Characterization of convex polyhedra with regular polygonal faces by minimal number of parameters

Jurij Kovič
Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia
amadea.kovic@siol.net

April 12th, 2012

Abstract

In order to characterize convex polyhedra with regular polygonal faces by minimal number of parameters we first introduce some new parameters, then we analyze a table of their values to see how well different sets of parameters tell these solids apart and finally we present their characterization by four parameters.

1 Introduction

For any given class of elements we can formulate the following »characterization problem«: »*Find a minimal set of parameters (describing some properties of the elements of the given class) such that the sequences of these parameters* $(p_1, p_2, ..., p_k)$ *are different for each element of the given class.*« If we find such parameters, then the sequences $(p_1, p_2, ..., p_k)$ uniquely determine or »code« the elements of the given class. This does not mean that we can reconstruct the elements from their parameters, it means only that we can tell them apart by them. In other words, the code we are looking for need not be »generic«, it should only be a »separating« one.

The problem we discuss in this paper: »*Find a minimal set of parameters characterizing convex polyhedra with regular polygonal faces.*« falls into this category of problems. We believe that »*the method of evaluating different sets of parameters with respect to their capacity to tell the elements of the given class apart*« used for solving this particular problem is applicable to other characterization problems.

JCMCC **89** (2014), pp. 249-263

On the other hand, our characterization problem belongs to a great family of problems with the same general theme: *how well various sets of parameters determine convex polyhedra.* Several categories of such problems were presented in detail by Alexandrov in his book on convex polyhedra [1]. There, for example, the questions of uniqueness and existence of convex polyhedra with prescribed development, or with prescribed face directions, are discussed. Another similar yet different problem of establishing combinatorial equivalence of polyhedra, which remains unsolved since Euler's times, is discussed by Grünbaum ([4], p. 294).

2 Basic notions

2.1 Convex polyhedra with regular polygonal faces

Let C denote the class of convex polyhedra with regular polygonal faces. The subclass of *uniform convex polyhedra* consist of the 5 *Platonic solids*, the 13 *Archimedean solids* and two infinite families of *prisms* and *antiprisms* ([1]). Since they are uniform, they may be denoted by the vertex-type symbol, for example $3 \cdot 4^2$, ([3] p. 107, 130 ff., [5], p. 170). The numbers 3, 4, 5, ... denote regular polygons, as in ([5], p. 169).

The subclass of *non-uniform convex polyhedra* consists of 92 solids, as it was conjectured by Johnson [5] and proved in a series of paper by Johnson, Grünbaum and Zallgaler [9]. These solids, called *Johnson solids*, are denoted J1, J2, ... , J92; their numbers are the same as in Johnson's list [5]. All the polyhedra from the class C have at least one symmetry.

2.2 Symmetry-type graphs and related parameters

Here we sketch the basic concepts of the theory of the symmetry-type graphs, a useful tool for studying maps, polyhedra and tilings ([7],[8]).

FLAG GRAPH. A *flag* is an ordered triple (v, e, f) consisting of a vertex v, an edge e incident with that vertex and a face f incident with that flag. If all the faces f of a polyhedron $\mathcal{P}$ are regular polygons, we can identify each flag $\Phi = (v, e, f)$ with the corresponding *characteristic triangle* $\Delta_\Phi = (v, e_c, f_c)$ whose vertices are the vertex v, the midpoint e_c of the edge e, and the center f_c of the face f face. Each flag Φ has three *adjacent flags*, sharing an edge with Φ: the *0-adjacent flag* Φ^0 lies in the same face f as Φ and along the same edge of f; the *1-adjacent flag* Φ^1 lies in the same face f as Φ, but not along the same edge of f; the *2-adjacent flag* Φ^2 lies along the same edge of f, but not in the same face as Φ. The *flag graph* $G_\mathcal{P}$ of a polyhedron $\mathcal{P}$ is a graph whose vertex set correspond to the flags of $\mathcal{P}$. The edges connecting pairs of adjacent flags (Φ, Φ^0), (Φ, Φ^1), (Φ, Φ^2) are labeled $0, 1$ and 2, respectively.

AUTOMORPHISMS OF THE FLAG GRAPH. Let $Aut(G_{\mathcal{P}})$ denote the group of automorphisms of the flag graph $G_{\mathcal{P}}$, preserving not only adjacency of vertices of $G_{\mathcal{P}}$ but also the labels 0, 1, 2 of edges.
ORBIT OF A FLAG. The *orbit* $T(\Phi)$ of a flag Φ is a set of all flags into which Φ is carried by all the rotations and reflections preserving the polyhedron.
SYMMETRY-TYPE GRAPH $T(\mathcal{P})$. The quotient graph of $G_{\mathcal{P}}$ under the action of the group of rotations and reflections preserving the polyhedron is called the *symmetry-type graph* of the polyhedron $\mathcal{P}$ and is denoted by $T(\mathcal{P})$. Its vertices are orbits of flags of $G_{\mathcal{P}}$ and its edges labeled 0, 1 and 2 correspond to the labeled edges of their representatives. It is very useful to know that the symmetry-type graph corresponds to a part of the polyhedron surface whose reflected and rotated copies produce the whole polyhedron surface.
POLYHEDRON NET. Symmetry-type graph $T(\mathcal{P})$ of any solid $\mathcal{P} \in C$ can be very easily deduced from the planar representation of its net:

i) If $\mathcal{P}$ has a rotation symmetry we first make a parallel projection of its net on the surface of the cylinder (the poles of the principal rotation axis of $\mathcal{P}$ correspond to the top and bottom circle of the cylinder). Cutting the cylinder along a vertical line we get a planar picture of a polyhedron net.

ii) If $\mathcal{P}$ has only one reflection symmetry (and no rotation symmetries), we make a parallel projection of (any) half of its surface projected onto the reflection plane of $\mathcal{P}$. From this projection we easily get the flag graph and identify the orbits of flags, thus obtaining the symmetry-type graph $T(\mathcal{P})$.

We can use also the Schlegel diagrams of the solids ([4], pp. 42–46) showing their »stretched and twisted« 1-skeletons from a »bird's eye view«.

The number of flag orbits equals the number of vertices of the symmetry-type graph $T(\mathcal{P})$ (see Figure 1). The numbers of orbits of vertices, edges and faces correspond to the numbers of connected components of the graphs, obtained from $T(\mathcal{P})$, if we delete its 0-, 1- or 2-edges, respectively ([8]).

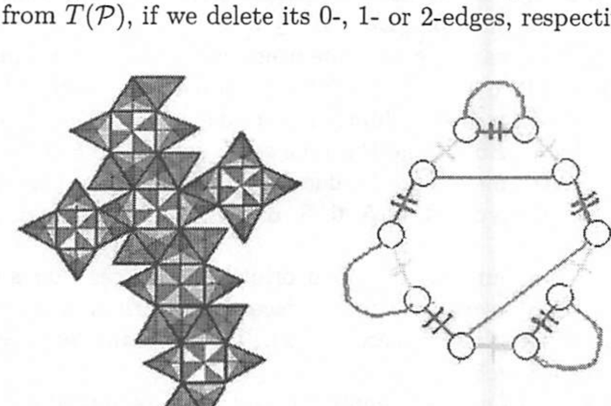


Figure 1: A net of a Snub cube ($3^4.4$) with 10 orbits of flags (left) and the corresponding symmetry-type graph with 10 vertices (right).

3 Review of parameters

In Section 4 we give the values of 16 parameters of convex polyhedra with regular polygonal faces for which we believed they could help us solve our characterization problem. We could not know in advance which of them will be useful and which redundant; this became more clear only after a long analysis of the gathered data.

We first give for each solid $\mathcal{P} \in C$ the numbers of its vertices, edges and faces, denoted by v, e, f, respectively. These three parameters, satisfying the Euler formula $v - e + f = 2$ ([6]), are obviously not enough to tell the solids of the class C apart (since, for example, the solids J72, J73, J74, J75 and J75 all have $v = 60$, $e = 120$ and $f = 62$).

Then we give the numbers of flag orbits flo, and the numbers of symmetries s of $\mathcal{P}$. Since there are four flags along each edge the total number of flags in the flag graph is $4e$; and since the number of flags in each orbit is the same, the number of symmetries is obtained by the formula ([2]):

$$s(\mathcal{P}) = 4e/flo.$$

Then follow the numbers vo, eo, fo of orbits of vertices, edges and faces, and the »Euler orbit characteristic«, which we define by the formula:

$$Eo = vo - eo + fo.$$

From the Table 1 we can easily see that for each $P \in C$ an unexpected inequality holds:

$$0 \leq Eo \leq 5.$$

Hence for each $P \in C$ the value of Eo differs from the Euler characteristic $E = v - e + f$ for at most 3 : $|Eo - E| \leq 3$.
From the Table 1 we can calculate the average value of Eo for the class of Johnson solids: $\frac{1}{92}(0 \cdot 6 + 1 \cdot 17 + 2 \cdot 32 + 3 \cdot 16 + 4 \cdot 15 + 5 \cdot 6 = 2.391$. The average value of Eo for the Archimedean solids is $\frac{1}{13}(1 \cdot 9 + 4 \cdot 2) = 1.307$. The average value of Eo for the Platonic solids is $\frac{1}{5}(1 \cdot 5) = 1$.

The next two parameters are N, denoting the number of maximal faces, and n, denoting the type (3, 4, 5, 6, 8, or 10) of the maximal faces in a given solid.

Then we give the numbers of »polar orbits« of vertices, edges and faces (i.e. the orbits, whose vertices, edges or faces are invariant for a non-trivial rotation for the angle $2\pi/a$, where $a \geq 2$). These parameters are denoted vo^p, eo^p, fo^p, respectively.

The last two parameters, denoted r and q, were introduced after it became clear that all the previous ones do not separate the solids from the class C. They were found by observing such pairs of solids (e. g. J76 and J77).

Parameter r is defined as the maximal sum $f_1 + \ldots + f_k$ of all the faces in the sequences $(f_1, \ldots, f_k)$ with the following properties:
i) the faces $f_1, \ldots, f_k$ intersect the same reflection plane of the given solid;
ii) any two consecutive faces f_i, f_{i+1} share either an edge or a vertex;
iii) they are of the form (a^p) or $(a^p.b^q)$ (they consist of $p \geq 2$ equal faces of the type a, and they may have also $q \geq 2$ faces of the type b).

If no such sequence exists, we put $r = 0$.

Parameter q is defined as the maximal sum $f_1 + \ldots + f_k$ of all the faces in the sequences $(f_1, \ldots, f_k)$ with the following properties:
i) the faces $f_1, \ldots, f_k$ intersect the same reflection plane of the given solid;
ii*) any two consecutive faces f_i, f_{i+1} share an edge;
iii*) at least two consecutive faces in this sequence are the same.

If no such sequence exists, we put $q = 0$.

For some spherical polyhedra the value of their Euler orbit characteristic $Eo = vo - eo + fo$ can be obtained from a simpler parameter:

Proposition 1 *If a spherical polyhedron $\mathcal{P}$ has only one rotation axis and no reflection symmetries then $Eo = 2 - eo^p$. Hence:*
$Eo = 0$, if (vo^p, eo^p, fo^p) equals (2,0,0) or (1,0,1) or (0,0,2);
$Eo = 1$, if (vo^p, eo^p, fo^p) equals (1,1,0) or (0,1,1);
$Eo = 2$, if (vo^p, eo^p, fo^p) equals (0,2,0).

Proof. For any spherical polyhedron its Euler characteristic is $E = v - e + f = 2$. If $\mathcal{P}$ is spherical and has no reflection symmetries, then it has exactly two polar orbits. If $I(\mathcal{P}) = \mathbb{Z}_m$, then all other orbits (of vertices, edges and faces) contain exactly m elements. Let vo_m, eo_m, fo_m denote the number of orbits of vertices, edges, or faces with with m elements. Then $Eo = (vo^p - eo^p + fo^p) + (vo_m - eo_m + fo_m)$. And from the Euler formula follows the relation: $2 = E = (vo^p - eo^p + fo^p).1 + (vo_m - eo_m + fo_m).m$, implying: $(vo_m - eo_m + fo_m) = (2 - (vo^p - eo^p + fo^p))/m$. Hence: $Eo = (vo^p - eo^p + fo^p) + (2 - (vo^p - eo^p + fo^p))/m$. The only possible values of $vo^p - eo^p + fo^p$ are 2, 0 and -2. If $vo^p - eo^p + fo^p = 2$, then $Eo = vo^p - eo^p + fo^p = 2$. If $vo^p - eo^p + fo^p = 0$, then $m = 2$ and $Eo = 0 + 2/2 = 1$. If $vo^p - eo^p + fo^p = -2$, then $m = 2$ and $Eo = -2 + 4/2 = 0$. □

In Figure 2 it is shown how some of more complicated parameters are determined (for the Johnson solid $J36$). There are 12 different flag orbits, 2 vertex orbits, 5 edge orbits and 4 face orbits, no polar orbit of vertices, 1 polar orbit of edges and 1 polar orbit of faces. The values of parameters r and q for this solid are both 14, but they are not calculated in the same way: r is obtained from the sequence 4-4-3-3 of two squares followed by two triangles, while q is obtained from the sequence 3-4-4-3.

In the Figure 2 (top left) we can see the representative flags of the 12 orbits of flags. They form a fundamental domain whose rotated and reflected copies cover all the surface of J36. The number of these copies is the same as the number of symmetries of J36.

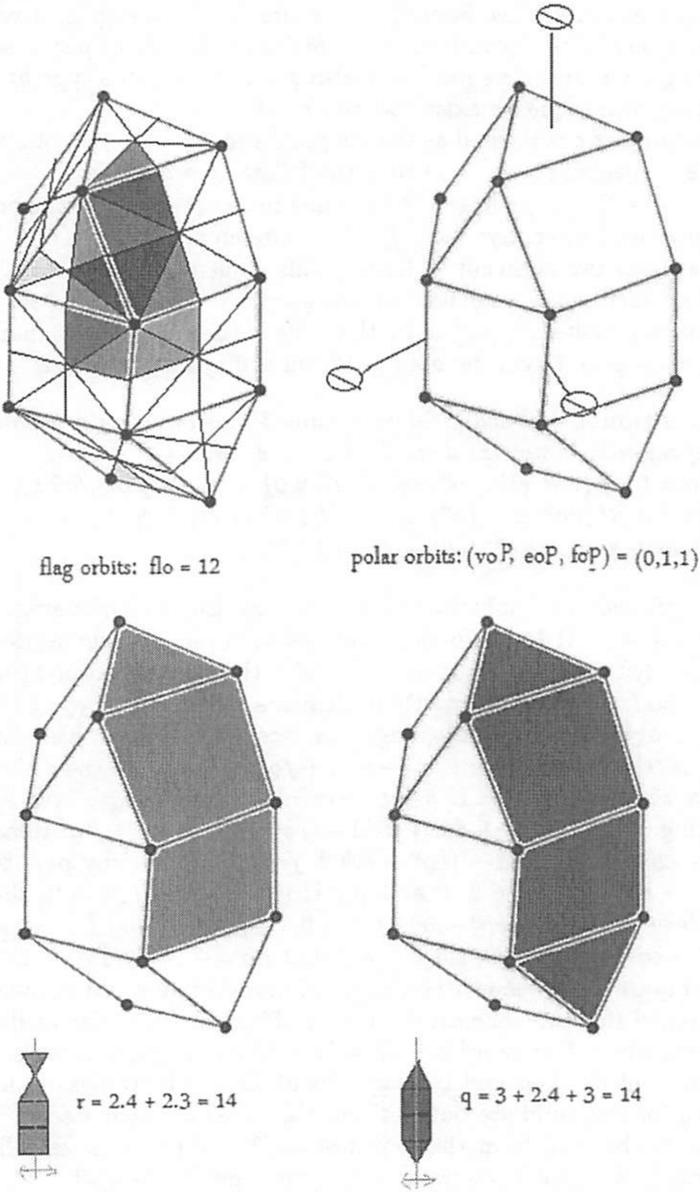


Figure 2: Some parameters of the Johnson solid J36.

4 Values of parameters

Jxx	v, e, f	flo	s	v_o, e_o, f_o	E_o	N,n	v_o^p, e_o^p, f_o^p	r	q
J1	5, 8, 5	4	8	2, 2, 2	2	1, 4	1, 0, 1	0	0
J2	6, 10, 6	4	10	2, 2, 2	2	1, 5	1, 0, 1	0	0
J3	9, 15, 8	10	6	2, 4, 4	2	1, 6	0, 0, 2	6	0
J4	12, 20, 10	10	8	2, 4, 4	2	1, 8	0, 0, 2	12	20
J5	15, 25, 12	10	10	2, 4, 4	2	1, 10	0, 0, 2	0	0
J6	20, 35, 17	14	10	3, 5, 5	3	1, 10	0, 0, 2	16	0
J7	7, 12, 7	8	6	3, 4, 3	2	3, 4	1, 0, 1	0	0
J8	9, 16, 9	8	8	3, 4, 3	2	5, 4	1, 0, 1	12	18
J9	11, 20, 11	8	10	3, 4, 3	2	1, 5	1, 0, 1	0	0
J10	9, 20, 13	10	8	3, 4, 4	3	1, 4	1, 0, 1	12	6
J11	11, 25, 16	10	10	3, 4, 4	3	1, 5	1, 0, 1	6	6
J12	5, 9, 6	3	12	2, 2, 1	1	6, 3	2, 1, 0	6	6
J13	7, 15, 10	3	20	2, 2, 1	1	10, 3	2, 1, 0	6	6
J14	8, 15, 9	5	12	2, 3, 2	1	3, 4	1, 1, 1	12	12
J15	10, 20, 12	5	16	2, 3, 2	1	4, 4	1, 1, 1	16	16
J16	12, 25, 15	5	20	2, 3, 2	1	5, 4	1, 1, 1	20	20
J17	10, 24, 16	6	16	2, 3, 2	1	16, 3	1, 1, 0	12	6
J18	15, 27, 14	18	6	3, 7, 6	2	1, 6	0, 0, 2	8	16
J19	18, 36, 20	18	8	3, 7, 6	2	1, 8	0, 0, 2	20	28
J20	25, 45, 22	18	10	3, 7, 6	2	1, 10	0, 0, 2	8	21
J21	30, 55, 27	22	10	4, 8, 7	3	1, 10	0, 0, 2	16	0
J22	15, 33, 20	22	6	4, 7, 7	4	1, 6	0, 0, 2	9	6
J23	20, 44, 26	22	8	4, 7, 7	4	1, 10	0, 0, 2	12	18
J24	25, 55, 32	22	10	4, 7, 7	4	1, 10	0, 0, 2	6	6
J25	30, 65, 37	26	10	5, 8, 8	5	1, 10	0, 0, 2	9	6
J26	8, 14, 8	7	8	2, 3, 2	1	4, 4	1, 1, 0	8	14
J27	12, 24, 14	8	12	2, 4, 3	1	6, 4	0, 2, 1	20	14
J28	16, 32, 18	8	16	2, 4, 3	1	10, 4	0, 2, 1	24	24
J29	16, 32, 18	8	16	2, 4, 3	1	10, 4	1, 0, 1	12	18
J30	20, 40, 22	8	20	2, 4, 3	1	2, 5	0, 2, 1	8	18
J31	20, 40, 22	8	20	2, 3, 3	2	2, 5	1, 0, 1	0	0
J32	25, 50, 27	20	10	4, 7, 7	4	7, 5	0, 0, 2	16	0
J33	25, 50, 27	20	10	4, 7, 7	4	7, 5	0, 0, 2	19	6
J34	30, 60, 32	12	20	3, 5, 4	2	12, 5	0, 2, 1	32	10
J35	18, 36, 20	12	12	2, 5, 5	2	12, 4	0, 0, 3	24	24
J36	18, 36, 20	12	12	2, 5, 4	1	12, 4	0, 1, 1	14	14
J37	24, 48, 26	12	16	2, 5, 4	1	18, 4	0, 1, 1	20	26
J38	30, 60, 32	12	20	2, 5, 5	2	2, 5	0, 0, 3	40	40
J39	30, 60, 32	12	20	2, 5, 4	1	2, 5	0, 1, 1	8	16
J40	35, 70, 37	28	10	5,10, 9	4	7, 5	0, 0, 2	8	16
J41	35, 70, 37	28	10	5, 10, 9	4	7, 5	0, 0, 2	18	18
J42	40, 80, 42	16	20	3, 6, 6	3	10, 5	0, 0, 3	40	40
J43	40, 80, 42	16	20	3, 6, 5	2	12, 5	0, 1, 1	20	0
J44	18, 42, 26	28	6	3, 8, 5	0	6, 4	0, 2, 1	0	0
J45	24, 56, 34	28	6	3, 8, 5	0	10, 4	0, 2, 1	0	0
J46	30, 70, 42	28	10	3, 8, 5	0	2, 5	0, 2, 1	0	0
J47	35, 80, 47	64	5	7, 16, 11	2	7, 5	0, 0, 2	0	0
J48	40, 90, 52	36	10	4, 10, 6	0	12, 5	0, 2, 1	0	0

Jxx	v, e, f	flo	s	v_o, e_o, f_o	E_o	N, n	v_o^p, e_o^p, f_o^p	r	q
J49	7, 13, 8	13	4	3, 5, 4	2	2, 4	1, 1, 0	14	14
J50	8, 17, 11	17	4	3, 6, 5	2	1, 4	0, 1, 1	12	6
J51	9, 21, 14	7	12	2, 3, 3	2	14, 3	1, 1, 1	12	6
J52	11, 19, 10	19	4	4, 6, 5	3	2, 5	1, 1, 0	22	22
J53	12, 23, 13	23	4	4, 8, 6	2	2, 5	0, 1, 1	14	14
J54	13, 22, 11	22	4	4, 8, 6	2	2, 6	1, 0, 1	26	26
J55	14, 26, 14	13	8	3, 5, 4	2	2, 6	1, 1, 1	14	14
J56	14, 26, 14	26	4	4, 9, 7	2	2, 6	0, 0, 2	18	18
J57	15, 30, 17	10	12	2, 4, 4	2	2, 6	1, 0, 2	6	0
J58	21, 35, 16	14	10	5, 6, 4	3	11, 5	1, 0, 1	10	10
J59	22, 40, 20	8	20	3, 4, 2	1	10, 5	1, 1, 0	0	0
J60	22, 40, 20	40	4	8, 10, 7	5	10, 5	0, 2, 0	10	10
J61	23, 45, 24	30	6	7, 10, 6	3	9, 5	2, 0, 0	10	10
J62	10, 20, 12	20	4	4, 7, 5	2	2, 5	0, 2, 0	16	10
J63	9, 15, 8	10	6	3, 4, 4	3	3, 5	2, 0, 0	6	6
J64	10, 18, 10	12	6	4, 5, 4	3	3, 5	1, 0, 1	6	6
J65	15, 27, 14	18	6	4, 7, 6	3	3, 6	0, 0, 2	6	0
J66	28, 48, 22	24	8	5, 9, 7	3	5, 8	0, 0, 2	24	30
J67	32, 60, 30	15	16	3, 6, 5	2	4, 8	0, 1, 2	32	32
J68	65, 105, 24	42	10	9, 15, 10	4	11, 5	0, 0, 2	16	22
J69	70, 120, 52	24	20	5, 9, 6	2	10, 10	0, 1, 1	0	0
J70	70, 120, 52	120	4	20, 35, 18	3	10, 10	0, 2, 0	16	22
J71	75, 135, 62	90	6	15, 27, 16	4	9, 10	0, 0, 2	16	22
J72	60, 120, 62	48	10	8, 16, 13	5	12, 5	0, 0, 2	8	16
J73	60, 120, 62	24	20	4, 8, 7	3	12, 5	0, 0, 2	8	16
J74	60, 120, 62	120	4	17, 34, 22	5	12, 5	0, 0, 2	12	22
J75	60, 120, 62	80	6	12, 24, 17	5	12, 5	0, 0, 2	8	16
J76	55, 105, 52	42	10	7, 14, 11	4	1, 10	0, 0, 2	0	0
J77	55, 105, 52	42	10	7, 14, 11	4	1, 10	0, 0, 2	8	16
J78	55, 105, 52	210	2	29, 56, 31	4	1, 10	0, 0, 0	8	23
J79	55, 105, 52	210	2	29, 56, 31	4	1, 10	0, 0, 0	0	0
J80	50, 90, 42	18	20	3, 6, 5	2	2,10	0, 0, 2	0	0
J81	50, 90, 42	90	4	14, 26, 16	4	2, 10	0, 0, 2	0	0
J82	50, 90, 42	180	2	27, 49, 27	5	2, 10	0, 0, 0	8	16
J83	45, 75, 32	50	6	9, 16, 11	4	3, 10	0, 0, 2	0	0
J84	8, 18, 12	9	8	2, 4, 2	0	12, 3	0, 3, 0	6	6
J85	16, 40, 26	10	16	3, 4, 3	2	2, 4	0, 1, 1	0	0
J86	10, 22, 14	22	4	4, 8, 5	1	2, 4	0, 2, 0	8	14
J87	11, 26, 17	52	2	7, 15, 11	3	1, 4	0, 0, 0	12	6
J88	12, 28, 18	28	4	5, 10, 6	1	2, 4	0, 2, 0	8	14
J89	14, 33, 21	33	4	5, 11, 8	2	2, 4	0, 1, 1	12	18
J90	16, 38, 24	19	8	3, 7, 4	0	4, 4	0, 2, 0	8	14
J91	14, 26, 14	13	8	3, 5, 4	2	4, 5	1, 1, 1	20	10
J92	18, 36, 20	24	6	4, 8, 7	3	1, 6	0, 0, 2	6	0

Table 1a: Parameters of Johnson solids.

solid	v, e, f	flo	s	v_o, e_o, f_o	E_o	N, n	v_o^p, e_o^p, f_o^p	r	q
3^3	4, 6, 4	1	24	1, 1, 1	1	4, 3	1, 1, 1	6	6
3^4	6, 12, 8	1	48	1, 1, 1	1	8, 3	1, 1, 1	12	6
4^3	8, 12, 6	1	48	1, 1, 1	1	6, 4	1, 1, 1	16	16
3^5	12, 30, 20	1	120	1, 1, 1	1	20, 3	1, 1, 1	6	6
5^3	20, 30, 12	1	120	1, 1, 1	1	12, 5	1, 1, 1	10	10
3.4.3.4	12, 24, 14	2	48	1, 1, 2	2	6, 4	1, 0, 2	16	0
3.5.3.5	30, 60, 32	2	120	1, 1, 2	2	12, 5	1, 0, 2	16	0
3.6^2	12, 18, 8	3	24	1, 2, 2	1	4, 6	0, 1, 2	12	18
3.8^2	24, 36, 14	3	48	1, 2, 2	1	6, 8	0, 1, 2	32	32
4.6^2	24, 36, 14	3	48	1, 2, 2	1	8, 6	0, 1, 2	12	20
3.10^2	60, 90, 32	3	120	1, 2, 2	1	12, 10	1, 0, 2	20	26
5.6^2	60, 90, 32	3	120	1, 2, 2	1	20, 6	1, 0, 2	12	22
3.4^3	24, 48, 26	4	48	1, 2, 3	2	18, 4	0, 0, 3	32	32
3.4.5.4	60, 120, 62	4	120	1, 2, 3	2	12, 5	0, 0, 3	0	0
4.6.8	120, 180, 62	6	48	1, 3, 3	1	6, 8	0, 0, 3	0	0
4.6.10	24, 60, 38	6	120	1, 3, 3	1	12, 10	0, 0, 3	0	0
$3^4.4$	16, 38, 24	10	24	1, 3, 3	1	6, 4	0, 1, 2	0	0
$3^4.5$	60, 150, 92	10	60	1, 3, 3	1	12, 5	0, 1, 2	0	0
$4^2.n$	$2n, 3n, n+2$	3	4n	1, 2, 2	1	2,n	1, 1, 1	n^2	n^2
$3^3.n$	$2n, 4n, 2n+2$	4	4n	1 , 2, 1	1	2,n	0, 2, 1	0	0

Table 1b: Parameters of the 5 Platonic solids, the 13 Archimedean solids, and of nfinite families of n-prisms 4^2n. and n-antiprisms 3^3n.

5 Analysis of solids and parameters

The information gathered in the Table 1 allows us not only to compare and classify the solids with respect to different sets of parameters, but also to compare different sets of parameters with respect to their capacity to separate the solids (into as many different equivalent classes as possible)!

5.1 Two ratios by which we can compare sets of parameters

If a set of parameters $\{p_1, \ldots, p_k\}$ divides a class with m objects into c equivalent classes (two objects are in the same equivalent class if they have the same values of all the parameters) and if it isolates i objects (an object is isolated if it is alone in its equivalence class), this gives us two ratios: m/c, *the average number of objects in equivalence classes*, and $0 \leq i/m \leq 1$, called *the isolating quotient of the given set of parameters* (with respect to the classified class). Objects are completely separated by the set of parameters $\{p_1, \ldots, p_k\}$ if and only if its isolating quotient is 1.

For example, for $\{s, Eo\}$ the first ratio is $m/c = 92/31$ and the isolating quotient with respect to the Johnson solids is $i/m = 10/92$ (see Table 2 below). The Table 2 shows equivalence classes of Johnson solids obtained by two parameters: the Euler orbit characteristic Eo and the number of symmetries s. We can see that the most common value of Euler orbit characteristic for the class of Johnson solids is 2.

From Table 2 we can calculate the average number of the symmetries of Johnson solids: it is $\frac{1}{92}(2 \cdot 4 + 4 \cdot 14 + 5 \cdot 1 + 6 \cdot 12 + 8 \cdot 12 + 10 \cdot 20 + 12 \cdot 7 + 16 \cdot 7 + 20 \cdot 13) = 9.51$.

Eo / s	2	4	5	6	8	10	12	16	20	Σ
0				J44	J84	J46				
0				J45	J90	J48				6
1		J86			J26		J12	J15	J13	
1		J88					J14	J17	J16	
1							J27	J28	J30	
1							J36	J29	J39	
1								J37	J59	17
2		J49	J47	J3	J1	J2	J35	J67	J31	
2		J50		J7	J4	J5	J51	J85	J34	
2		J53		J18	J8	J9	J57		J38	
2		J54			J19	J20			J43	
2		J56			J55				J69	
2		J62			J91				J80	
2		J89								32
3	J87	J52		J61	J10	J6			J42	
3		J70		J63	J66	J11			J73	
3				J64		J21				
3				J65		J58				
3				J92						16
4	J78	J81		J22	J23	J24				
4	J79			J71		J32				
4				J83		J33				
4						J40				
4						J41				
4						J68				
4						J76				
4						J77				15
5	J82	J60		J75		J25				
5		J74				J72				6
Σ	4	14	1	12	12	20	7	7	13	92

Table 2: Classification of Johnson solids by parameters Eo and s.

The Table 3 shows how by increasing the number of observed parameters we can gradually increase the number of equivalence classes defined by those parameters. Finally, when we use all the 16 parameters from the Table 1 each Johnson solid is separated from all the others in its own equivalence class.

set of parameters	c	i
{Eo}	6	0
{n}	6	0
{s}	9	1
$\{v_o^p, e_o^p, f_o^p\}$	9	1
{n}	14	3
{s, Eo}	31	10
{N, n}	34	3
{flo}	36	16
{flo, Eo}	53	33
{flo, Eo, s}	72	60
{flo, Eo, N, n}	86	80
{v, e, f, flo, s, vo, eo, fo, Eo, N, n, v_o^p, e_o^p, f_o^p}	87	82

Table 3: The numbers of equivalence classes c and of the isolated solids i.

5.2 Two simple tricks for finding equivalence classes: summation of parameters and lexicographic ordering

A simple trick how to find the values c and i (or how to find equivalence classes of solids) for a chosen set of parameters $\{p_1, \ldots, p_k\}$ is to form a sum $\{p_1 + \ldots + p_k\}$ of these parameters. If for two solids these sums are different, some of parameters must be different, and the solids must be in different equivalence classes (with respect to these parameters). Then we have to check only for the solids with the same sum $\{p_1 + \ldots + p_k\}$ if they have some parameters different. This we can do by lexicographic ordering of these solids by the values of parameters $\{p_1, \ldots, p_k\}$.

6 The coding problem

Some polyhedra (J28, J29), (J32, J33), (J40, J41), (J76, J77), (J78, J79) differ only in the values of parameters r and q (see Table 1). That was the reason why we introduced parameters r and q. *Which of our 16 parameters are redundant for the purpose of coding the solids of the class C?* Using the summation trick (described in 5.2) and lexicographic ordering of parameters we can see that the parameters $(v, e, f, flo, s, n, m, q)$ suffice to separate the solids of the class C. *What is the minimal number of parameters by which we can code the solids of the class C?*

The next counterexamples show why certain combinations of three parameters do not separate the class C:
(v, e, f) = (14,26,14) for J55 and J56, (f, m, q) = (8,4,14) for J26 and J49, (v, s, r) = (9,6,6) for J3 and J63, (s, q, m) = (10,0,10) for J5 and J6, (e, m, q) = (80,5,0) for J31 and J59, (v, s, q) = (25,10,6) for J33 and J24, (s, q, e) = (20,0,20) for J31 and J59, (s, q, f) = (10,0,27) for J21 and J32, (s, q, flo) = (6,0,28) for J44 and J45, (v, m, r) = (9,4,12) for J10 and J8, (v, m, q) = (40,5,0) for J48 and J43, $(flo, N \cdot n, q) = (4, 60, 0)$ for the Archimedean solid 3.4.5.4 and the 30-gonal antiprism 3^3.30.

By a simple »multiplication trick« we can get from two parameters N and n just one parameter $N \cdot n$:

Proposition 2 *The sequences of parameters $(flo, N \cdot n, q)$ are different for each solid of the class consisting of the 92 Johnson solids, the 5 Platonic solids and the 13 Archimedean solids.*

Proof. This is easily seen by arranging these sequences in the lexicographic order (Table 4). □

Proposition 3 *The sequences of four parameters (flo, N, n, q) are different for each solid of the class C. Likewise, the sequences of four parameters (q, s, f, e) and (q, s, f, n) are different for each solid of the class C.*

Proof. From the Tables 1b and 4 we see that the sequences of parameters $(flo,\ N \cdot n, q)$ are different for each solid of the class C except for the Archimedean solid 3.4.5.4 and the 30-gonal antiprism 3^3.30 which both have the values of these parameters $(flo, N \cdot n, q) = (4, 60, 0)$. Since these two solids differ in parameters N and n, the sequences of four parameters (flo, N, n,q) are different for each solid of the class C. Likewise we can easily see that the parameters (q, s, f) separate all the Johnson, Archimedean and Platonic solids except two pairs: J21 and J22, which both have $(q, s, f) = (0, 10, 27)$, and J48 and J76, which both have $(q, s, f) = (0, 10, 52)$ and that the values of e and n are different also for these pairs of solids. □

Table 4, together with values $flo = 3$ for prisms and $flo = 4$ for antiprisms from Table 1c, gives us also the classification of convex polyhedra with respect to the numbers of flag orbits:

Proposition 4 *The class C of convex polyhedra with regular polygonal faces is divided into 38 equivalent classes with respect to the number of flag orbits.*

$(flo, N\cdot n, q)$: solid	$(flo, N\cdot n, q)$: solid	$(flo, N\cdot n, q)$: solid	$(flo, N\cdot n, q)$: solid
(1, 12, 6) : 3^3	(8,10,18): J30	(14,10,0): J6	(26,12,18): J56
(1, 24, 6) : 3^4	(8,12,0): J7	(14,55,10): J58	(28,8,14): J88
(1, 24, 16) : 4^3	(8,20,18): J8	(15,32,32): J67	(28,10,10): J46
(1, 60, 6) : 3^5	(8,24,14): J27	(16,50,40): J42	(28,24,0): J44
(1, 60, 10) : 5^3	(8,40,18): J29	(16,60,0): J43	(28,35,16): J40
(2, 24, 0) : 3.4.3.4	(8,40,24): J28	(17,4,6): J50	(28,35,18): J41
(2, 60, 0) : 3.5.3.5	(8,50,0): J59	(18,6,16): J18	(28,40,0): J45
(3, 18, 6) : J12	(9,36,6): J84	(18,8,28): J19	(30,45,10): J61
(3, 24, 18) : 3.6^2	(10,4,6): J10	(18,10,21): J20	(33,8,18): J89
(3, 30, 6) : J13	(10,5,6): J11	(18,18,0): J65	(36,60,0): J48
(3, 48, 20) : 4.6^2	(10,6,0): J3	(18,20,0): J80	(40,50,10): J60
(3, 48, 32) : 3.8^2	(10,8,0): J85	(19,10,22): J52	(42,10,0): J76
(3, 120, 22) : 5.6^2	(10,8,20): J4	(19,16,14): J90	(42,10,16): J77
(3, 120, 26) : 3.10^2	(10,10,0): J5	(20,10,0): J32	(42,55,22): J68
(4, 4, 0) : J1	(10,12,0): J57	(20,10,10): J62	(48,60,16): J72
(4, 5, 0) : J2	(10,15,6): J63	(20,35,6): J33	(50,30,0): J83
(4, 60, 0) : 3.4.5.4	(10, 24, 0) : $3^4.4$	(22,6,6): J22	(52,4,6): J87
(4, 72, 32) : 3.4^3	(10, 60, 0) : $3^4.5$	(22,8,14): J86	(64,35,0): J47
(5, 12, 12) : J14	(12,10,16): J39	(22,10,0): J21	(80,60,16): J75
(5, 16, 16) : J15	(12,10,40): J38	(22,10,6): J24	(90,20,0): J81
(5, 20, 20) : J16	(12,15,6): J64	(22,10,18): J23	(90,90,22): J71
(6, 48, 0) : 4.6.8	(12,48,14): J36	(22,12,26): J54	(120,60,22): J74
(6, 48, 6) : J17	(12,48,24): J35	(23,10,14): J53	(120,100,22): J70
(6, 120, 0) : 4.6.10	(12,60,10): J34	(24,6,0): J92	(180,20,16): J82
(7, 6, 14) : J26	(12,72,26): J37	(24,40,30): J66	(210,10,0): J79
(7, 42, 6) : J51	(13,8,4): J49	(24,60,16): J73	(210,10,23): J78
(8, 5, 0) : J9	(13,12,14): J55	(24,100,0): J69	
(8, 10, 0) : J31	(13,20,10): J91	(26,10,6): J25	

Table 4: Lexicographic ordering of solids by three parameters flo, $N \cdot n$ and q.

Finally, there is a simple trick, based on the unique factorization theorem for primes, by which we can separate the solids of C by a single parameter:

Proposition 5 *If the solids of any class of objects are separated into different equivalent classes by k parameters* $q_1, q_2, \ldots, q_k$*, then they can be separated by just one composed parameter* $2^{q_1} \cdot 3^{q_2} \ldots \cdot p_k^{q_k}$*, where* $2, 3, 5, \ldots, p_k$ *are the first k prime numbers.*

7 Summary: why, how and what

A few words explaining the justification of this project and its results might be appropriate. WHY: The idea was to find an algebraic description of Johnson solids, similar to a description of uniform polyhedra by their vertex-type symbol. Instead of using a long word for a uniform polyhedron such as Rhombicosidodecahedron (which does not tell us much) or the corresponding uniform notation number U27 (which tells us even less) we can use a vertex-type symbol (3.4.5.4) (which contains some information about the structure of the polyhedron). The selected data we collected (Table 4) enable us to describe each Johnson solid with just three parameters. For example, Triaugmented Truncated Dodecahedron J71 is described by the ordered triple $(flo, N \cdot n, q) = (90, 90, 22)$.

HOW: For each polyhedron its net had to be drawn (this was difficult!), as in Figure 3. Then the values of parameters could be easily detected.

WHAT: To obtain a characterization of convex polyhedra with regular polygonal faces by minimal number of parameters we have analyzed 16 parameters (Table 1, Section 4). Some of them are related to the symmetry-type graphs ([7], [8]) of these solids. We have introduced a parameter called »Euler orbit characteristic« and proved some of its properties (Proposition 1). Comparing various sets of parameters with respect to how well they separate the solids we have found a characterization of the class C of convex polyhedra with regular polygonal faces by four parameters (Proposition 3). Likewise, the solids of the subclass of C consisting of the Johnson solids ([5], Archimedean solids and Platonic solids (Proposition 2) can be separated by just three parameters. We have classified the solids of the class C by the numbers of their flag orbits (Proposition 4). We have seen that the average number of symmetries of Johnson solids is 9.51 and that the average number of their Euler orbit characteristic is 2.391. Finally, we see (by combining Proposition 3 and Proposition 5) that the solids of the infinite class C can be separated by just one composed parameter: $2^{flo} \cdot 3^N \cdot 5^n \cdot 7^q$.

References

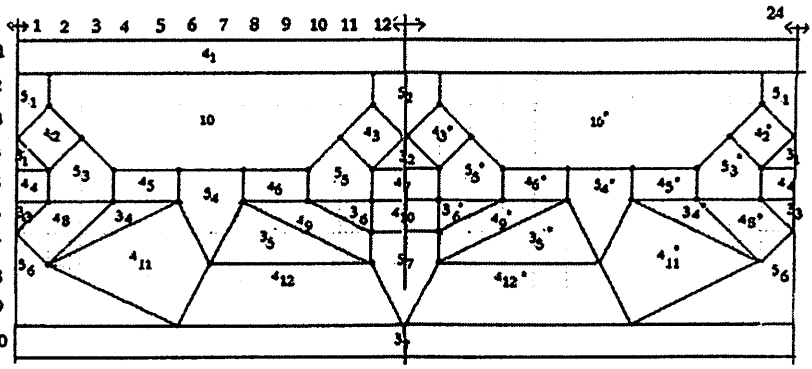
[1] A. D. Alexandrov, *Convex Polyhedra* (Springer-Verlag, Berlin Heidelberg 2005).

[2] A. T. Balaban and T. Pisanski, *Flag graphs and their applications in mathematical chemistry for benzenoids*, in preparation.

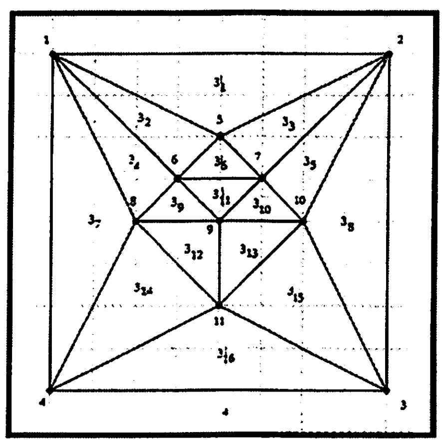
[3] W. W. R. Ball, *Mathematical recreations and essays*, 11th ed., revised by H.S.M Coxeter (London, 1939).

[4] B. Grünbaum, *Convex polytopes* (John Wiley & Sons, London, 1967).

[5] N. W. Johnson, *Convex polyhedra with regular faces*, Canad. J. Math., **18** (1966), 169–200.



J(82)



J87

Figure 3: The nets of the Johnson solids J82 and J87.

[6] I. Lakatos, *Proofs and Refutations* (Cambridge University Press, New York, 1976).

[7] S. Lins, *Graph-encoded maps*, J.Combin. Theory Ser. B **32** (1982), 171–181.

[8] A. Orbanić, D. Pellicer, A. I. Weiss: *Map operations and k-orbit maps*, J. Combin. Theory Ser. A **117** (2010) 411–429.

[9] V. A. Zalgaller, *Convex Polyhedra with Regular Faces* (Consultants Bureau, New York, 1969).