A SHORT PROOF OF q-BINOMIAL THEOREM

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ABSTRACT. In this paper, we show a short proof of q-binomial theorem by Schützenberger's identity with q-commuting variables.

1. INTRODUCTION

The q-binomial theorem is a fundamental result in the theory of q-series. Many proofs of this theorem have appeared in the literature [1]. In this section, we give a new proof of it via Schützenberger's identity.

We first recall some definitions, notations and known results in [3] which will be used in the proof. Throughout the whole paper, it is supposed that $0 < |q| < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.1}$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{1.2}$$

Tannery's Theorem [2] or [5] is a special case of Lebesgues dominated convergence theorem on the sequence space L^1.

Tannery's Theorem: If $s(n) = \sum_{k\geq 0} f_k(n)$ is a finite sum (or a convergent series) for each n, $\lim\limits_{n\to\infty} f_k(n) = f_k$, $|f_k(n)| \leq M_k$, and $\sum_{k=0}^{\infty} M_k < \infty$, then

$$\lim_{n\to\infty} s(n) = \sum_{k=0}^{\infty} f_k. \tag{1.3}$$

Schützenberger's identity with q-commuting variables [4] is very interesting and useful. It can be read as: let A and B be two linear operators

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satisfying $BA = qAB$, then we have

$$(A+B)^n = \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] A^k B^{n-k}. \tag{1.4}$$

Now we give a new proof of q-binomial theorem.

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1. \tag{1.5}$$

Proof. Let $A = z(1-a)\eta_1$ and $B = (1-z)\eta_2$, where η_1, η_2 are two linear operators defined by

$$\eta_1\{f(a,z)\} = f(qa,z), \quad \eta_2\{f(a,z)\} = f(a,qz).$$

It is easy to know $BA = qAB$. Let $A = z(1-a)\eta_1$ and $B = (1-z)\eta_2$ in (1.4) and both sides of (1.4) act on 1, we get

$$(A+B)^n\{1\} = \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] A^k B^{n-k}\{1\}. \tag{1.6}$$

By direct calculation, we gave

$$(A+B)^n\{1\} = (az;q)_n, \quad A^k B^{n-k}\{1\} = (a;q)_k z^k (z;q)_{n-k}. \tag{1.7}$$

Substituting (1.7) into (1.6), we get

$$(az;q)_n = \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] (a;q)_k z^k (z;q)_{n-k}. \tag{1.8}$$

After let $n \to \infty$ in (1.8) and use Tannerys theorem, we get (1.5). □

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