$. Thus by the above observation, $2^4\,1^t$ with $t \geq 3$ is flexible for all $1 \leq g \leq 4$. Additionally if $g = 2$ and $t = 2$ we have that $M(2^4\,1^2, 2) - b2(12,6,2) = 148 > 0$, so $2^4\,1^2$ with gap g is flexible for all $2 \leq g \leq 4$. Also, if $g = 2$ and $t = 1$ we have that $M(2^4\,1, 2) - b2(11,5,2) = -2$ while $M(2^4\,1, 3) - b2(12,5,3) = 90$, so $2^4\,1$ is flexible if $g = 3$ or 4.

Similarly, if $\bar{\tau} = 2^5$ with $g = 1$ then by direct computation $M(2^5\,1^t, 1) - b2(t+11, t+5, t) = (55t^4 + 510t^3 + 1445t^2 + 750t - 1584)/12 > 0$ for all $t \geq 1$. Finally, if $\bar{\tau} = 2^6$ with $g = 1$, $M(2^6\,1^t, 1) - b2(t+13, t+6, t) = (354t^5 + 5235t^4 + 28640t^3 + 68625t^2 + 57826t - 12240)/60 > 0$ for all $t \geq 1$.□

Note that for partition types $\tau = 2^3\,1^t$ and $2^2\,1^t$ with a non-zero gap g do not satisfy the Generalized Edge Inequality $M(\tau, g) \geq b2(n,r,t)$, indeed, we have that $M(2^3\,1^t, g) - b2(t+g+6, t+3, t) = \binom{g+2}{g}\frac{(t+1)!}{t!} - 2\binom{t+2}{t}\binom{g+2}{2} - 2(t+2)(g+2) - 2 < 0$. Similarly $M(2^2\,1^t, g) - b2(t+g+4, t+2, t) < 0$.

Lemma 5.5 *Partition types $\tau = d\,2^s\,1^t$ with a non-zero gap g and $s,t \geq 1$ satisfy the Generalized Edge Inequality $M(\tau,g) \geq b(d+2s+t+g, s+t+1, t)$ and are flexible unless $g=1$ and the partition type is one of the following:*

1. $3\,2\,1^t$ *with* $t \geq 1$ *or*

2. $4\,2\,1$.

Proof. Let $\tau = d\,2^s\,1^t$ with gap $g \geq 1$, and write $\tau = \bar{\tau} \oplus 1^t$ where $\bar{\tau} = d\,2^s$ with gap g. If $\bar{\tau}$ satisfies the Generalized Edge Inequality $M(\bar{\tau},g) \geq m(d+2s+g, s+1)$ then by Lemma 5.1 the partition type $\bar{\tau} \oplus 1^t$ is flexible. Thus in view of Theorem 4.2 we may assume that $\bar{\tau}$ is one of the following: $3\,2$ with $g=1$ or 2; or $3\,2^2$ with $g=1$; or $4\,2$ or $5\,2$ with $g=1$. We show that unless τ is as stated in (1)-(2) above, we have that $M(\bar{\tau} \oplus 1^t, g) \geq b(n,r,t)$, where $n = d+2s+t+g$ and $r = s+t+1$. Observe that

$$M(\bar{\tau} \oplus 1^t, g) = \binom{d+s+g-2}{g} \frac{(d+s-1)!(s+t)!}{(d-1)!s!t!},$$

and

$$b(n,r,t) = \sum_{k=t}^{s+t} \left(2\binom{s+t}{k}\binom{d+s+g-2}{s+t-k} + \binom{s+t}{k}\binom{d+s+g-2}{s+t+1-k} \right).$$

If $\bar{\tau} = 3\,2$ with $g=1$ or 2 then $d=3, s=1$ and we have that $M(\bar{\tau} \oplus 1^t, g) = 3(t+1)\binom{g+2}{g}$ and $b(t+g+5, t+2, t) = 2(t+1)(g+2) + (t+1)\binom{g+2}{2} + g + 4$, so if $g=2$, $M(\bar{\tau} \oplus 1^t, 2) - b(t+7, t+2, t) = 4t - 2 > 0$ for all $t \geq 1$. However if $g=1$ we have that $M(\bar{\tau} \oplus 1^t, 1) - b(t+6, t+2, t) = -5$ for all t.

In the remainder of the proof we may assume that $g=1$. If $\bar{\tau} = 3\,2^2$, we have that $M(\bar{\tau} \oplus 1^t, 1) = 24(t+1)(t+2)$ and $b(t+8, t+3, t) = 8t^2 + 38t + 50$, so $M(\bar{\tau} \oplus 1^t, 1) > b(t+8, t+3, t)$ for all $t \geq 1$. Now assume that $\bar{\tau} = d\,2$ with $d = 4$ or 5. If $d=4$, $M(\bar{\tau} \oplus 1^t, 1) = 16(t+1)$ and $b(t+7, t+2, t) = 14t+20$, so $M(\bar{\tau} \oplus 1^t, 1) > b(t+7, t+2, t)$ for all $t \geq 2$. If $d=5$, $M(\bar{\tau} \oplus 1^t, 1) = 25(t+1)$ and $b(t+8, t+2, t) = 20t + 27$, so $M(\bar{\tau} \oplus 1^t, 1) > b(t+9, t+2, t)$ for all $t \geq 1$. □

It is easy to check that neither $\tau = 3^2\,1^t$ with gap $g=1$ nor $\tau = d\,1^t$ with either $d \geq 4$ and $g=1$, or $g = 2,3$ satisfy the Generalized Edge Inequality $M(\tau,g) \geq b(n,r,t)$. We show that partition types $d\,1^t$ and $d\,2\,1^t$ are flexible using counting and constructive arguments in Section 6.

5.1 Partition types with zero gap: a proof of Theorem 1.2

The following result presents a proof of the first and second parts of Theorem 1.2.

Lemma 5.6 *Let $s \geq 4$ and $t \geq 1$.*

1. *The partition type $2^s\,1^t$ with zero gap is flexible unless it is one of the following:*
   (a) $2^4\,1^t$ *and* $t = 1, 2, 3$ *or* 4;
   (b) $2^5\,1$ *or* $2^5\,1^2$;
   (c) $2^6\,1$.
2. *$2^s\,1^t$ with zero gap is flexible when $2s + t \geq 14$.*

Proof. If $s \geq 9$ then 2^s satisfies the Edge Inequality (Theorem 4.1), and so by Lemma 5.1, $2^s\,1^t$ is flexible. We assume that $s \leq 8$ and show that if s and t are outside of the limits imposed by the conditions (1a)-(1c) above, then $2^s\,1^t$ satisfies the Generalized Edge Inequality $M(2^s\,1^t, 0) \geq m2(n, r))$, where $n = 2s + t$ and $r = s + t$. We have $M(2^s\,1^t, 0) = (s + t - 1)!/t!$ and $m2(2s + t, s + t) = 2\binom{2s+t-2}{s+t-1}$. Let

$$f(s,t) = \frac{M(2^s\,1^t, 0)}{m2(2s + t, s + t)} = \frac{(s-1)!((s+t-1)!)^2}{2t!(2s+t-2)!}$$

We show that then $f(s,t) \geq 1$ unless $s = 4$ and $t = 1, 2, 3, 4$, or $s = 5$ and $t = 1, 2$, or $s = 6$ and $t = 1$ as imposed by one of the conditions (1a)-(1c). Indeed, since $f(s, t+1)/f(s,t) = \frac{t^2+2st+s^2}{t^2+2st+2s-1} \geq 1$ for all $s, t \geq 1$, the function $f(s,t)$ is increasing in t. Moreover, direct calculations show that if $s = 8$ we have that $f(8,1) = 448/143 > 1$ so $f(8,t) > 1$ for all $t \geq 1$. Similarly, $f(7,1) = 210/143 > 1$, so $f(7,t) > 1$. However, $f(6,1) < 1$, while $f(6,2) = 35/22$, so $f(6,t) > 1$ for $t \geq 2$. Similarly, $f(5,2) < 1$ while $f(5,3) = 14/11 > 1$, $f(4,5) = 56/55$, while $f(4,4) < 1$. □

For the partition types in $\mathcal{E}$ (Theorem 4.1) having a class with more than 2 elements only $\tau = 3^2\,2$ is such that $\tau \oplus 1^t$ satisfies the Generalized Edge Inequality. In particular, $M(3^2\,2\,1^t, 0) = 15(t+1)(t+2)$ and $b(t+8, t+3, t) = 8t^2+38t+50$, and so $M(3^2\,2\,1^t, 0) \geq b(8, 3, t)$ when $t \geq 2$. Thus the following result holds.

Lemma 5.7 *The partition type $3^2\,2\,1^t$ with zero gap is flexible if $t \geq 2$.*

The result above provides a proof for the third part of Theorem 1.2.

6 Partition types $d\,1^t$ and $d\,2\,1^t$ with non-zero gap and $t \geq 1$

Note that the graph $J(n,1)$ is a complete graph. The next results demonstrates that certain partition types of weight one are flexible.

Proposition 6.1 *Let $d \geq 4$, then the partition type d with a gap $g \geq 2$ satisfies the Edge Inequality.*

Proof. Observe that $n = d + g$, $r = 1$, $M(d,g) = \binom{n-2}{d-2}$ and $m(n,r) = n$. It is easy to see that the Edge Inequality holds for $d = 4$ and $g \geq 2$. Using induction on g we have that $M(d+1,g) = M(d,g)\frac{n-1}{d-1} \geq n\frac{n-1}{d-1} = n + \frac{ng}{d-1} \geq n+1$. □

In many cases $\tau = d\,1^t$ does not satisfy any form of the Generalized Edge inequality. However we can show that τ is flexible.

Proposition 6.2 *A non-trivial partition type $d\,1^t$ with a non-zero gap g and $t \geq 0$ is flexible.*

Proof. Since by Proposition 6.1 a partition type d with gap $g \geq 2$ satisfies the Edge Inequality when $d \geq 4$, Lemma 5.1 implies that the partition type $d\,1^t$ is flexible for $d \geq 4, g \geq 2$. Thus we may assume that either $g = 1$ or $d = 2, 3$.

Let $\mathcal{H}$ be a Hamiltonian cycle in $J(n,r)$, where $n = d+g+t$ and $r = t+1$. For an edge AB in $\mathcal{H}$ let $C = A \cap B$ be the *core* of the edge, and let $\mathcal{H}(C)$ be the list of all edges in $\mathcal{H}$ with the core C appearing in the same relative order as they appear in $\mathcal{H}$. It was shown in [8] that if $r \geq 2$, then $\mathcal{H}(C)$ has at most $n - r$ edges, and it has $n - r$ edges precisely when the edges are consecutive. Note that if $t \geq 1$ and γ is a suitable label for an edge $DE \in \mathcal{H}(C)$, then the set of singletons of γ coincides with the set C, and so τ-labels associated with edges with distinct cores are distinct. Therefore we can show that $\tau = d\,1^t$ is flexible by identifying suitable τ-labels for each $\mathcal{H}(C)$.

Assume first that $\tau = 2\,1^t$, $g \geq 1$ and $t \geq 0$. For each edge DE of $\mathcal{H}(C)$ there is a unique τ-label with the unique non-singleton class coinciding with the symmetric difference of D and E, the set of singletons coinciding with C. (Note that if $t = 0$ then $C = \emptyset$ is the only core of an edge in $\mathcal{H}$.) Thus in this case the existance of τ-labeling is guaranteed by the existence of the Hamiltonian cycle. In what follows we assume that $d \geq 3$.

Consider the case when τ consists of one non-singleton class, $\tau = d$, so $n = d + g$, $r = 1$ and either $d = 3, 4$, or $d \geq 5$ with $g = 1$. Write $\mathcal{H} = \{a_1\}, \{a_2\}, \ldots, \{a_n\}$. For $i = 1, 2, \ldots, n$, let γ_i have a unique class $\{a_i, a_{i+1}, \ldots, a_{i+d-1}\}$ if $i+d-1 \leq n$, and let it have a unique class $\{a_i, a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i+d-n-1}\}$ otherwise. Then γ_i is a τ-label for the edge $\{a_i\}, \{a_{i+1}\}$ for $i = 1, \ldots n-1$, and γ_n is a τ-label for the edge $\{a_n\}, \{a_1\}$, with $\gamma_i \neq \gamma_j$ if $i \neq j$. Thus in what follows we may assume that $t > 0$.

Let $\tau = 3\,1^t$ and $g, t \geq 1$. For a given core C associated with an edge in $\mathcal{H}$, we label $\mathcal{H}(C)$ by partitions with singleton classes consisting of the elements of C. If $|\mathcal{H}(C)| = n - r$ then the edges with the core C are consecutive in $\mathcal{H}$, and so the corresponding path in $\mathcal{H}$ can be written as $\{a_1\} \cup C, \{a_2\} \cup C, \ldots, \{a_{n-r}\} \cup C, \{a_{n-r+1}\} \cup C$ for distinct $a_1, a_2, \ldots, a_{n-r}, a_{n-r+1} \in X_n$. Note that $n - r + 1 = g + 3 \geq 4$, and for $i = 1, 2, \ldots, n - r - 1$, let γ_i have a unique 3-element class $\{a_i, a_{i+1}, a_{i+2}\}$, and let γ_{n-r} have a unique 3-element class $\{a_{n-r}, a_{n-r+1}, a_1\}$. Then γ_i is a τ-label for the edge $\{a_i\} \cup C, \{a_{i+1}\} \cup C$ for $i = 1, 2, \ldots, n - r$, and $\gamma_i \neq \gamma_j$ if $i \neq j$. If $|\mathcal{H}(C)| < n - r$ we can simply choose a suitable τ-label γ for an edge DE in $\mathcal{H}(C)$ that was not used previously to label the edges in $\mathcal{H}(C)$. Since the set of singletons of such a τ-label coincides with C, we only need to choose a single third element for the non-trivial 3-element class of γ, there are $|X_n \setminus (D \cup E)| = n - r - 1$ elements to choose from, and there are at most $n - r - 1$ elements in $\mathcal{H}(C)$, so we can choose an element that was not used previously in any 3-class of a τ-label in $\mathcal{H}(C)$.

Finally assume that $d \geq 4$, $g = 1$ and $t \geq 1$. As above, if $|\mathcal{H}(C)| = n - r$ then the edges with the core C are consecutive in $\mathcal{H}$, and so the corresponding path in $\mathcal{H}$ can be written as $\{a_1\} \cup C, \{a_2\} \cup C, \ldots, \{a_{n-r}\} \cup C, \{a_{n-r+1}\} \cup C$ for distinct $a_1, a_2, \ldots, a_{n-r}, a_{n-r+1} \in X_n$. Note that $n - r + 1 = d + 1 \geq 4$, and for $i = 1, 2, \ldots, n - r - 1$, let γ_i have the gap set $\{a_{i+2}\}$, and let γ_{n-r} have the gap set $\{a_1\}$. Then γ_i is a τ-label for the edge $\{a_i\} \cup C, \{a_{i+1}\} \cup C$ for $i = 1, 2, \ldots, n - r$, and $\gamma_i \neq \gamma_j$ if $i \neq j$. If $|\mathcal{H}(C)| < n - r$ we can simply choose a suitable τ-label γ for an edge DE in $\mathcal{H}(C)$ that was not used previously to label the edges in $\mathcal{H}(C)$. Indeed, we only need to choose the single element gap set of γ, there are $|X_n \setminus (D \cup E)| = n - r - 1$ elements to choose from, and there are at most $n - r - 1$ elements in $\mathcal{H}(C)$. $\square$

We use the above results to prove that any partition type $d\,2\,1^t$ with a non-zero gap and is flexible, regardless whether it has any non-singleton classes. In view of Proposition 4.7, Lemma 5.1 and Lemma 5.5 we only need to consider the cases outlined in the result below.

Lemma 6.3 *Partition types* $3\,2\,1^t$ *with gap* $g = 1$ *or* 2, $4\,2\,1^t$ *with gap* $g = 1$, *and* $5\,2\,1^t$ *with gap* $g = 1$ *are flexible for* $t \geq 0$.

Proof. Let $d = 3, 4, 5$, $g = 1, 2$ and let $\tau = d\,2\,1^t$ with $t \geq 0$. Let $\mathcal{H}$ be a Hamiltonian cycle in $J(n, r)$, where $n = d + 2 + t + g$, and $r = t + 2$. As usual, we τ-label $\mathcal{H}$ sequentially, starting at an edge and following the cycle in a fixed direction.

First observe that if $q = d + t + g$, $s = t + 1$ and $C_1, C_2, \ldots, C_{\binom{q}{s}}$ is a listing of all s-element subsets of X_q then there exist distinct partitions $\pi_1, \pi_2, \ldots, \pi_{\binom{q}{s}}$ of type $d\,1^t$ such that C_j and π_j are orthogonal to each other for all $j = 1, 2, \ldots, \binom{q}{s}$. Indeed, let $\mathcal{B} = B_1, B_2, \ldots, B_{\binom{q}{s}}$ be a Hamiltonian cycle in $J(q, s)$. By Lemma 6.2, $\mathcal{B}$ can be $d\,1^t$-labeled, so there exist distinct partitions $\gamma_1, \gamma_2, \ldots, \gamma_{\binom{q}{s}}$ of type $d\,1^t$ such that γ_j is orthogonal to B_j (as well as B_{j+1}). Let χ be a permutation of the set $\{1, 2, \ldots, \binom{q}{s}\}$ such that $C_i = B_{\chi(i)}$ for all i, and let $\pi_i = \gamma_{\chi(i)}$. Then C_j and π_j are orthogonal to each other for all j as required.

Now let $\mathcal{H}$ be a Hamiltonian cycle in $J(n, r)$ as above, and suppose $\{a, b\}$ is the symmetric difference of the sets H_i, H_{i+1} forming an edge in $\mathcal{H}$. Let $D(a, b)$ be the listing of all the edges H_j, H_{j+1} in $\mathcal{H}$ for which $\{a, b\}$ is the symmetric difference of the sets H_j and H_{j+1} (appearing in the same relative order as they appear in $\mathcal{H}$. Note that $|D(a, b)| \leq \binom{n-2}{r-1}$ as any edge H_j, H_{j+1} in $D(a, b)$ is uniquely determined by its core $C_j = H_j \cap H_{j+1}$. We define a τ-label σ_j for an edge in H_j, H_{j+1} in $D(a, b)$ by letting the two-element class of σ_j be the symmetric difference $\{a, b\}$ and the rest of the classes coincide with π_j defined as above. Then σ_j is a τ-label for the edge H_j, H_{j+1}, the τ-labels for a given $D(a, b)$ are distinct since the partitions $\pi_1, \pi_2, \ldots, \pi_{\binom{q}{s}}$ are distinct, and the τ-labels for $D(a, b)$ and $D(c, d)$ with $\{a, b\} \neq \{c, d\}$ are distinct as they have different two-element classes. □

7 Proof of Theorem 1.1

Take $n \geq 3$ and let τ be a partition type defined on X_n with a non-zero gap g and weight $r \geq 1$. Since Proposition 6.2 shows that a partition type $d\,1^t$ is flexible for $d \geq 2$ and $t \geq 0$, we may assume that τ has at least two non-singleton classes.

To prove the first part of Theorem 1.1, suppose first that τ has at least one class of size greater than two. If $n \geq 9$ and τ has no singleton classes, then by Theorem 4.2, τ is flexible. If $3 \leq n \leq 8$ and τ has no singleton classes, it is either of the form $d\,2$ or $3\,2^2$ or 3^2 (Theorem 4.2). If $\tau = d\,2$ then by Lemma 6.3, τ is flexible. That leaves $\tau = 3\,2^2$ or 3^2 with gap $g = 1$.

Thus assume that $\tau = \bar{\tau} \oplus 1^t$, where $t \geq 1$ and $\bar{\tau}$ is a partition type with no singletons and gap g defined on a $(n-t)$-element set. If $\bar{\tau}$ satisfies the Generalized Edge Inequality $M(\bar{\tau}, g) \geq |\mathcal{C}(n,r)|$ then by Lemma 5.1, $\tau = \bar{\tau} \oplus 1^t$ is flexible. So assume that $\bar{\tau}$ does not satisfy the Generalized Edge Inequality, thus by Theorem 4.2 we have that $n - t \leq 8$ and either $\bar{\tau} = d\,2^{r-1}$ (with restrictions on d, r and g as stated in Theorem 4.2) or $\bar{\tau} = 3^2$ with $g = 1$. Lemma 5.5 shows that if $\bar{\tau} = d\,2^{r-1}$ then $\tau = \bar{\tau} \oplus 1^t$ with $t \geq 1$ is flexible unless $\tau = 3\,2\,1^t$ with $g = 1$ or $\tau = 4\,2\,1$ with $g = 1$. Lemma 6.3 shows that $3\,2\,1^t$ and $4\,2\,1$ are flexible. It is easy to check that $3^2\,1^t$ does not satisfy the Generalized Edge Inequality $M(3^2\,1^t, 1) \geq |\mathcal{C}(t+7, t+2)|$, and this completes a proof of the first part of Theorem 1.1.

To prove the second part of the theorem, assume that $\tau = 2^{r-t}\,1^t$ with $r - t \geq 2$ and $t \geq 0$. If $t = 0$, Theorem 4.2 shows that τ is flexible unless r is between 2 and 6 and the gap that satisfies the stated conditions. If $t > 0$, Lemma 5.4 shows that τ is flexible unless it is of the form $2^4\,1$ with $g = 1, 2$ or $2^4\,1^2$ with $g = 1$. □

In conclusion we list of the partition types with any gap $g \geq 0$ that may not be flexible.

1. **Partition types with zero gap**
   (a) $d\,3\,1^t$ with $d > 3, t \geq 0$;
   (b) $d\,2^s\,1^t$ with $d \geq 3$, $s, t \geq 1$;
   (c) $3^2\,2\,1$;
   (d) $d^2\,1^t$ with $d = 3, 4, 5$ and $t \geq 0$;
   (e) $3^3\,1^t$ with $t \geq 0$;
   (f) $2^4\,1^t$ with $t = 1, 2, 3, 4$;
   (g) $2^5\,1^t$ with $t = 1, 2$;
   (h) $2^6\,1$.
2. **Partition types with non-zero gap** g
   (a) $3^2\,1^t$ with $t \geq 0$ with $g = 1$;
   (b) $3\,2^2$ with $g = 1$;
   (c) $2^s\,1^t$ with $s = 2, 3$, $t \geq 0$, $g \geq 1$;
   (d) 2^4 with $1 \leq g \leq 4$; 2^5 with $1 \leq g \leq 3$; 2^6 with $g = 1$;
   (e) $2^4\,1$ with $g = 1, 2$ or $2^4\,1^2$ with $g = 1$.

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