On primal graphs with maximum degree 2

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Abstract

In 1969, Dewdney introduced the set Γ of *primal graphs*, characterized by the following two properties: every finite, simple graph G is the union of non-isomorphic, edge-disjoint subgraphs of G so that each of the subgraphs is in Γ; and, if G is in Γ, then the only such union consists of G itself. In the period around 1990, several works concerning the determination of the graphs in Γ were published and one Ph.D. thesis written. However, the classification of the members of Γ remains elusive. The main point of this work is to simplify and unify some of the principal results of Preen's Ph.D. thesis that generalize earlier results about primal graphs with maximum degree 2.

1 Introduction

As an analogue of a basis of a vector space over the 2-element field, Dewdney introduced the notion of sets of "primal graphs" [5]. This begins with the following notion.

Definition 1.1 *1. A* decomposition *of a graph G is a set of non-isomorphic, edge-disjoint subgraphs of G whose union is G.*

2. If Λ is a set of graphs and Φ is a decomposition of a graph G, then Φ is a decomposition of G over Λ *if $\Phi \subseteq \Lambda$.*

A graph typically has many different decompositions. In this work, we are interested in finding decompositions with additional structure.

Definition 1.2 *Let $\mathcal{G}$ be a set of finite, simple graphs. A subset Γ of $\mathcal{G}$ is* primal relative to $\mathcal{G}$ *if:*

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1. *each $G \in \mathcal{G}$ has a decomposition Φ so that $\Phi \subseteq \Gamma$; and*
2. *if $G \in \Gamma$, then the only decomposition Φ of G with $\Phi \subseteq \Gamma$ is $\Phi = \{G\}$.*

Dewdney's main result was to show that every set of graphs has a unique primal subset.

Theorem 1.3 **([5])** *If $\mathcal{G}$ is a set of finite, simple graphs, then there is a unique subset Γ of $\mathcal{G}$ that is primal relative to $\mathcal{G}$.*

This theorem is easily proved by recursively constructing Γ. The graphs in $\mathcal{G}$ with fewest edges are necessarily in Γ. Given the current Γ, we add to it all the graphs in $\mathcal{G}$ having fewest edges and no decomposition over Γ. In this way, we eventually determine, for each graph $G \in \mathcal{G}$, whether or not G is in Γ. It is an easy exercise to prove that the resulting Γ is primal relative to G.

By considering a graph with fewest edges that is in one, but not the other, of distinct primal sets, uniqueness is also quite straightforward.

When $\mathcal{G}$ is the set of all (finite, simple) graphs, the primal set relative to G is the set Γ of *primal graphs*. The determination of the primal graphs with maximum degree 2 or the primal forests with maximum degree 3 is far from complete.

Some effort has gone into classifying the members of Γ; see for example [1, 3, 5]. If G is a graph and n is a positive integer, then nG is the graph consisting of n disjoint copies of G. For non-negative integers i and j, the graphs $2^i K_{1,2^j}$ and $2^i K_{2,2}$ are known to be primal [1]. In the context of graphs with maximum degree at most 2, the problem then becomes: what are the primal graphs with maximum degree 2 other than the graphs in $\Gamma_0 := \{2^i K_2 \mid i \geq 0\} \cup \{2^i K_{1,2} \mid i \geq 0\}$ and the graphs of the form $2^i K_{2,2}$? In considering graphs with maximum degree 2, we may assume that no component is a $K_{2,2}$, as all such components can be decomposed using $2^i K_{2,2}$'s, without affecting the decomposition of the remainder of the graph.

For a graph G, $\Omega(G)$ consists of those graphs all of whose components are subgraphs of G. It follows from [3, Theorem 4] that $\Omega(C_5)$ has infinitely many primal graphs not in Γ_0 (obviously none of these is of the form $2^i K_{2,2}$).

For two graphs H and K, $H + K$ denotes the disjoint union of H and K. Chinn, Richter, and Truszczyński [2] introduce a natural parameter $d(G)$ and prove that, for graphs with maximum degree at most 2, $C_5 + K_2$ and $7C_5 + K_2$ are the only primal graphs with maximum degree 2 and d between 1 and 42. Every graph G with maximum degree 2 and $d(G) \leq 42$ decomposes over $\Gamma_0 \cup \{C_5 + K_2\,,\, 7C_5 + K_2\}$. Unfortunately, several hundred other primal examples with maximum degree 2 were determined, all having

$d = 43$ and 296 edges (the smallest possible to have $d \geq 43$), so the simple answer for $d \leq 42$ does not continue.

This result was generalized in one direction by Preen in his doctoral work [7]. An important innovation in Preen's work is the introduction of the class of binary graphs: a *(p, q)-binary graph* is the edge-disjoint union of pK_2 and $qK_{1,2}$. A graph G is *binary* if, for some non-negative integers p and q, G is a (p, q)-binary graph. Binary graphs trivially decompose over Γ_0: use the primal graphs of the form $2^i K_{1,2}$ to deal with the $qK_{1,2}$ portion and, independently, use those of the form $2^i K_2$ to deal with the pK_2.

A principal result in Preen's thesis is that if B is a binary graph, T has maximum degree 2 and does not contain either C_4 or C_5, and n is an integer with $0 \leq n \leq 42$, then $B + T + nC_5$ is primal if and only if $B + T + nC_5$ either is in Γ_0, or is $C_5 + K_2$, or is $7C_5 + K_2$. These results are obtained by complicated calculations.

Let B be a (p, q)-binary graph and let G be a graph. Then $\alpha^*(B + G)$ is the largest integer i such that $B + G$ contains an isomorph H of $2^i K_2$ so that $H \cap B$ is contained in the pK_2. Similarly, $\beta^*(B + G)$ is the largest integer j so that $B + G$ contains an isomorph K of $2^j K_{1,2}$ so that $H \cap B$ is contained in the $qK_{1,2}$. These numbers give a natural upper bound on the largest elements of Γ_0 we can necessarily use in a decomposition of $B + G$.

It follows that a decomposition of $B + G$ respecting the description of B as a (p, q)-binary graph can cover at most $2^{\alpha^*(B+G)+1} - 1$ edges with primal graphs of the form $2^i K_2$ and at most $2^{\beta^*(B+G)+2} - 2$ edges with primal graphs of the form $2^j K_{1,2}$.

Thus, the parameter $D(B+G) := |E(B+G)| - (2^{\alpha^*(B+G)+1} + 2^{\beta^*(B+G)+2} - 3)$ naturally records how far $B+G$ is from decomposing over Γ_0. Our main results are summarized in the following theorem, simultaneously generalizing the above-mentioned results for graphs with maximum degree 2 and $d \leq 42$ and Preen's theorem. We believe the methods we present here to be simpler than those employed in the earlier arguments.

Theorem 1.4 *Let B be a binary graph, and T a graph with maximum degree 2 that does not contain $K_{2,2}$. Then:*

1. *if $D(B + T) \leq 0$, then $B + T$ has a decomposition over Γ_0;*
2. *if $D(B+T) \leq 6$, then $B+T$ has a decomposition over $\Gamma_0 \cup \{C_5+K_2\}$; and*
3. *if $D(B + T) \leq 42$, then $B + T$ has a decomposition over $\Gamma_0 \cup \{C_5 + K_2, 7C_5 + K_2\}$.*

It is important to understand that there are at least several hundred primal graphs with maximum degree at most 2 and $D(G) = 43$. Our methods are inadequate to go beyond this point. On the other hand, unlike

Preen's work, our theorem includes many graphs that contain nC_5 with $n > 42$. As long as there are other components to balance the extra C_5's so that $D \leq 42$, our theorem guarantees a decomposition.

The next section sets out some preliminary observations that will be useful in later work. In Section 3 is the proof of Theorem 1.4 (1), while Section 4 proves (2) and (3). In Section 5 we give some commentary on the family of primal graphs G with maximum degree 2 and $D(G) = 43$, as well as some additional related work and future directions.

2 Basic facts

In this section, we prove some basic lemmas that we will need for Theorem 1.4.

Let $\mathfrak{B}$ consist of all the binary graphs and let $\mathfrak{T}^4$ consist of those graphs with maximum degree 2 that do not contain $K_{2,2}$. Because cycles of length a multiple of 3 and all paths are in $\mathfrak{B} \cap \mathfrak{T}^4$, a graph in $\mathfrak{B} + \mathfrak{T}^4$ may have multiple representations in the form $B + T$, with $B \in \mathfrak{B}$ and $T \in \mathfrak{T}^4$. Initially we shall be concerned as to which graphs $B + T$ in $\mathfrak{B} + \mathfrak{T}^4$ decompose over Γ_0. Since we know nothing much about the structure of B, we will typically only be able to use our limited knowledge to decompose it as in the three paragraphs preceding Theorem 1.4.

Definition 2.1 *Let $T \in \mathfrak{T}^4$ and let B be the edge-disjoint union of $B_1 = pK_2$ and $B_2 = qK_{1,2}$. A decomposition Φ of $B + T$ is* conventional *if, for each edge e of B and for the element H of Φ containing e, the component of either B_1 or B_2 that contains e is a component of H.*

In other words, a conventional decomposition of $B + T$ uses, in a natural way, the K_2's and $K_{1,2}$'s in the description of B as a (p, q)-binary graph. Note that, for a conventional decomposition, p and q must be specified in advance; we reserve p and q as the parameters for $B \in \mathfrak{B}$.

We now introduce several parameters that are central to the remainder of this work (items (5)–(7) were already mentioned in the introduction).

Definition 2.2 *Let G be an arbitrary graph, let B be a (p, q)-binary graph, and let $T \in \mathfrak{T}^4$. Then:*

1. *$m_1(G)$ is the largest m so that $mK_2 \subseteq G$;*
2. *$m_1^*(B + T) = p + m_1(T)$;*
3. *$m_2(G)$ is the largest m so that $mK_{1,2} \subseteq G$;*
4. *$m_2^*(B + T) = q + m_2(T)$;*
5. *$\alpha^*(B + T)$ is the largest integer i so that $2^i \leq m_1^*(B + T)$;*

6. $\beta^*(B+T)$ *is the largest integer* i *so that* $2^i \leq m_2^*(B+T)$*; and*
7. $D(B+T) = |E(B+T)| - 2^{\alpha^*(B+T)+1} - 2^{\beta^*(B+T)+2} + 3$.

As we mentioned in the introduction, $D(G)$ counts the excess of $E(G)$ over the number $2^{\alpha^*+1} + 2^{\beta^*+2} - 3$, giving an indication of how far G is from conventionally decomposing over Γ_0.

We need some simple observations.

Observation 2.3 *1. If* C *is a cycle of length* n*, then* $m_1(C) = \lfloor n/2 \rfloor$ *and* $m_2(C) = \lfloor n/3 \rfloor$.

2. *If* P *is a path of length* n*, then* $m_1(P) = \lfloor (n+1)/2 \rfloor$ *and* $m_2(P) = \lfloor (n+1)/3 \rfloor$.
3. *If* G *and* H *are graphs, then, for* $i = 1, 2$, $m_i(G+H) = m_i(G) + m_i(H)$.
4. *If* G *and* H *are in* $\mathfrak{B} + \mathfrak{T}^4$*, then* $G + H \in \mathfrak{B} + \mathfrak{T}^4$ *and, for* $i = 1, 2$,

$$m_i^*(G+H) = m_i^*(G) + m_i^*(H).$$

■

The following lemma relates $D(B+T)$ and the components of T that are 5-cycles.

Lemma 2.4 *1. If* K *is any path or cycle other than* C_5*, then*

$$m_1(K) + 2m_2(K) \geq |E(K)|.$$

2. *If* $B \in \mathfrak{B}$ *and* $T \in \mathfrak{T}^4$*, then* T *has at least* $D(B+T)$ *components that are 5-cycles.*

Proof. The first item is a direct consequence of Observation 2.3.

As for the second item, let r be the largest integer so that $rC_5 \subseteq T$. Then the first item implies the second line in the sequence below, while the definitions of α^* and β^* imply the third.

$$\begin{aligned} |E(B+T)| &= p + 2q + |E(T)| \\ &\leq p + m_1(T)) + 2(q + m_2(T)) + r \\ &\leq (2^{\alpha^*+1} - 1) + 2(2^{\beta^*+1} - 1) + r. \end{aligned}$$

It follows that $D(B+T) \leq r$, as claimed. ■

Our final preliminary observation is the following.

Lemma 2.5 *Let $T \in \mathfrak{T}^4$ and let B be the edge-disjoint union of $B_1 = pK_2$ and $B_2 = qK_{1,2}$. Let $T' = pK_2 + T$. Then $B + T$ has a conventional decomposition if and only if $B_2 + T'$ has a conventional decomposition.*

Proof. Let Φ be a conventional decomposition of $B + T$. Fix a bijection between the edges of B_1 and the pK_2 edges in T'. Replacing each edge e of B_1 with its bijective image turns Φ into a conventional decomposition of $B_2 + T'$, yielding a conventional decomposition Φ' of $B_2 + T'$. The same argument works in reverse to convert Φ' into Φ. ■

The impact of Lemma 2.5 is that, as far as conventional decompositions go, we may assume B is simply $qK_{1,2}$ and that B must be covered by $K_{1,2}$-components of the decomposition. Not every graph of the form $qK_{1,2} + T$ that we consider has a conventional decomposition. However, these are quite exceptional and, in particular, for them, no component of T is K_2. Therefore, the original graph $B+T$ with B a (p,q)-binary graph must have $p = 0$ anyway. The upshot is that, for us, there is no loss of generality in assuming $B = qK_{1,2}$.

3 Graphs in $\mathfrak{B} + \mathfrak{T}^4$ that decompose over Γ_0

The main result in this section is the following observation.

Theorem 3.1 *Let $B = qK_{1,2}$ and $T \in \mathfrak{T}^4$. If $D(B + T) \leq 0$, then either $B+T$ has a conventional decomposition over Γ_0 or there exist non-negative integers i and b, with $b \leq 2^i$, so that $T = (2^i - b)C_8 + (2b)P_4$. In the latter case, the graph $B + T$ decomposes over Γ_0.*

We make two remarks about the exceptional case. Firstly, note that no component of T is K_2. Therefore, if B is a (p,q)-binary graph and $T \in \mathfrak{T}^4$ are such that $D(B + T) \leq 0$, then either $B + T$ has a conventional decomposition over Γ_0 or $p = 0$, $B = qK_{1,2}$ and, for some integers i and b, $T = (2^i - b)C_8 + (2b)P_4$.

Secondly, we can be more precise about the exceptional case; we provide in the appendix a complete description. This greater precision is not required to prove Theorem 1.4.

For the proof of Theorem 3.1, we partition $\mathfrak{B} + \mathfrak{T}^4$ into several subsets.

Definition 3.2 *Let $c \geq -1$ be an integer.*

1. *The set $(\mathfrak{B} + \mathfrak{T}^4)_{even}$ consists of those graphs $B + T$ with $B = qK_{1,2}$ and $T \in \mathfrak{T}^4$ each of whose components has an even number of edges.*
2. *The set $(\mathfrak{B} + \mathfrak{T}^4)_c$ consists of those graphs $B + T$ for which:*

(a) T has at least one component with an odd number of edges;

(b) for some q, $B = qK_{1,2}$ and T has at most $2^{c+1} - q - 1$ cycles of odd length;

(c) $|E(B+T)| \leq 2^{\alpha^*(B+T)+1} + 2^{c+2} - 3$; and

(d) $|E(B+T)| \leq 2^{\alpha^*(B+T)+1} + 2^{\beta^*(B+T)+2} - 3$.

The constraint $|E(B+T)| \leq 2^{\alpha^*(B+T)+1} + 2^{c+2} - 3$ allows for the possibility of a conventional decomposition over Γ_0 that uses only those $2^i K_{1,2}$ with $i \leq c$. We let PDΓ_0 denote the set of graphs $B + T$ in $\mathfrak{B} + \mathfrak{T}^4$ for which $D(B+T) \leq 0$. (These graphs are **P***otentially* **D***ecomposable over* Γ_0.)

We note that $(\mathfrak{B} + \mathfrak{T}^4)_{even} \subseteq \mathrm{PD}\Gamma_0$.

It is a triviality that, if $D(B+T) \leq 0$, then either $B+T \in (\mathfrak{B}+\mathfrak{T}^4)_{even}$ or there is a $c \leq \beta^*(B+T)$ so that $B+T \in (\mathfrak{B}+\mathfrak{T}^4)_c$. In order to prove Theorem 3.1, we begin with the following. This is the key insight that simplifies the decomposition over Γ_0 of graphs in PDΓ_0.

Theorem 3.3 *For each integer $c \geq -1$, each graph in $(\mathfrak{B} + \mathfrak{T}^4)_c$ has a conventional decomposition over Γ_0 that does not use any $2^i K_{1,2}$ with $i > c$.*

The proof will be by induction on c. The implication is that in the base case $c = -1$, no $2^i K_{1,2}$ is used in the conventional decomposition. This fact will be used to prove that (most) graphs in $(\mathfrak{B}+\mathfrak{T}^4)_{even}$ have conventional decompositions.

Proof. Let $B = qK_{1,2}$. As mentioned, we proceed by induction on c. For the base case $c = -1$, $T \in \mathfrak{T}^4$ is such that $B + T \in (\mathfrak{B} + \mathfrak{T}^4)_{-1}$. Since T has at most $2^{(-1)+1} - q - 1$ odd cycles, T has no odd cycles and $q = 0$; in particular, B is empty. We write m_1 and α^* instead of $m_1(T)$ and $\alpha^*(T)$, respectively.

We claim that we may choose an isomorph H of $2^{\alpha^*} K_2$ so that $T \setminus H$ is of the form nK_2. To this end, let M be an isomorph of $m_1 K_2$. Then M is a maximum matching in T.

Each component of T is either a path or an even cycle. The complement of M in each of these components is a matching and, therefore, $T \setminus M$ is a matching. Furthermore, the augmenting paths of $T \setminus M$ are precisely the components of T that are odd paths.

Since $|E(T)| \leq 2^{\alpha^*+1} + 2^{-1+2} - 3 = 2^{\alpha^*+1} - 1$, we see that $T \setminus M$ has at most $2^{\alpha^*} - 1$ edges. As $|M| = m_1(T) \geq 2^{\alpha^*(T)}$, $T \setminus M$ has fewer edges than M has. Thus, there are at least $2^{\alpha^*} - |E(T \setminus M)|$ augmenting paths for $T \setminus M$.

These augmenting paths are all components of T that are odd paths. Thus, we can use them to find an isomorph H of $2^{\alpha^*} K_2$ so that $G \setminus H$

is a matching. As above with M, $T \setminus H$ has at most $2^{\alpha^*} - 1$ edges, so it decomposes over Γ_0 without using an isomorph of H, yielding the desired conventional decomposition of T.

We now proceed to the inductive step. We may assume $B + T \in (\mathfrak{B} + \mathfrak{T}^4)_c \setminus (\mathfrak{B} + \mathfrak{T}^4)_{c-1}$. Thus, either T has at least $2^c - q$ odd cycles or $|E(B+T)| > 2^{\alpha^*+1} + 2^{c+1} - 3$.

Case 1: *T has at least $2^c - q$ odd cycles.*

In this case, let H be an isomorph of $2^c K_{1,2}$ selected as follows:

1. select one $K_{1,2}$ from each of as many as possible odd cycles of length at least 5 in T;
2. if T has fewer than 2^c odd cycles of length at least 5, then additionally select one $K_{1,2}$ from as many of the remaining odd cycles in T as possible;
3. if T has fewer than 2^c odd cycles, then select the remaining $K_{1,2}$'s from $B = qK_{1,2}$.

Since $T \notin \mathfrak{T}^4_{even}$, some component C of T has an odd number of edges. Therefore, some component of $C \setminus H$, and so a component of $T \setminus H$, has an odd number of edges. This is Condition 2a for $(B+T) \setminus H \in (\mathfrak{B} + \mathfrak{T}^4)_{c-1}$.

Let $\sigma(T)$ denote the number of odd cycles in T. If $\sigma(T) \geq 2^c$, then $B \setminus H = B$ and $T \setminus H$ has 2^c fewer odd cycles than T has. That is, $T \setminus H$ has at most $(2^{c+1} - q - 1) - 2^c$ odd cycles. This is Condition 2b for $(B+T) \setminus H \in (\mathfrak{B} + \mathfrak{T}^4)_{c-1}$ in this instance.

So suppose, on the other hand, that T has fewer than 2^c odd cycles. Then $\sigma(T \setminus H) = 0$ and $B \setminus H$ is $(q-(2^c-\sigma(T)))K_{1,2}$. Set $q' = q - 2^c + \sigma(T)$; to get Condition 2b for $(B+T) \setminus H \in (\mathfrak{B} + \mathfrak{T}^4)_{c-1}$ in this case, we need to know that $T \setminus H$ has at most $2^c - q' - 1$ odd cycles. Since $\sigma(T \setminus H) = 0$, it suffices to show $2^c - q' - 1 \geq 0$.

We know by assumption that $\sigma(T) \leq 2^{c+1} - q - 1$ and, by definition, $q' = q - 2^c + \sigma(T)$. Thus, $2^c - q' - 1 = 2^c - q + 2^c - \sigma(T) - 1 = 2^{c+1} - q - 1 - \sigma(T) \geq 0$, as required.

The choice of H implies that $m_1(T \setminus H) = m_1(T)$; therefore $\alpha^*((B+T) \setminus H) = \alpha^*(B+T)$. Consequently,

$$\begin{aligned} |E((B+T) \setminus H)| &= |E(B+T)| - 2^{c+1} \\ &\leq \left(2^{\alpha^*(B+T)+1} + 2^{c+2} - 3\right) - 2^{c+1} \\ &= 2^{\alpha^*((B+T)\setminus H)+1} + 2^{c+1} - 3\,. \end{aligned}$$

This is Condition 2c for $(B+T) \setminus H$ being in $(\mathfrak{B} + \mathfrak{T}^4)_{c-1}$.

As a first possibility in establishing that the final condition holds, suppose $c < \beta^*(B+T)$. Then the choice of H implies there is an isomorph K

of $2^{\beta^*(B+T)}K_{1,2}$ in $B+T$ containing H. Therefore, $K \setminus H \subseteq (B+T) \setminus H$ shows $\beta^*((B+T)\setminus H) \geq \beta^*(B+T) - 1 \geq c$. Combined with the preceding paragraph, we conclude

$$|E((B+T)\setminus H)| \leq 2^{\alpha^*((B+T)\setminus H)+1} + 2^{\beta^*((B+T)\setminus H)+1} - 3,$$

which is Condition 2d for $(B+T)\setminus H$ being in $(\mathfrak{B}+\mathfrak{T}^4)_{c-1}$.

In the remaining possibility for Condition 2d, $c \geq \beta^*(B+T)$. As one subcase, suppose there are at least 2^c odd cycles in T having length at least 5. Then $2^c K_{1,2} \subseteq T \setminus H$, and so $\beta^*((B+T)\setminus H) \geq c$. Again,

$$|E((B+T)\setminus H)| \leq 2^{\alpha^*((B+T)\setminus H)+1} + 2^{c+1} - 3,$$

so

$$|E((B+T)\setminus H)| \leq 2^{\alpha^*((B+T)\setminus H)+1} + 2^{\beta^*((B+T)\setminus H)+2} - 3,$$

which is Condition 2d for being in $(\mathfrak{B}+\mathfrak{T}^4)_{c-1}$.

In the other subcase, T has fewer than 2^c odd cycles of length at least 5. Therefore, $T \setminus H$ has no 5-cycles; it follows from Lemma 2.4 that

$$|E((B+T)\setminus H)| \leq 2^{\alpha^*((B+T)\setminus H)+1} + 2^{\beta^*((B+T)\setminus H)+2} - 3,$$

as required for Condition 2d.

Thus, in all possibilities for Case 1, we have $(B+T)\setminus H \in (\mathfrak{B}+\mathfrak{T}^4)_{c-1}$.

Case 2: *T has fewer than $2^c - q$ odd cycles.*

In this case, the fact that $B+T \notin (\mathfrak{B}+\mathfrak{T}^4)_{c-1}$ implies

$$|E(B+T)| > 2^{\alpha^*(B+T)+1} + 2^{c+1} - 3.$$

Let $\sigma(T)$ and $\rho(T)$ denote, respectively, the numbers of odd cycles and odd paths in T. By Condition 2a of the definition of $(\mathfrak{B}+\mathfrak{T}^4)_c$, $\sigma(T)+\rho(T) > 0$. Let $s = m_1^*(B+T) - 2^{\alpha^*(B+T)}$. This "slack" is the amount by which $m_1(T)$ may be reduced by deleting $K_{1,2}$'s from T in a conventional decomposition without reducing α^*. Obviously, the deletion of a $K_{1,2}$ from T reduces $m_1^*(B+T)$ by at most 1.

We aim to prove that $q+s+\sigma(T) \geq 2^c$. As a start, note that Observation 2.3 implies, for each component K of T, $|E(K)| = 2m_1(K) + \sigma(K) - \rho(K)$; therefore, $|E(T)| = 2m_1(T) + \sigma(T) - \rho(T)$. Consequently, we have

$$\begin{aligned}
2^{\alpha^*(B+T)+1} + 2^{c+1} - 2 &\leq |E(B+T)| \\
&= |E(B)| + |E(T)| \\
&= 2q + (2m_1(T) + \sigma(T) - \rho(T)) \\
&= 2(2^{\alpha^*(B+T)} + s) + 2q + \sigma(T) - \rho(T) \\
&= 2^{\alpha^*(B+T)+1} + 2(q+s+\sigma(T)) - (\sigma(T)+\rho(T)).
\end{aligned}$$

Cancelling the $2^{\alpha^*(B+T)+1}$ and recalling that $\sigma(T)+\rho(T) \geq 1$, it follows that

$$2^{c+1} - 2 \leq 2(q+s+\sigma(T)) - 1,$$

which implies that $q+s+\sigma(T) \geq 2^c$, as claimed.

Condition 2d for $B+T \in (\mathfrak{B}+\mathfrak{T}^4)_c$ implies

$$|E(B+T)| \leq 2^{\alpha^*(B+T)+1} + 2^{\beta^*(B+T)+2} - 3,$$

while the fact that we are in Case 2 implies that

$$|E(B+T)| \geq 2^{\alpha^*(B+T)+1} + 2^{c+1} - 2.$$

These inequalities combine to show $\beta^*(B+T) \geq c$. Thus, we can choose an isomorph H of $2^c K_{1,2}$ in $B+T$ by first choosing an isomorph H_1 of $\sigma(T)K_{1,2}$ so that each odd cycle of T has a copy of $K_{1,2}$ in H_1. Now extend H_1 to H using at most s other $K_{1,2}$'s in T. Since $T \setminus H$ has no odd cycles, Lemma 2.4 implies Condition 2d for membership of $(B+T)\setminus H$ in $(\mathfrak{B}+\mathfrak{T}^4)_{c-1}$.

Since $(B+T) \notin (\mathfrak{B}+\mathfrak{T}^4)_{even}$, we see that $(B+T)\setminus H \notin (\mathfrak{B}+\mathfrak{T}^4)_{even}$. Thus, Condition 2a holds for $(B+T)\setminus H$ being in $(\mathfrak{B}+\mathfrak{T}^4)_{c-1}$. The fact we are in Case 2 implies T has at most $2^c - q - 1$ odd cycles, showing that $2^c - q - 1 \geq 0$. Since $T \setminus H$ has no odd cycles, Condition 2b holds for $(B+T)\setminus H$ being in $(\mathfrak{B}+\mathfrak{T}^4)_{c-1}$.

The choice of H implies that $m_1(T\setminus H) \geq m_1(T) - s$, because $m_1(T \setminus H_1) = m_1(T)$ and we chose at most s other $K_{1,2}$'s in H from T. It follows that $m_1(T\setminus H) \geq 2^{\alpha^*(B+T)}$. We conclude $\alpha^*((B+T)\setminus H) = \alpha^*(B+T)$.

Now we can get Condition 2c for $(B+T)\setminus H$ being in $(\mathfrak{B}+\mathfrak{T}^4)_{c-1}$:

$$\begin{aligned} |E((B+T)\setminus H)| &= |E(B+T)| - 2^{c+1} \\ &\leq \left(2^{\alpha^*(B+T)+1} + 2^{c+2} - 3\right) - 2^{c+1} \\ &= 2^{\alpha^*((B+T)\setminus H)+1} + 2^{c+1} - 3, \end{aligned}$$

as required.

Therefore $(B+T)\setminus H$ is in $(\mathfrak{B}+\mathfrak{T}^4)_{c-1}$, as required. ■

Theorem 3.3 goes a long way to proving Theorem 3.1; however, there remain the graphs in $(\mathfrak{B}+\mathfrak{T}^4)_{even}$ to consider.

Theorem 3.4 *Let $B+T \in (\mathfrak{B}+\mathfrak{T}^4)_{even}$. Then $B+T$ has a conventional decomposition into Γ_0, except in the case, for some non-negative integers i and b with $b < 2^i$, $T = (2^i - b)C_8 + (2b)P_4$. In the exceptional case, $B+T$ has a decomposition over Γ_0.*

Proof. Let q be that non-negative integer so that $B = qK_{1,2}$. We may assume T has at least one cycle C that is not of length a multiple of 3, as otherwise $B + T$ is binary and trivially has a conventional decomposition over Γ_0. Since C has even length and is not C_4, C has length at least 8.

As we did earlier, we let the slack s be defined by $s = m_1(T) - 2^{\alpha^*(B+T)}$. Set $\alpha = \alpha^*(B+T)$; we have $2^\alpha \leq m_1(T) < 2^{\alpha+1}$ and $s < 2^\alpha$.

Subcase 1: *either* $s > 0$ *or* $m_2(T) > 2^{\alpha-1}$.

If $s > 0$, then we claim that $m_2(T) \geq s+1$. Since a component K of T has an even number of edges, it is neither K_2 nor C_5. Thus, Observation 2.3 implies that $4m_2(K) \geq |E(K)|$; consequently, $4m_2(T) \geq |E(T)|$.

On the other hand, because every component of T is even, $|E(T)| = 2m_1(T)$, so

$$|E(T)| = 2(2^\alpha + s) > 4s\,,$$

the inequality because $s < 2^\alpha$. Putting these together, we have $4m_2(T) \geq |E(T)| > 4s$, which obviously implies $m_2(T) \geq s+1$, as claimed.

Now let H be an isomorph in $B+T$ of either

$$\begin{cases} (q+s+1)K_{1,2} & \text{if } s > 0, \text{ or} \\ (q+2^{\alpha-1}+1)K_{1,2} & \text{if } m_2(T) > 2^{\alpha-1}\,, \end{cases}$$

including B in H, and, furthermore, requiring H to contain two copies of $K_{1,2}$ in C that are separated by a single edge in C.

Suppose first that $s > 0$. With this choice of H, we claim that $m_1(T \setminus H) \geq m_1(T) - s$. This can be seen by ordering the choices for H. When we choose the first $K_{1,2}$ in C, m_1 drops by 1. We may take our next choice as the $K_{1,2}$ to also be in C, separated from the first one by a single edge. At this point, m_1 does not change. Each remaining $K_{1,2}$ in $T \cap H$ reduces m_1 by at most 1. Moreover, some component of $(B+T) \setminus H$ is a single edge, so $(B+T) \setminus H \notin (\mathfrak{B} + \mathfrak{T}^4)_{even}$.

It follows that $m_1(T \setminus H) \geq 2^\alpha$. Also,

$$|E((B+T) \setminus H)| = |E(T)| - 2(s+1) = 2^{\alpha+1} - 2\,.$$

Thus, $(B+T) \setminus H \in (\mathfrak{B} + \mathfrak{T}^4)_{-1}$.

In the case $m_2(T) > 2^{\alpha-1}$, the same argument shows that $m_1(T \setminus H) \geq 2^{\alpha-1}$ and that $|E((B+T) \setminus H| = 2^\alpha - 2$. Therefore, also in this case, $(B+T) \setminus H \in (\mathfrak{B} + \mathfrak{T}^4)_{-1}$.

Theorem 3.3 implies $T \setminus H$ has a decomposition over $\{2^i K_2 \mid i \geq 0\}$. Combining this with an obvious decomposition of H over $\{2^i K_{1,2} \mid i \geq 0\}$ yields the desired conventional decomposition of $B+T$.

Subcase 2: $s = 0$ *and* $m_2(T) \leq 2^{\alpha-1}$.

As $s = 0$, $m_1(T) = 2^\alpha$. On the other hand, for each $n \geq 3$, $\lfloor n/2 \rfloor \leq 2\lfloor n/3 \rfloor$, so Observation 2.3 implies $m_1(T) \leq 2m_2(T)$. Therefore, $m_2(T) \geq 2^{\alpha-1}$. It follows that, in this subcase, $m_2(T) = 2^{\alpha-1}$ and $m_1(T) = 2^\alpha$.

We claim that there is a non-negative integer $b < 2^\alpha - 2$ so that $T = (2^{\alpha-2} - b)C_8 + (2b)P_4$. If K is a path or cycle of even length and is not one of C_4, C_8, and P_4, then Observation 2.3 implies $4m_2(K) > |E(K)|$. Since every component of T has an even number of edges, $|E(T)| = 2m_1(T)$, so $|E(T)| = 2^{\alpha+1} = 4m_2(T)$. Since no component of T is C_4, every component of T is either C_8 or P_4.

Let m and n be the non-negative integers so that $T = mC_8 + nP_4$. We noted at the beginning of the proof that T has a cycle, so $m > 0$. Then $m_1(T) = 4m + 2n$, so $4m + 2n = 2^\alpha$. As the left hand side is even, $\alpha \geq 1$. Thus, $2m + n = 2^{\alpha-1}$, whence $n = 2^{\alpha-1} - 2m$. If $n = 0$, then $m = 2^{\alpha-2}$ and $b = 0$. If $n > 0$, then either

- $n = 1$, $\alpha = 1$, and $m = 0$, a contradiction, or
- $n > 1$, $\alpha > 1$, and n is even. Set $b = n/2$ and notice that $m = 2^{\alpha-2} - b$.

Thus, the graph we are considering is $qK_{1,2} + (2^{\alpha-2} - b)C_8 + (2b)P_4$, with $b < 2^{\alpha-2}$. If $q > 0$, then, allowing for an unconventional decomposition, we may interpret $qK_{1,2} + (2^{\alpha-2} - b)C_8 + (2b)P_4$ as a graph in $\mathfrak{T}^4$. Every component is even, but now not every component is either a C_8 or P_4. It follows that this graph decomposes over Γ_0 by some earlier subcase.

Thus, we may assume $q = 0$. In this case, there is a subgraph H of $(2^{\alpha-2} - b)C_8 + (2b)P_4$ that is an isomorph of $2^{\alpha-1}K_{1,2}$ so that, for each C_8, $C_8 \cap H = 2K_{1,2}$ and $C_8 \setminus H = K_2 + P_3$, while, for each of the $2b$ P_4's, $P_4 \cap H = K_{1,2}$ and $P_4 \setminus H = 2K_2$. We next pick an isomorph H' of $(2^{\alpha-2} - b)K_{1,2}$ so that $H' \cap P_3 = K_{1,2}$. We see that $((2^{\alpha-2} - b)C_8 + (2b)P_4) \setminus H) \setminus H'$ is just rK_2, showing that $(2^{\alpha-2})C_8 + (2b)P_4$ has a (conventional) decomposition over Γ_0, as required. ■

4 Going further

In this section, we identify two additional primal graphs in $\mathfrak{B} + \mathfrak{T}^4$ and use these with Γ_0 to decompose many members of $\mathfrak{B} + \mathfrak{T}^4$. We have already seen in the arguments that 5-cycles play an important role, as these are the only connected graphs K in $\mathfrak{T}^4$ for which $m_1(K) + 2m_2(K) < |E(K)|$.

In particular, we have the following.

Theorem 4.1 *Among all the graphs $B + T \in \mathfrak{B} + \mathfrak{T}^4$ with $1 \leq D(B + T) \leq 42$, only $C_5 + K_2$ and $7C_5 + K_2$ are primal. In particular, every graph in $\mathfrak{B} + \mathfrak{T}^4$ with $1 \leq D \leq 42$ has a conventional decomposition over $\Gamma_0 \cup \{C_5 + K_2, 7C_5 + K_2\}$.*

Proof. Suppose $B = qK_{1,2}$ and $T \in \mathfrak{T}^4$ are such that $1 \le D(B+T) \le 42$. Lemma 2.4 implies T contains a 5-cycle. Since $D(C_5) = 0$, $B + T$ has another component and, therefore, $C_5 + K_2 \subseteq B + T$.

Since a proper subgraph K of $C_5 + K_2$ has $D(K) \le 0$, K decomposes over Γ_0. Since $D(C_5 + K_2) \ge 1$, $C_5 + K_2$ does not decompose over Γ_0, so $C_5 + K_2$ is primal.

We have argued above that there is an isomorph H of $C_5 + K_2$ contained in $B + T$ so that the C_5 is contained in T. We note that $qK_{1,2} + C_5$ has a conventional decomposition via $(q+1)K_{1,2}$, $2K_2$, and K_2. In particular, $D(qK_{1,2} + C_5) \le 0$, so $B + T \ne qK_{1,2} + C_5$. That is, T has a component other than the one C_5. Thus, we may choose H so that the K_2 is also in a component of T; if possible, choose this component to be a C_5.

Suppose first that $T \setminus H$ has no 5-cycle. Lemma 2.4 implies $D((B + T) \setminus H) \le 0$. As long as $(T \setminus H)$ is not $(2^{\alpha} - b)C_8 + (2b)P_4$, Theorem 3.1 implies $(B + T) \setminus H$ has a conventional decomposition over Γ_0.

The following observation will allow us to deal with the exceptional case.

Claim 1 *Let S be a graph with maximum degree 2 and let e be an edge of S so that there are non-negative integers i and b with $b < 2^i$ and $S - e = (2^i - b)C_8 + (2b)P_4$. Then there is another edge e' of S so that $S - e'$ has two components with an odd number of edges and $|E(S - e')| \le m_1(S - e') + 2m_2(S - e')$.*

Proof. If no component of S is C_5, then let e' be any edge from a C_8. Lemma 2.4 implies $m_1(S - e') + 2m_2(S - e') \ge |E(S - e')|$. Otherwise, $S = (2^i - b)C_8 + (2b - 1)P_4 + C_5$ and we may choose e' to be in a P_4. Thus, $m_1(S - e') = 4(2^i)$ and $m_2(S - e') = 2(2^i)$. Since $|E(S - e')| = 8(2^i)$, $|E(S - e')| \le m_1(S - e') + 2m_2(S - e')$. □

In the exceptional case, we use the claim to pick e' for H instead of e. Now $B + (T \setminus H)$ has a conventional decomposition over Γ_0, as required.

In the remaining case, we may assume $3C_5 \subseteq T$. In this case, $(B + T) \setminus H = B + (T \setminus H)$ and $(B + T) \setminus H \notin (\mathfrak{B} + \mathfrak{T}^4)_{even}$.

Claim 2 $D(B + (T \setminus H)) = D(B + T) - 6$.

Proof. It suffices to show that $\alpha^*(B + (T \setminus H)) = \alpha^*(B + T)$ and $\beta^*(B + (T \setminus H)) = \beta^*(B + T)$. We start with the latter.

The choice of H implies $m_2^*(B + (T \setminus H)) = m_2^*(B + T) - 1$. Thus, if $\beta^*(B + (T \setminus H)) < \beta^*(B + T)$, we must have $m_2^*(B + T) = 2^{\beta^*(B+T)}$.

Observation 2.3 shows that, for every component K of T, $|E(K)| \le m_1(K) + 3m_2(K)$, so $m_1(T) + 3m_2(T) \ge |E(T)|$. Letting $\beta^* = \beta^*(B + T)$

and $\alpha^* = \alpha^*(B+T)$, it follows that

$$\begin{aligned}(2^{\alpha^*+1}-1)+3(2^{\beta^*}) &\geq m_1(T)+3(q+m_2(T)) \text{ (because } \\ &\qquad 2^{\beta^*}=q+m_2(T) \text{ and } 2^{\alpha^*+1}-1 \geq m_1(T)) \\ &\geq 2q+|E(T)|+q \\ &= |E(B+T)|+q \\ &\geq (2^{\alpha^*+1}+2^{\beta^*+2}-2)+q \text{ (because } \\ &\qquad D(B+T)\geq 1).\end{aligned}$$

Therefore, $1 \geq q + 2^{\beta^*}$, so $m_2^*(B+T) = 2^{\beta^*} \leq 1$. However, $3C_5 \subseteq T$, so $m_2^*(T) \geq 3$, a contradiction. Therefore, $\beta^*(B+(T \setminus H)) = \beta^*(B+T)$.

Now suppose $\alpha^*(B+(T\setminus H)) < \alpha^*(B+T)$. Since $m_1(T\setminus H) = m_1(T)-2$, we deduce that $m_1(T) \leq 2^{\alpha^*}+1$. Using the fact that $2m_1(T)+m_2(T) \geq |E(T)|$, we conclude that

$$\begin{aligned}2q+2m_1(T)+m_2(T) &\geq 2q+|E(T)| \\ &= |E(B+T)| \\ &\geq 2^{\alpha^*+1}+2^{\beta^*+2}-2 \text{ (because } D(B+T)\geq 1) \\ &\geq 2(m_1(T)-1)+2(q+m_2(T)+1)-2 \\ &\qquad \text{(because } 2^{\alpha^*} \geq m_1(T)-1 \text{ and} \\ &\qquad 2^{\beta^*+1} \geq q+m_2(T)+1) \\ &= 2m_1(T)+2q+2m_2(T)-2.\end{aligned}$$

Therefore, $2 \geq m_2(T)$. However, $3C_5 \subseteq T$, so $m_2(T) \geq 3$, a contradiction. □

An immediate consequence of Claim 2 and Theorem 3.1 is that, if $3C_5 \subseteq T$ and $D(B+T) \leq 6$, then $B+T$ has a conventional decomposition over $\Gamma_0 \cup \{C_5+K_2\}$, completing the proof of the theorem for $D(B+T) \leq 6$. We shall now repeat the preceding analysis to get the next step.

Let $B+T \in \mathfrak{B}+\mathfrak{T}^4$ be such that $D(B+T) \geq 7$. Lemma 2.4 implies that T contains $D(B+T)$ 5-cycles, and, therefore, $7C_5 \subseteq T$. However, $D(7C_5) = 6$, and so $D(B+T) \geq 7$ implies $7C_5+K_2$ is contained in $B+T$. As every proper subgraph K of $7C_5+K_2$ has $D(K) \leq 6$, all proper subgraphs of $7C_5+K_2$ decompose over $\Gamma_0 \cup \{C_5+K_2\}$. Since $D(7C_5+K_2) = 7$, $7C_5+C_2$ does not decompose over $\Gamma_0 \cup \{C_5+K_2\}$, and therefore $7C_5+K_2$ is primal.

If $B = qK_{1,2}$ and $T = 7C_5$, we see that $\alpha^*(B+T) = 3$ and $2^{\beta^*(B+T)+1} > q+7$. It follows that

$$2^{\alpha^*(B+T)+1}+2^{\beta^*(B+T)+2}-3 \geq 16+2(q+8)-3 = 2q+29.$$

Since $|E(B+T)| = 2q+35$, $D(B+T) \leq 6$. Therefore, T has a component other than $7C_5$. In exactly the same way as for the case $D \leq 6$, we can use Claim 1 to choose the extra edge making $7C_5 + K_2$ so as not to leave $(2^i - b)C_8 + (2b)P_4$. Thus, if $9C_5 \not\subseteq T$, then $B + (T \setminus H)$ has a conventional decomposition over Γ_0, yielding a conventional decomposition of $B+T$ over $\Gamma_0 \cup \{7C_5 + K_2\}$.

In the remaining case, we can assume $7C_5 \subseteq T \setminus H$, as otherwise $D(B+(T \setminus H)) \leq 6$ and $B+T$ decomposes over $\Gamma_0 \cup \{C_5 + K_2, 7C_5 + K_2\}$, as required.

Thus, we may assume the K_2-component in H is contained in a C_5 in T and that there is an additional $7C_5$ contained in $T \setminus H$. That is, $15C_5 \subseteq T$.

Note that $m_1(T \setminus H) = m_1(T) - 14$ and $m_2(T \setminus H) = m_2(T) - 7$. Now we aim for the analogue of Claim 2.

Claim 3 $D(B + (T \setminus H)) = D(B+T) - 36$.

Proof. It suffices to prove the two equalities $\alpha^*(B+(T\setminus H)) = \alpha^*(B+T)$ and $\beta^*(B+(T\setminus H)) = \beta^*(B+T)$.

Suppose first by way of contradiction that $\beta^*(B+(T\setminus H)) < \beta^*(B+T)$. Then $m_2^*(B+T) \leq 2^{\beta^*(B+T)}+6$. Letting r be the number of C_5's contained in T, Lemma 2.4 implies $m_1(T) + 2m_2(T) + r \geq |E(T)|$. Since $15C_5 \subseteq T$, $r \geq 15$.

We note that $r \leq m_2(T)$, so that $r \leq m_2^*(B+T) \leq 2^{\beta^*(B+T)} + 6$, or $2^{\beta^*(B+T)} \geq r - 6$. Since $r \geq 15$, $2^{\beta^*(B+T)} \geq 9$, whence $\beta^*(B+T) \geq 4$.

On the other hand, letting $\beta^* = \beta^*(B+T)$,

$$
\begin{aligned}
2^{\alpha^*+1} + 3(2^{\beta^*}) &\geq 2^{\alpha^*+1} + 2(2^{\beta^*}) + r - 6 \\
&\geq (m_1(T)+1) + 2(q + m_2(T) - 6) + r - 6 \\
&\geq 2q + |E(T)| - 17 \\
&= |E(B+T)| - 17 \\
&\geq 2^{\alpha^*+1} + 2^{\beta^*+2} + 4 - 17 \qquad \text{(because } D(B+T) \geq 7\text{)}.
\end{aligned}
$$

Therefore, $13 \geq 2^{\beta^*}$, so $\beta^* \leq 3$, contradicting the earlier conclusion that $\beta^* \geq 4$.

In order to prove the remaining equality $\alpha^*(B+(T\setminus H)) = \alpha^*(B+T)$, suppose by way of contradiction that $\alpha^*(B+(T\setminus H)) < \alpha^*(B+T)$. Since $m_1(T\setminus H) = m_1(T) - 14$, we deduce that $m_1(T) \leq 2^{\alpha^*} + 13$.

We know that $7C_5 + P_4 \subseteq T \setminus H$, so $\alpha^*((B+T)\setminus H) \geq 4$. Since $\alpha^*(B+T) > \alpha^*((B+T)\setminus H)$, we deduce that $\alpha^*(B+T) \geq 5$. Again we let r be the number of C_5's in T, $\alpha^* = \alpha^*(B+T)$, and $\beta^* = \beta^*(B+T)$ to get:

$$\begin{aligned} 2^{\alpha^*} + 2^{\beta^*+2} + r &\geq (m_1(T) - 13) + 2(q + m_2(T) + 1) + r \\ &\geq |E(B+T)| - 11 \\ &\geq 2^{\alpha^*+1} + 2^{\beta^*+2} + 4 - 11 \qquad \text{(because } D(B+T) \geq 7\text{)}. \end{aligned}$$

It follows that $7+r \geq 2^{\alpha^*} \geq m_1(T) - 13$, so $m_1(T) \leq 20 + r$. Evidently, $m_1(T) \geq 2r$, so $r \leq 20$. On the other hand, $\alpha^* \geq 5$, so $2^{\alpha^*} \geq 32$, so $7 + r \geq 32$, or $r \geq 25$, a contradiction. □

The proof is completed by noting that if $7 \leq D(B+T) \leq 42$, then Claim 3 and the lead-up to it implies there is an isomorph H of $7C_5 + K_2$ in T so that $D((B+T) \setminus H) \leq 6$. Claim 2 and the lead-up to it implies $(B+T) \setminus H$ conventionally decomposes over $\Gamma_0 \cup \{C_5 + K_2\}$ and the result follows. ■

5 Consequences and future work

One might hope from the main theorem that one can continue on indefinitely finding the next primal graphs in $\mathfrak{B} + \mathfrak{T}^4$. Unfortunately, this fails at the next step.

Suppose $B+T \in \mathfrak{B} + \mathfrak{T}^4$ is minimal with respect to having $D(B+T) \geq 43$. It is easy to see that $D(B+T) = 43$ and so $|E(B+T)| = 2^{\alpha^*+1} + 2^{\beta^*+2} + 40$. Moreover, $43C_5 \subseteq B+T$, so $\alpha^* \geq 6$ and $\beta^* \geq 5$, whence $|E(B+T)| \geq 128 + 128 + 40 = 296$. It is not hard to find graphs (even $T \in \mathfrak{T}^4$) with $D(B+T) = 43$ and $|E(B+T)| = 296$. One example is $59C_5 + K_2$, but there are hundreds of others. An interesting one (because it has a long cycle) is $C_{41} + 49C_5 + K_{1,2} + 8K_2$.

Where we spent some effort of detail showing that $D \leq 42$ implies the existence of a decomposition, one might hope for at least some start at a general theory. However, we do not know how to prove the following, which should be easy.

Conjecture 5.1 *If $T \in \mathfrak{T}^4$ has $D(T) \geq 43$, then T has a subgraph T' with $D(T') = 43$ and $|E(T')| = 296$.*

The proof of Theorem 4.1 seems adhoc to us; the proofs of Claims 2 and 3 seem to come from nowhere. Perhaps unifying these results would help in understanding Conjecture 5.1.

There are other avenues to explore. The graphs A and P are obtained from a triangle, the former by adding two new degree 1 vertices adjacent to different vertices of the triangle, and the latter identifying one end of a path of length 2 with one of the vertices of the triangle. Both of these graphs have $|E(G)| = 5$ and $m_1(G)+2m_2(G) = 4$. In the graphs in $\mathfrak{B}+\mathfrak{T}^4$, replacing all the C_5's with either all A's or all P's yields the same theorems. For example, $A + K_2$ and $7A + K_2$ are both primal. There should be a more general class of graphs to allow these so that they naturally come out as the primal examples, but we have not determined one, other than $\Omega(A)$ and $\Omega(P)$.

In the class of graphs with maximum degree 3, there are many primal graphs that we have not mentioned yet. The graph Y obtained by subdividing once each edge of $K_{1,3}$ is one example. The family $\Omega(Y)$ has infinitely many primal graphs, the next one being $7Y + 10K_2$.

For an integer $k \geq 3$, the *k-sun* is the graph obtained from a k-cycle by adding k new degree 1 vertices, each joined to a different one of the vertices of the k-cycle. In [4] it is proved that the k-sun is primal if and only if there is an integer $i \geq 2$ so that $k = 2^i - 1$. That is, the class of graphs with maximum degree 3 not containing a Y seems to be a rich class for primal graphs. It is known [2, 3] that a primal forest in this class (which is a forest of caterpillars with maximum degree 3) is in Γ_0.

6 Appendix

In this section we prove the following.

Lemma 6.1 *Let i, b, and q be non-negative integers for which $b < 2^i$. Then $qK_{1,2} + (2^i - b)C_8 + (2b)P_4$ has no conventional decomposition over Γ_0 if and only if $b = 0$ and q is of the form $2^{i+k} - 2^{i+1} - 1$, for some integer $k \geq 2$.*

Proof. If, for some integer $k \geq 2$, $q = 2^{i+k} - 2^{i+1} - 1$, then $\beta^* = i + k - 1$ and $\alpha^* = i + 2$. We first show we must use $2^{i+2}K_2$ in a conventional decomposition.

Otherwise, we can use at most

$$2^{\alpha^*} + 2^{\beta^*+2} - 3 = 2^{i+2} + 2^{i+k+1} - 3$$

edges in a conventional decomposition. On the other hand, $qK_{1,2} + (2^i - b)C_8 + (2b)P_4$ has $2(2^{i+k} - 2^{i+1} - 1) + 8(2^i)$ edges, which is $2^{i+2} + 2^{i+k+1} - 2$, too large for such a decomposition.

However, if $b = 0$, then $2^iC_8 \setminus 2^{i+2}K_2$ is always isomorphic to $2^{i+2}K_2$, and so $2^{i+2}K_2$ cannot be used in a conventional decomposition of $qK_{1,2} + 2^iC_8$.

Now for the converse. Suppose first that we are attempting to conventionally decompose $qK_{1,2} + (2^i - b)C_8 + (2b)P_4$, with $b > 0$. Begin by choosing $H = 2^{i+2}K_2$ in $(2^i - b)C_8 + (2b)P_4$ so that, for every P_4, $P_4 \setminus H = K_{1,2}$. Theorem 3.4 implies we can conventionally decompose the remainder. Since $2^{i+2}K_2$ is not contained in $((2^i - b)C_8) + (2b)P_4) \setminus H$, this completes a conventional decomposition of $qK_{1,2} + (2^i - b)C_8 + (2b)P_4$.

Thus, we may assume $b = 0$. The hypothesis implies q is not of the form $2^{i+k} - 2^i - 1$, with $k \geq 2$. In this case, we proceed by induction on q, the base case $q = 0$ being trivial: Theorem 3.4 implies there is a decomposition of 2^iC_8 over Γ_0 and every such decomposition is conventional.

There are three cases to consider: $q \geq 2^{\beta^*}$, $2^{i+1} \leq q < 2^{\beta^*}$, and $q < 2^{i+1}$. We start with $q \geq 2^{\beta^*}$. In this case $q \geq 2^{\beta^*} \geq 2^{i+1}$, so $\beta^* \geq i+2$. Let $H = 2^{\beta^*}K_{1,2} \subseteq qK_{1,2}$. Since $m_2^*((q - 2^{\beta^*})K_{1,2} + 2^iC_8) = m_2^*(qK_{1,2} + 2^iC_8) - 2^{\beta^*}$, a conventional decomposition of $(q - 2^{\beta^*})K_{1,2} + 2^iC_8$ completes a conventional decomposition of $qK_{1,2} + 2^iC_8$.

However, there may be an integer $k \geq 2$ so that $q = 2^{\beta^*} + 2^{i+k} - 2^{i+1} - 1$; in this circumstance, there is no conventional decomposition of $(q - 2^{\beta^*})K_{1,2} + 2^iC_8$. Clearly, $\beta^* \geq i+k$. If $\beta^* = i+k$, then $q = 2^{i+k+1} - 2^{i+1} - 1$ is a forbidden form. Thus, $\beta^* > i+k$.

Modify H so that all but one component of H is in $qK_{1,2}$ and one component is in a C_8. In this case,

$$(qK_{1,2} + 2^iC_8) \setminus H = (2^{i+k} - 2^{i+1})K_{1,2} + (2^i - 1)C_8 + P_6 .$$

Theorem 3.4 implies this has a conventional decomposition. Since $m_2^*((2^{i+k} - 2^{i+1})K_{1,2} + (2^i - 1)C_8 + P_6) = 2^{i+k}$, this conventional decomposition does not use $2^{\beta^*}K_{1,2}$ and so completes a conventional decomposition of $qK_{1,2} + 2^iC_8$.

In the second case, $2^{i+1} \leq q < 2^{\beta^*}$. Since $m_2^*(2^iC_8) = 2^{i+1}$, $\beta^* \geq i+2$. Choose $H = 2^{\beta^*}K_{1,2}$ so that $qK_{1,2} \subseteq H$. Then $(qK_{1,2} + 2^iC_8) \setminus H$ is isomorphic to some proper subgraph of 2^iC_8. Theorem 3.1 shows this has a conventional decomposition. Since $m_2^*(2^iC_8) = 2^{i+1}$ and $\beta^* \geq i+2$, this completes a conventional decomposition of $qK_{1,2} + 2^iC_8$.

In the remaining case, $q < 2^{i+1}$. Since $q \neq 2^{i+2} - 2^{i+1} - 1$, $q \neq 2^{i+1} - 1$, so $q \leq 2^{i+1} - 2$. Clearly, $\beta^* = i+1$. Let $H_1 = 2^{i+1}K_{1,2} \subseteq 2^iC_8$ be such that $2^iC_8 \setminus H_1 = 2^iP_3 + 2^iK_2$. Now choose $H_2 = 2^{i+1}K_2 \subseteq 2^iP_3 + 2^iK_2$ so that $(2^iP_3 + 2^iK_2) \setminus H_2 = (2^{i+1} - 2)K_2 + K_{1,2}$ (that is, choose two edges from all but one of the P_3's to go into H_2 and chose an end edge from the last P_3). Thus, $(qK_{1,2} + 2^iC_8) \setminus (H_1 \cup H_2) = qK_{1,2} + (2^{i+1} - 2)K_2 + K_{1,2}$.

We complete the conventional decomposition of $qK_{1,2} + 2^iC_8$ by using $2^jK_{1,2}$'s to cover the remaining $(q+1)K_{1,2}$ (since $q \leq 2^{i+1} - 2$, we will not use $2^{i+1}K_{1,2}$ in this decomposition) and 2^jK_2's to cover the $(2^{i+1} - 2)K_2$ (this will not use $2^{i+1}K_2$). ■

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