On the Number of 6×6 Sudoku Grids

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Abstract

Partially filled 6×6 Sudoku grids are categorised based on the arrangement of the values in the first three rows. This categorisation is then employed to determine the number of 6×6 Sudoku grids.

1 Introduction

A Sudoku grid, $S^{x,y}$, is a $n \times n$ array subdivided into n mini-grids of size $x \times y$ (where $n = xy$); the values $1, \ldots, n$ are contained within the array in such a way that each value occurs exactly once in every *row*, *column* and *mini-grid*.

A Sudoku grid can be thought of as a constrained Latin square, therefore these grids share many interesting properties and similar questions about these structures can be explored. In this paper the number of 6×6 Sudoku (Rodoku) grids is determined through mathematical means. The only known mathematical enumeration of the number of Sudoku grids is for the trivial case of 4×4 Sudoku (Shidoku), which using the results of [8] can be shown to have only 3 grids up to isomorphism. For the Sudoku-related structures 'NRC-Sudoku' [12] and '2-Quasi-Magic Sudoku' [9] the number of grids has been established. Other mathematical research on Sudoku has mainly concentrated on aspects of graph colouring, isomorphisms and their links to integral graphs and cayley tables [2, 3, 4, 6, 7, 14, 15].

A Rodoku grid, $S^{3,2}$ consists of two *bands*, each composed of three horizontally-consecutive mini-grids, and three *stacks*, each composed of two vertically-consecutive mini-grids. Each 3×2 mini-grid possesses three sub-rows, or *tiers* and two sub-columns, or *pillars*. Let ${S^{3,2}}_{a,b}$ represent the

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mini-grid in band a and stack b for $a \in \{1,2\}$ and $b \in \{1,2,3\}$. Each tier in ${S^{3,2}}_{a,b}$ contains a pair of values, considered as a set. Let $T_{a,b}$ be the set of all such sets of values, then $T_{a,b}$ can be thought of as the set of *tier-pairs* (of values). Each pillar in ${S^{3,2}}_{a,b}$ contains three values, considered as a set. Let $P_{a,b}$ be the set of all such sets of values, then $P_{a,b}$ can be thought of as the set of *pillar-triples* (of values).

A Rodoku grid will be called *reduced*, and labelled $s^{3,2}$ if: the values in ${S^{3,2}}_{1,1}$ are in canonical form, $[{S^{3,2}}_{1,1}]_{i,j} = (i-1)2 + j$; for each mini-grid ${S^{3,2}}_{1,b}$, with $b = 2,3$, the values in $[{S^{3,2}}_{1,b}]_{1,j}$ for $j = 1,2$ are increasing; $[{S^{3,2}}_{1,2}]_{1,1} < [{S^{3,2}}_{1,3}]_{1,1}$; and for ${S^{3,2}}_{2,1}$, the values in $[{S^{3,2}}_{2,1}]_{i,1}$ with $i = 1,2,3$ are increasing [8].

The strategy employed here for the enumeration of Rodoku grids is based on the classification of partially filled grids, which is similar to the method described for the enumeration of the number of Latin squares [1, 10, 11, 16]. The method of calculation for Latin squares is based on a k-regular bipartite graph $G = (S, C, E)$ where S is the set of value vertices, $s_j \in S$, and C the set of column vertices, $c_i \in C$. An edge, $e \in E$, is incident to c_i and s_j if and only if the value j does not appear in column i. Each row is successively calculated by computing the number of one-factorizations in G producing a complete set of reduced $k \times n$ Latin rectangles in which equivalent rectangles are eliminated as they appear. The number of Latin squares is obtained by calculating the product of the number of ways in which a rectangle is formed and the number of ways each rectangle can be completed to a square. Due to the restrictive position of the values in the mini-grids of a Rodoku grid only five cases occur for the top band (first three rows) of the Rodoku grid (see Section 2) and in this paper these are explicitly described and the number of occurences of each is counted. For each of the cases, the number of ways of arranging the values in the bottom band is determined (see Section 4).

A band of a Rodoku grid, comprising 1×3 contiguous mini-grids of size 3×2 is termed $S^{3,2}_{1,3}$, and its reduced form labelled $s^{3,2}_{1,3}$, (the stated constraints of Sudoku relating to the structure hold). The number of ways, $\mathrm{S}^{3,2}_{1,3}(6)$, of arranging the values within $S^{3,2}_{1,3}$, is given by Equation (1) which was first presented in [5].

$$\mathrm{S}^{3,2}_{1,3}(6) = (3 \times 2)! \times 2!^6 \sum_{i=0}^{2} \binom{2}{i}^3 = 460800. \qquad (1)$$

The top band of a Rodoku grid will also be assumed to be in its *reduced* form [8] such that: the values in ${S^{3,2}}_{1,1}$ are in canonical form; for each minigrid ${S^{3,2}}_{1,b}$ with $b = 2, 3$ the values in $[{S^{3,2}}_{1,b}]_{1,j}$ for $j = 1, 2$ are increasing; and $[{S^{3,2}}_{1,2}]_{1,1} < [{S^{3,2}}_{1,3}]_{1,1}$. Henceforth all top bands of a Rodoku grid will be in reduced form. The relationship between the number of Rodoku band arrangements, $\mathrm{S}^{3,2}_{1,3}(6)$, and the number of reduced Rodoku band arrangements, $\mathrm{s}^{3,2}_{1,3}(6)$, is given in Equation (2) which was first presented in [8]:

$$\mathrm{S}^{3,2}_{1,3}(6) = 460800 = 6! \times 2^{(3-1)} \times (3-1)! \times \mathrm{s}^{3,2}_{1,3}(6). \qquad (2)$$

Therefore the number of reduced Rodoku band arrangements is $\mathrm{s}^{3,2}_{1,3}(6) = 80$.

2 Classification of Arrangements of Values in the First Band

The set R contains all arrangements of the values in a band of a Rodoku grid. The classification of the arrangements of the values in the top band of $S^{3,2}$ is defined by the relationships between $P_{1,1}$, $P_{1,2}$ and $P_{1,3}$, which partition the set R into five disjoint subsets $R_1, \ldots, R_5 \subset R$ such that $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 = R$ and $R_i \cap R_j = \emptyset \ \forall i, j \ i \neq j$. The properties of the elements of these subsets are described explicitly in the following ways:

- $r_1 \in R$ is an element in R_1 if $P_{1,2} = P_{1,3}$.
- $r_2 \in R$ is an element in R_2 if $T_{1,1} = T_{1,2} = T_{1,3}$ and $P_{1,2} \neq P_{1,3}$.
- $r_3 \in R$ is an element in R_3 if: neither triple of $P_{1,2}$ contains both values from any $t \in T_{1,1}$, and one triple of $P_{1,3}$ contains both values from some $t \in T_{1,1}$; or if neither triple of $P_{1,3}$ contains both values from any $t \in T_{1,1}$, and one triple of $P_{1,2}$ contains both values from some $t \in T_{1,1}$.
- $r_4 \in R$ is an element in R_4 if for $t_i, t_j, t_k \in T_{1,1}$ (where i, j, k are all different) the triples of $P_{1,2}$ and $P_{1,3}$ each contain exactly one value from t_i and both values from either t_j or t_k.
- $r_5 \in R$ is an element in R_5 if for $t_i, t_j, t_k \in P_{1,1}$ (where i, j, k are all different) the triples of $P_{1,2}$ each contain exactly one value from t_i and both values from either t_j or t_k and the triples of $P_{1,3}$ each contain one value from t_j and both values from either t_i or t_k.

Examples of a representative element, r_i, for each R_i are given in Figure 1.

1	4	2	5	3	6
2	5	3	6	4	1
3	6	4	1	2	5

(a) r_1

1	4	2	5	3	6
2	5	3	6	4	1
3	6	4	1	5	2

(b) r_2

1	4	3	5	2	6
2	5	1	6	3	4
3	6	2	4	5	1

(c) r_3

1	4	3	5	2	6
2	5	6	1	4	3
3	6	4	2	5	1

(d) r_4

1	4	2	6	3	5
2	5	3	4	6	1
3	6	5	1	4	2

(e) r_5

Figure 1: Examples of Representative Elements for the Top Band of a Rodoku Grid

Lemma 1. *The cardinality of* R_1 *is* 8.

Proof. There is one way of arranging the values in ${S^{3,2}}_{1,1}$. Two subcases occur: Firstly there is one way of arranging the tier-pairs from $T_{1,1}$ in $T_{1,2}$ and $T_{1,3}$ such that $T_{1,1} = T_{1,2} = T_{1,3}$. There are four ways of assigning the values from each tier-pair to $P_{1,2}$ and then one way of assigning the values to $P_{1,3}$ such that $P_{1,2} = P_{1,3}$. Secondly there are four ways of arranging the values in $T_{1,2}$ and $T_{1,3}$ such that $T_{1,1}$, $T_{1,2}$ and $T_{1,3}$ are all different. In each arrangement there is one way of assigning the values to the pillar-triples such that $P_{1,1} = P_{1,2} = P_{1,3}$. □

Lemma 2. *The cardinality of* R_2 *is* 12.

Proof. There is one way of arranging the values in ${S^{3,2}}_{1,1}$. There is one way of arranging the tier-pairs from $T_{1,1}$ in $T_{1,2}$ and $T_{1,3}$ such that $T_{1,1} = T_{1,2} = T_{1,3}$. There are four ways of assigning the values from each tier-pair to $P_{1,2}$, and three ways of assigning the values to $P_{1,3}$ such that $P_{1,2} \neq P_{1,3}$. □

Lemma 3. *The cardinality of* R_3 *is* 24.

Proof. There is one way of arranging the values in ${S^{3,2}}_{1,1}$. There are four ways of arranging the tier-pairs from $T_{1,1}$ in $T_{1,2}$ and $T_{1,3}$ such that $T_{1,1}$, $T_{1,2}$ and $T_{1,3}$ are all different. There is one way of assigning the values of the tier-pairs of $T_{1,2}$ such that the two values from every $t \in T_{1,1}$ are in different pillar-triples in $P_{1,2}$ (or equivalently in $P_{1,3}$) and then three ways of assigning the values to $P_{1,3}$ (or $P_{1,2}$) such that $P_{1,2} \neq P_{1,3}$. □

Lemma 4. *The cardinality of R_4 is* 12.

Proof. There is one way of arranging the values in ${S^{3,2}}_{1,1}$. There are four ways of arranging the values in $T_{1,2}$ and $T_{1,3}$ such that $T_{1,1}$, $T_{1,2}$ and $T_{1,3}$ are all different. Any one of the three tier-pairs $t \in T_{1,1}$ may be chosen such that the values of t are in different pillar-triples in $P_{1,2}$, thus determining the positions of the remaining values. Since the values from the same tier-pair are also in different pillar-triples in $P_{1,3}$ then there is one way of arranging the values in ${S^{3,2}}_{1,1}$ such that $P_{1,1}$, $P_{1,2}$ and $P_{1,3}$ are all different. □

Lemma 5. *The cardinality of R_5 is* 24.

Proof. There is one way of arranging the values in ${S^{3,2}}_{1,1}$. There are four ways of arranging the values in $T_{1,2}$ and $T_{1,3}$ such that $T_{1,1}$, $T_{1,2}$ and $T_{1,3}$ are all different. Any one of the three tier-pairs $t \in T_{1,1}$ may be chosen such that the values of t are in different pillar-triples in $P_{1,2}$, thus determining the position of the remaining values. There remain two candidate tier-pairs $t \in T_{1,1}$ which are in different pillar-triples in $P_{1,3}$; once selected there is one way of assigning the values to $P_{1,3}$ such that $P_{1,1}$, $P_{1,2}$ and $P_{1,3}$ are all different. □

3 Properties of the Second Band

Consider the top band of a Rodoku grid to have a given arrangement of values. For $p_i = \{[{S^{3,2}}_{1,3}]_{1,2}, [{S^{3,2}}_{1,3}]_{2,2}, [{S^{3,2}}_{1,3}]_{3,2}\}$ and for

$p_j = \{[{S^{3,2}}_{2,3}]_{1,2}, [{S^{3,2}}_{2,3}]_{2,2}, [{S^{3,2}}_{2,3}]_{3,2}\}$, $p_j = \{1, \ldots, 6\}/p_i$ and therefore there are 3! ways of arranging the values in ${S^{3,2}}_{2,3}$. Once the values are assigned to ${S^{3,2}}_{2,3}$ then for $p_i \in P_{1,2}$ and $p_j \in P_{2,2}$, $p_j = \{1, \ldots, 6\}/p_i$ and there are 2 ways of arranging the values in each pillar of ${S^{3,2}}_{2,2}$. There are therefore 24 ways of arranging the values in ${S^{3,2}}_{2,2}$ and ${S^{3,2}}_{2,3}$ (since the grid is reduced), while ${S^{3,2}}_{2,1}$ is not completed.

The mini-grid ${S^{3,2}}_{1,1}$ contains a predetermined arrangement of the values (they are in canonical form). Relaxing the canonical form of ${S^{3,2}}_{1,1}$, consider the values in ${S^{3,2}}_{1,1}$ to be assigned to the tier-pairs, $T_{1,1}$, but not to specific cells. This alteration results in the top band being not entirely reduced (though since they meet all other conditions they will still be referred to as reduced) and the number of reduced Rodoku grids is increased by a factor of 2^3. This alteration to ${S^{3,2}}_{1,1}$ simplifies the enumeration process since if values are allocated to $P_{2,1}$ but unassigned to specific cells (a process which will be used later) there always exist at least one way of assigning the values to the cells of ${S^{3,2}}_{1,1}$ and ${S^{3,2}}_{2,1}$.

For ${S^{3,2}}_{1,b}$ for $b = 2, 3$ consider the values in each row in these mini-grids to each form a set. Let Λ be the set of all such sets. Similarly for ${S^{3,2}}_{2,b}$ for $b = 2, 3$ consider the values in each row in these mini-grids to each form a set. Let Ω be the set of all such sets. The tier-pairs in $T_{1,1}$ and $T_{2,1}$ are resolved since for every t_i in $T_{1,1}$ and $\alpha_i \in \Lambda$ (for $i = 1, 2, 3$), $t_i = \{1, \ldots, 6\} \setminus \alpha_i$, and for every $t_j \in T_{2,1}$ and $\beta_j \in \Omega$ (for $j = 1, 2, 3$), $t_j = \{1, \ldots, 6\} \setminus \beta_j$.

Definition 6. *If $\alpha = \beta$ for any $\alpha \in \Lambda$ and any $\beta \in \Omega$ then the top two bands are said to contain a* twin-row.

If ${S^{3,2}}_{a,b}$ for $a = 1, 2$ and $b = 2, 3$ contains a valid arrangement of the values then the values have effectively been assigned to $T_{1,1}$ and $T_{2,1}$. The number of ways of assigning each of the values in the tier-pairs of $T_{1,1}$ and $T_{2,1}$ to the cells of ${S^{3,2}}_{1,1}$ and ${S^{3,2}}_{2,1}$ is dependent on the relationship between $T_{1,1}$ and $T_{2,1}$. Three situations occur: $|T_{1,1} \cap T_{2,1}| = 3$, $|T_{1,1} \cap T_{2,1}| = 1$ and $|T_{1,1} \cap T_{2,1}| = 0$.

Lemma 7. *For $|T_{1,1} \cap T_{2,1}| = x$ there are 2^x ways of arranging the values to the cells of ${S^{3,2}}_{1,1}$ and ${S^{3,2}}_{2,1}$ if $x = 3$, and $2^{(x+1)}$ otherwise.*

Proof. Each tier-pair of values in $T_{1,2} \cap T_{2,1}$ represents a pair of 'orientable' values, for which the assignment of these values to the pillars of both of ${S^{3,2}}_{1,1}$ and of ${S^{3,2}}_{2,1}$ is independent of any of the other values in these mini-grids. This gives x pairs of orientable values and therefore 2^x arrangements. The remaining values from the tier-pairs not in $T_{1,1} \cap T_{1,2}$ form a set of values for which the placement of any one value determines the assignment to pillars of all the remaining values in both ${S^{3,2}}_{1,1}$ and ${S^{3,2}}_{2,1}$. Thus if $|T_{1,1} \cap T_{2,1}| = x$ then there are x pairs of orientable values if $x = 3$ and $x + 1$ sets of orientable values if $x \neq 3$. □

There is a direct relationship between the number of twin-rows in ${S^{3,2}}_{a,b}$, for $a = 1, 2$ and $b = 2, 3$, and $T_{1,1} \cap T_{2,1}$ such that: if ${S^{3,2}}_{a,b}$ for $a = 1, 2$ and $b = 2, 3$ contains three twin-rows then $|T_{1,1} \cap T_{2,1}| = 3$, and there are eight ways of arranging the values in ${S^{3,2}}_{1,1}$ and ${S^{3,2}}_{2,1}$ (Lemma 7). If ${S^{3,2}}_{a,b}$ for $a = 1, 2$ and $b = 2, 3$ contains one twin-row then $|T_{1,1} \cap T_{2,1}| = 1$ and there are four ways of arranging the values in ${S^{3,2}}_{1,1}$ and ${S^{3,2}}_{2,1}$ (Lemma 7). If ${S^{3,2}}_{a,b}$ for $a = 1, 2$ and $b = 2, 3$ contains no twin-rows then $|T_{1,1} \cap T_{2,1}| = 0$ and there are two ways of arranging the values in ${S^{3,2}}_{1,1}$ and ${S^{3,2}}_{2,1}$ (Lemma 7).

4 Enumeration of Complete Grids

For each of the representative elements $r_1, \ldots, r_5$ the number of twin-rows is now calculated for each of the 24 arrangements of the values in ${S^{3,2}}_{2,2}$ and ${S^{3,2}}_{2,3}$. In Lemmas 8 to 12 the number of ways of forming a complete Rodoku grid for each of the representative elements $r_1, \ldots, r_5$ is calculated and used in Theorem 13 to calculate the total number of Rodoku grids.

Lemma 8. *An element of R_1 can be used to form a complete Rodoku grid in* 96 *ways.*

Proof. Let ${S^{3,2}}_{1,2}$ and ${S^{3,2}}_{1,3}$ have an arrangement of the values satisfying the properties of the elements of R_1. Consider the six ways of arranging the values in ${S^{3,2}}_{2,3}$. Three cases arise: $P_{2,3} = P_{1,3}$ (one arrangement), $|P_{2,3} \cap P_{1,3}| = 1$ (three arrangements) and $|P_{2,3} \cap P_{1,3}| = 0$ (two arrangements).

If $P_{2,3} = P_{1,3}$, two of the four arrangements for ${S^{3,2}}_{2,2}$ result in three twin-rows, otherwise there are no twin-rows. If $|P_{2,3} \cap P_{1,3}| = 1$ all arrangements of the values in ${S^{3,2}}_{2,2}$ result in one twin-row. If $|P_{2,3} \cap T_{1,3}| = 0$ one arrangement of the values in ${S^{3,2}}_{2,2}$ results in three twin-rows; since this is the only arrangement for which a twin-row may be formed.

In total there are therefore 4 arrangements with three twin-rows, 12 arrangements with two twin-rows and 8 arrangements with no twin-rows. An element of R_1 can be used to form a complete Rodoku grid in $4 \times 8 + 12 \times 4 + 8 \times 2 = 96$ ways (Lemma 7). □

Lemma 9. *An element of R_2 can be used to form a complete Rodoku grid in* 80 *ways.*

Proof. Let ${S^{3,2}}_{1,2}$ and ${S^{3,2}}_{1,3}$ have an arrangement of the values satisfying the properties of the elements of R_2. Consider the six ways of arranging the values in ${S^{3,2}}_{2,3}$. Four cases arise: $T_{2,3} = T_{1,3}$ (one arrangement, equivalent to the same case for elements of R_1 in Lemma 8); $|T_{2,3} \cap T_{1,3}| = 0$ (two arrangements); $|T_{2,3} \cap T_{1,3}| = 1$ and $P_{2,3} = P_{1,2}$ (one arrangement, equivalent to the same case for elements of R_1 in Lemma 8); and $|T_{2,3} \cap T_{1,3}| = 1$ and $P_{2,3} \neq P_{1,2}$ (two arrangements).

If $|T_{2,3} \cap T_{1,3}| = 0$ only one tier-pair in ${S^{3,2}}_{2,3}$ contains values which are in different pillars in ${S^{3,2}}_{1,2}$ and thus only the row containing this tier-pair can be used to a form twin-row. If $|T_{2,3} \cap T_{1,3}| = 1$ and $P_{2,3} \neq P_{1,2}$, two arrangements of the values in ${S^{3,2}}_{2,2}$ result in one twin-row (in the row for which a tier of ${S^{3,2}}_{2,3}$ contains the same values as a tier of ${S^{3,2}}_{1,3}$); since the other two rows cannot form a twin-row then in all the remaining arrangements of the values in $S^{3,2}{}_{2,2}$ there are no twin-rows.

In total there are therefore 2 arrangements with three twin-rows, 10 arrangements with two twin-rows and 12 arrangements with no twin-rows. An element of R_2 can be used to form a complete Rodoku grid in $2 \times 8 + 10 \times 4 + 12 \times 2 = 80$ ways (Lemma 7). □

Lemma 10. *An element of R_3 can be used to form a complete Rodoku grid in 72 ways.*

Proof. Let $S^{3,2}{}_{1,2}$ and $S^{3,2}{}_{1,3}$ have an arrangement of the values satisfying the properties of the elements of R_3. Consider the six ways of arranging the values in $S^{3,2}{}_{2,3}$. Four cases arise: $|T_{2,3} \cap T_{1,1}| = 1$ and $|T_{1,3} \cap T_{2,3}| = 1$ (one arrangement, equivalent to the same case for elements of R_1 in Lemma 8); $|T_{2,3} \cap T_{1,1}| = 1$ and $|T_{1,3} \cap T_{2,3}| = 0$ (one arrangement, equivalent to the same case for elements of R_2 in Lemma 9); $|T_{2,3} \cap T_{1,1}| = 0$ (three arrangements, equivalent to the same case for elements of R_2 in Lemma 9); and $|T_{2,3} \cap T_{1,3}| = 0$ (one arrangement).

If $|T_{2,3} \cap T_{1,3}| = 0$ then in this case a twin-row is formed when $T_{2,3} = T_{2,2}$ and $T_{2,2} = T_{1,3}$; this is the only way in which a twin-row may be formed and three twin-rows are formed.

In total there are therefore 1 arrangement with three twin-rows, 9 arrangements with two twin-rows and 14 arrangements with no twin-rows. An element r_3 can be used to form a complete Rodoku grid in $1 \times 8 + 9 \times 4 + 14 \times 2 = 72$ ways (Lemma 7). □

Lemma 11. *An element of R_4 can be used to form a complete Rodoku grid in 96 ways.*

Proof. Let $S^{3,2}{}_{1,2}$ and $S^{3,2}{}_{1,3}$ have an arrangement of the values satisfying the properties of the elements of R_4. Consider the six ways of arranging the values in $S^{3,2}{}_{2,3}$. Two cases arise: $|T_{2,3} \cap T_{1,1}| = 1$ (two arrangements) and $|T_{2,3} \cap T_{1,1}| = 0$ (four arrangements).

If $|T_{2,3} \cap T_{1,1}| = 1$ and $|T_{1,3} \cap T_{2,3}| = 1$ then in this case a twin-row can only be formed with the row for which $S^{3,2}{}_{1,2}$ does not contain a tier-pair of values t where $t \in T_{1,1}$. There are therefore two arrangements with two twin-rows and two arrangements with no twin-rows. If $|T_{2,3} \cap T_{1,1}| = 0$ then $|T_{1,3} \cap T_{2,3}| = 0$ and in this case there is one way of forming three twin-rows. This is the only way of forming a twin-row with the rows which do not contain either of the values from the tier-pair t for $t \in T_{1,1}$ where there does not exist a $p \in P_{1,2}$ such that $t \subset p$. Therefore in one arrangement three twin-rows are formed, in two arrangements two twin-rows are formed and in the final arrangement no twin-rows are formed.

In total there are therefore 4 ways of forming three twin-rows, 12 ways of forming one twin-row and 8 ways of forming no twin-rows. An element of R_4 can therefore be used to form a complete Rodoku grid in $4\times8+12\times4+8\times2 = 96$ ways (Lemma 7). □

Lemma 12. *An element of R_5 can be used to form a complete Rodoku grid in 80 ways.*

Proof. Let ${S^{3,2}}_{1,2}$ and ${S^{3,2}}_{1,3}$ have an arrangement of the values satisfying the properties of the elements of R_5. Consider the six ways of arranging the values in ${S^{3,2}}_{2,3}$. Three cases arise: $|T_{2,3} \cap T_{1,1}| = 1$ and as such $|T_{2,3} \cap T_{1,2}| = 1$ (two arrangements); $|T_{2,3} \cap T_{1,1}| = 3$ and for every tier-pair $t \in T_{2,3}$ there does not exist a $p \in P_{1,2}$ such that $t \subset p$ (two arrangements); and $|T_{2,3} \cap T_{1,1}| = 3$ and there exists one tier-pair $t \subset T_{2,3}$ such that $t \in p$ for a $p \in P_{1,2}$.

If $|T_{2,3} \cap T_{1,1}| = 1$ then $|T_{2,3} \cap T_{1,2}| = 1$ and it is not possible to form three twin-rows. One twin-row may be formed in two ways, in the row of ${S^{3,2}}_{2,3}$ which does not contain one value from the tier-pair t for $t \in T_{1,1}$ where there does not exist a $p \in P_{1,2}$ such that $t \subset p$. If $|T_{2,3} \cap T_{1,1}| = 3$ and for every tier-pair $t \in T_{2,3}$ there does not exist a $p \in P_{1,2}$ such that $t \subset p$, there is one way of forming a twin-row (with the row for which a tier of ${S^{3,2}}_{2,3}$ does not contain either of the two values from t where $t \in T_{1,1}$ and there does not exist a $p \in P_{1,2}$ such that $t \subset p$). If $|T_{2,3} \cap T_{1,1}| = 3$ and there exists one tier-pair $t \in T_{2,3}$ such that $t \subset p$ for a $p \in P_{1,2}$ there are three twin-rows formed if $T_{2,3} = T_{1,2}$ and $T_{2,2} = T_{1,3}$. This is the only way in which a twin-row may be formed for the row for which the tier of ${S^{3,2}}_{2,3}$ contains neither value from the tier-pair t where $t \in T_{1,1}$ and $t \notin T_{1,2}$.

In total there are therefore 2 ways of forming three twin-rows, 10 ways of forming one twin-row and 12 ways of forming no twin-rows. An element of R_5 can therefore be used to form a complete Rodoku grid in $2 \times 8 + 10 \times 4 + 12 \times 2 = 80$ ways (Lemma 7). □

The number reduced Rodoku grids is now calculated in Theorem 13.

Theorem 13. *The number of reduced Rodoku grids, $\mathrm{s}^{3,2}(6)$, is 816.*

Proof. The set R contains all reduced arrangements of the values in the top band of a Rodoku grid such that $|R| = \mathrm{s}^{3,2}_{1,3}(6) = 80$. (Additionally the values are assigned to $T_{1,1}$ but not assigned to specific cells in $S^{3,2}{}_{1,1}$.) The set R is partitioned into five subsets R_i for $i = 1, \ldots, 5$, the cardinality of each given by Lemmas 1 to 5. For each of the elements of the subsets the number of ways of forming a complete Rodoku grid is given in Lemmas 8 to 12. If $n(R_i)$ is the number of ways of completing each representative element $r_i \in R_i$ to a Rodoku grid then the total number of reduced Rodoku grids, $s^{3,2}(6)$, is

$$\begin{aligned} s^{3,2}(6) &= \frac{1}{2^3}\sum_{i=1}^{5} |R_i| \times n(R_i) \\ &= \frac{1}{2^3}(8 \times 96 + 12 \times 80 + 24 \times 72 + 12 \times 96 + 24 \times 80) = 816. \end{aligned}$$

□

Since the number of reduced Rodoku grids has been calculated the total number of Rodoku grids, $S^{3,2}(6)$, can be calculated by Equation 3 from [8], thus proving analytically the exhaustive computational count provided by Pettersen on the Sudoku Forum [13].

$$\begin{aligned} S^{3,2}(6) &= (n)! \times (y!)^{(x-1)} \times (x!)^{(y-1)} \times (x-1)! \times (y-1)! \times s^{x,y}(n)(3) \\ &= 6! \times 2!^2 \times 3!^1 \times 2! \times 1! \times 816 \\ &= 28200960. \end{aligned}$$

References

[1] S. E. Bammel, J. Rothstein, *The number of* 9×9 *Latin squares*, Discrete Math 11 (1975) 93–95.

[2] J. Carmichael, K. Schloeman, M. B. Ward, *Cosets and cayley-sudoku tables*, Mathematics Magazine 83 (2) (2010) 130–139.

[3] C. Elsholtz, A. Mütze, Sudoku im mathematikunterricht, Math. Semesterber. 39 (2007) 69–93.

[4] B. Felgenhauer, F. Jarvis, *Mathematics of Sudoku I*, Mathematical Spectrum 39 (2006) 15–22.

[5] Forum Contributors, *Su-doku's maths*, Available at: `http://www.sudoku.com/boards/viewtopic.php?t=44&start=412` (2005).

[6] F. Harary, W. Slany, O. Verbitsky, *On the computational complexity of the forcing chromatic number*, SIAM J. Comput. 37 (2007) 1–19.

[7] A. M. Herzberg, M. R. Murty, *Sudoku squares and chromatic polynomials*, Notices of the American Mathematical Society 54 (2007) 708–717.

[8] S. K. Jones, S. Perkins, P. A. Roach, *The Structure of Reduced Sudoku Grids and the Sudoku Symmetry Group*, Submitted.

[9] S. K. Jones, S. Perkins, P. A. Roach, *Properties, isomorphisms and enumeration of 2-Quasi-Magic Sudoku grids, Discrete Mathematics* 311 (1) (2009) 1098–1110. `http://dx.doi.org/10.1016/j.disc.2010.09.026` `doi:10.1016/j.disc.2010.09.026`.

[10] B. D. McKay, E. Rogoyski, *Latin squares of order 10, Electronic Journal of Combinatorics* 2(3) (1995) 1–4.

[11] B. D. McKay, I. M. Wanless, *On the number of Latin squares, Annals of Combinatorics* 9 (2005) 335–344.

[12] B. Michel, *Mathematics of NRC-Sudoku*, Tech. rep., Universiteit Utrecht, Netherlands (December 2007).

[13] K. F. Pettersen, *Enumeration of* 6 × 6 *Sudoku*, Available at: `http://forum.enjoysudoku.com/su-doku-s-maths-t44-405.html\#p11444` (2005).

[14] E. Russell, F. Jarvis, *Mathematics of Sudoku II*, Mathematical Spectrum 39 (2) (2007) 54–58.

[15] T. Sander, *Sudoku graphs are integral*, Electronic Journal of Combinatorics 16.

[16] M. B. Wells, *The number of Latin squares of order eight*, Journal of Combinatorial Theory 3 (1967) 98–99.