On the strongly regular unit distance graphs

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Abstract

A unit distance graph is a finite simple graph which may be drawn on the plane so that its vertices are represented by distinct points and the edges are represented by closed line segments of unit length. In this paper we show that the only primitive strongly regular unit distance graphs are (i) the pentagon, (ii) $K_3 \times K_3$, (iii) the Petersen graph, and (iv) possibly the Hoffman-Singleton graph.

Keywords: The unit distance graph, euclidean distance, Moore graphs, three dimensional hypercube, strongly regular graphs

All graphs in this paper are simple, i.e.undirected, without loops or multiple edges. *The unit distance graph* of the real euclidean plane $\mathbb{R}^2$, sometimes denoted by $G(\mathbb{R}^2)$, is the infinite graph whose vertices are all the points of $\mathbb{R}^2$, two of them being adjacent if and only if they are at euclidean distance 1. A long standing open problem due to Hadwiger and Nelson [6] is the exact determination of the vertex chromatic number $\chi(G)$ of $G = G(\mathbb{R}^2)$. It is well known that $4 \leq \chi(G) \leq 7$. Indeed, $\mathbb{R}^2$ may be tiled by congruent copies of a regular hexagon of appropriate size so that the plane has a colouring in seven colours in which the interior of each tile is monochromatic and no two points of the same colour are exactly at distance one. See, for example, figure VI.3, page 112 in [1]. This shows that $\chi(G) \leq 7$. Also, the graph known as Moser's spindle, displayed in Figure 1, is a subgraph of $G(\mathbb{R}^2)$ (as Figure 1 itself shows) with chromatic number 4 (as is easily verified). Hence $\chi(G) \geq 4$.

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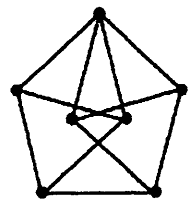


Figure 1: Moser's Spindle

A *unit distance representation* of a finite simple graph H with vertex set V is a one-one map $f : V \longrightarrow \mathbb{R}^2$ such that, whenever $x, y \in V$ are adjacent in H, we have $\| f(x) - f(y) \| = 1$ (Here $\| \cdot \|$ denotes the usual eucledian norm on $\mathbb{R}^2$). In other words a unit distance representation of H is an isomorphism of H with a subgraph of $G(\mathbb{R}^2)$. We say that H is a *unit distance graph* if it has a unit distance representation. Thus a unit distance graph is nothing but a finite subgraph of $G(\mathbb{R}^2)$, up to isomorphism.

If one accepts Zermelo's axiom of choice, then a well known compactness argument shows that the chromatic number of any infinite graph is just the supremum of the chromatic numbers of its finite subgraphs. In particular, $\chi(G(\mathbb{R}^2))$ is the maximum of the chromatic numbers of the unit distance graphs. Thus, the folklore conjecture that $\chi(G(\mathbb{R}^2)) = 4$ may be restated as:

Four Colour Conjecture for unit distance graphs: *The chromatic number of any unit distance graph is at most 4.*

It can be shown that Moser's spindle is the smallest unit distance graph with chromatic number four. In view of the above discussion, we have:

Proposition 1 *For any unit distance graph H, we have* $\chi(H) \leq 7$.

In looking for unit distance graphs which are counter-examples to the four colour conjecture, it is natural to try to restrict the search to special subclasses of simple graphs with a high degree of combinatorial regularity. One such subclass is that of *strongly regular graphs* (s.r.g's), which is of great importance in finite geometry and group theory. Recall that an s.r.g with parameters (v, k, λ, μ) is a regular graph of degree k on v vertices such that any two adjacent vertices have exactly λ common neighbours, while any two (distinct but) non-adjacent vertices have exactly μ common neighbours. An easy counting argument shows that the parameters of any s.r.g satisfy $k(k-\lambda-1) = \mu(v-k-1)$. Since, clearly, any connected s.r.g must have $\mu \geq 1$, this relation implies $v \leq k^2+1$ for any connected s.r.g (with equality only for $\lambda = 0, \mu = 1$). Thus there are only finitely many connected s.r.g's of any given degree. The s.r.g's with $\lambda = 0, \mu = 1$ (and hence $v = k^2+1$) are collectively known as the *Moore graphs.* These are precisely the finite graphs of diameter 2 and girth 5 (see[5, section 5.8]).

The complement $\bar{H}$ of an s.r.g H is again an s.r.g. An s.r.g H is said to be *primitive* if both H and $\bar{H}$ are connected. The only imprimitive s.r.g's are mK_n (the disjoint union of m copies of the n-vertex complete graph K_n) and the complement $K_{m,n}$ of mK_n (thus, $K_{m,n}$ is the complete multipartite graph with m parts of size n each). From this elementary observation, it is easy to see that the only connected but imprimitive strongly regular unit distance graphs are $K_n, n \leq 3$, and $K_{2,2} = C_4$, the 4-cycle.

Among the simplest examples of primitive s.r.g's are the *triangular graphs* $T_n, n \geq 5$, and the graphs $K_n \times K_n$, $n \geq 3$. By definition, T_n is the line graph of K_n and $K_n \times K_n$ is the line graph of the complete bipartite graph $K_{2,n}$. The parameters of T_n are $(\frac{n(n-1)}{2}, 2n-4, n-2, 4)$ and the parameters of $K_n \times K_n$ are $(n^2, 2n-2, n-2, 2)$. It is well known that for $n \neq 8$, T_n is uniquely determined (among s.r.g's) by its parameters, while for $n \neq 4$, $K_n \times K_n$ is uniquely determined by its parameters (see [2], for instance).

It is well known (see [3]) that a Moore graph of degree k (i.e. an s.r.g with parameters $(k^2+1, k, 0, 1)$) can exist only for $k = 2$, 3, 7 or 57. The *pentagon* C_5 is the unique Moore graph of degree 2. The unique Moore graph of degree 3 is called the *Petersen graph*; it may be described as the complement $\bar{T}_5$ of the triangular graph T_5. The unique Moore graph of degree 7 is known as the Hoffman-Singleton graph. The following simple description of this graph is due to N.Robertson. Its 50 vertices are named x_{ij} and y_{ij}, where the indices i, j vary over $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$. For $i, j, k, l \in \mathbb{Z}_5$, x_{ij} and x_{kl} are adjacent if and only if $i = k$ and $l - j = \pm 1$; y_{ij} and y_{kl} are adjacent if and only if $i = k$ and $l - j = \pm 2$; x_{ij} and y_{kl} are adjacent if and only if $l + j = ik$. Here all arithmetic operations on the indices are modulo 5. From this description, one sees a partition of vertex set of Hoffman-Singleton into ten pentagons which fall into two classes of size five each. The induced subgraph on the union of any two pentagons from the same class is $2C_5$ (disjoint union of two C_5 with no in-between edges), while the induced subgraph on the union of two pentagons from different classes is a copy of the Petersen graph. Thus, Hoffman-Singleton graph contains many (actually, exactly 525) copies of the Petersen graph. The existence of a Moore graph of degree 57 is one of the outstanding open problems in finite geometry.

Recall that if H_1 and H_2 are two graphs, with vertex sets V_1 and V_2 respectively, then their cartesian product $H_1 \times H_2$ is defined to be the graph, with vertex set $V_1 \times V_2$, where $(x_1, y_1), (x_2, y_2) \in V_1 \times V_2$ are adjacent if and only if either $x_1 = x_2$ and y_1, y_2 are adjacent in H_2 or else $y_1 = y_2$ and x_1, x_2 are adjacent in H_1. The following result is due to Horvat and Pisanski [7]. We include a simpler proof for the sake of completeness.

Theorem 1 *If H_1 and H_2 are any two unit distance graphs, then so is their cartesian product $H_1 \times H_2$.*

Proof: Let the points $x_1, x_2, \ldots, x_m$ (respectively $y_1, y_2, \ldots, y_n$) represent the vertices of H_1 (respectively H_2) in a unit distance representation. Suitably rotating the point set $\{y_1, y_2, \ldots, y_n\}$ (while holding $\{x_1, x_2, \ldots, x_m\}$ fixed) we may assume that the two sets $\{y_i - y_j : 1 \leq i \neq j \leq n\}$ and $\{x_i - x_j : 1 \leq i \neq j \leq m\}$ are disjoint. Then $\{x_i + y_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a set of mn distinct points in $\mathbb{R}^2$ providing a unit distance representation of $H_1 \times H_2$. ▮

Remark 1: It is easy to see that for any two graphs H_1 and H_2, we have $\chi(H_1 \times H_2) = max\{\chi(H_1), \chi(H_2)\}$. Indeed, both the graphs H_i occur as subgraphs of $H_1 \times H_2$, so that $\chi(H_1 \times H_2) \geq \chi(H_i), i = 1, 2$. Also, if $n = max\{\chi(H_1), \chi(H_2)\}$, and $f_i : V_i = V(H_i) \to \mathbb{Z}_n$ is a proper vertex colouring of H_i in n colours, then $f : V_1 \times V_2 \to \mathbb{Z}_n$ defined by $f(x, y) = f_1(x) + f_2(y) \pmod{n}$ is a proper vertex colouring of $H_1 \times H_2$ in n colours. Thus, $\chi(H_1 \times H_2) = max\{\chi(H_1), \chi(H_2)\}$. This shows that, not surprisingly, Theorem 1 does not lead to constructions of unit distance graphs with large chromatic numbers.

The *n-dimensional hypercube* Q_n is defined to be the cartesian product $K_2 \times \ldots \times K_2$ of n copies of K_2. Thus Q_n is a regular graph of degree n on 2^n vertices. Since K_2 is trivially a unit distance graph, Theorem 1 implies:

Corollary 1 *For each $n \geq 1$, the hypercube Q_n is a unit distance graph.*

Our main result is an almost complete classification of the (primitive) strongly regular unit distance graphs (except that we are unable to decide if the Hoffman-Singleton graph has a unit distance representation):

Theorem 2 *The only primitive strongly regular unit distance graphs are the pentagon, $K_3 \times K_3$, the Petersen graph, and possibly the Hoffman-Singleton graph.*

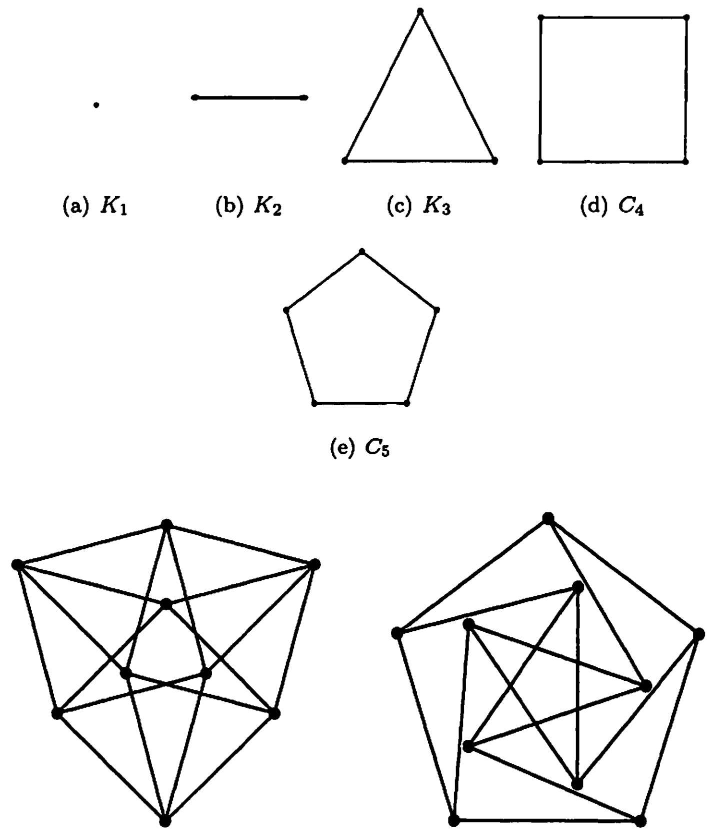
The regular pentagon (with unit side) is a unit distance representation of C_5. Since $C_4 = Q_2 = K_2 \times K_2$, Theorem 1 implies that these (as well as $K_3 \times K_3$) are unit distance s.r.g's.

Remark 2: (A unit distance representation of $K_3 \times K_3$) The best such representation may be obtained as follows. Begin with a unit square with sides AB, BC, CD, DA. Draw equilateral triangles ABE, BCF, CDG, DAH so that the points E, F are outside the square while the points G, H are inside the square. It turns out that the points E, F, G, H lie on a common unit circle. Let I be its center. Draw the edges EG, FH, IE, IF, IG, IH. This gives a unit distance representation of $K_3 \times K_3$ on the points $A, \ldots, I$. (The point I may be found as follows. Let O be the meeting point of the diagonals AC and BD. Let X and Y be the midpoints of EF and GH, respectively. Let Z be the midpoint of XY. Then I is the point such that Z is the midpoint of OI.)

Remark 3: (A unit distance representation of the Petersen Graph) For $1 \leq j \leq 5$, let $\omega_j = e^{\frac{2\pi i j}{5}}$ be the five fifth roots of unity and let $\omega'_j = i\omega_j$ be the other five tenth roots of unity. Here $i = \sqrt{-1}$. Also let τ be the golden ratio: $\tau = \frac{-1+\sqrt{5}}{2}$. Put $V = \{\frac{\omega_j}{\sqrt{2-\tau}} : 1 \leq j \leq 5\} \bigcup \{\frac{\omega'_j}{\sqrt{3+\tau}} : 1 \leq j \leq 5\}$. Then it is easy to see that V is the vertex set of Petersen graph in a unit distance representation of the latter. This representation was found by Erdös et. al. [4]. More generally, [8] shows that all generalized Petersen graphs are unit distance.

The usual picture of the Petersen graph (as seen in the front cover of the journal Discrete Mathematics, for instance) consists of a regular pentagon and a concentric regular pentagram, with the vertices of the pentagon joined to the corresponding vertices of the pentagram by five bridges. The construction given above shows that if both the pentagon and the pentagram are scaled to have unit sides, and then the pentagram is rotated around its center by a right angle, then a unit distance representation of Petersen results.

Thus all the graphs mentioned in Theorem 2 (except possibly Hoffman-Singleton) are indeed unit distance graphs. In Figure 2, we present unit distance representations of the connected unit distance s.r.g's. Notice that, with the trivial exception of the graphs K_n, $n \leq 3$, all these representations are flexible: they can be perturbed infinitesimally to get other (less beautiful) unit distance representations of these graphs.



(a) K_1 (b) K_2 (c) K_3 (d) C_4

(e) C_5

(f) $K_3 \times K_3$ (g) Petersen graph

Figure 2: Connected unit distance s.r.g's

Now we proceed to a proof of Theorem 2.

Lemma 1 *If H is a primitive unit distance s.r.g then its parameters λ and μ satisfy $(\lambda,\mu)=(0,1)$, $(0,2)$ or $(1,2)$.*

Proof: Note that, in the plane, at most two points are at distance 1 from any two given points. Thus we have $\lambda \leq 2, \mu \leq 2$. Also, primitivity implies $\mu \geq 1$. We first rule out the possibility $\lambda = 2$.

Suppose $\lambda = 2$. We identify the vertices of H with the points representing them (in a unit distance representation of H). We define two sequences $\{x_n : n = 1, 2, \ldots\}$ and $\{y_n : n = 1, 2, \ldots\}$ of vertices of H such that, for each $n \geq 1$, x_n and y_n are adjacent in H, and for each $n \geq 2$, y_n is adjacent to both x_{n-1} and y_{n-1}, while x_n is adjacent to x_{n-1}. The definition is inductive. Choose x_1 and y_1 to be any two adjacent vertices of H. Choose y_2 to be either of the two common neighbours of x_1 and y_1. Choose x_2 to be the common neighbour, other than y_1, of x_1 and y_2. For $n \geq 3$, choose y_n to be the common neighbour, other than x_{n-2}, of x_{n-1} and y_{n-1}; finally choose x_n to be the common neighbour, other than y_{n-1}, of x_{n-1} and y_n. Then it is easy to see that the euclidean distance between x_1 and x_n is $n-1$ for every n. Thus the vertices x_n are pairwise distinct. This is a contradiction since H is a finite graph.

Thus, $0 \leq \lambda \leq 1$ and $1 \leq \mu \leq 2$. To complete the proof, it suffices to note that there is no primitive s.r.g with $\lambda = \mu = 1$ (This is essentially Erdös' friendship theorem, and is immediate from the usual restrictions on the parameters of s.r.g's [3]). ∎

Recall that, given a connected graph H, the usual (or graphical) distance between two vertices x, y of H is the length of a shortest path in H joining x and y. The *graphical diameter* of a finite connected graph is the maximum of the pairwise distances between its vertices. Given a unit distance representation of a finite connected graph H, we define its *euclidean diameter* to be the maximum of the pairwise euclidean distances between the points representing the vertices of H. Clearly, the euclidean diameter of H is bounded above by its graphical diameter.

Lemma 2 *The euclidean diameter of any unit distance representation of the hypercube Q_3 is strictly bigger than 2.*

Proof: Fix a unit distance representation of Q_3 and identify the eight vertices of Q_3 with the eight points in $\mathbb{R}^2$ representing them. Choose a vertex z_0 and let z_1, z_2 and z_3 be its neighbours in Q_3. It is easy to see that there are two choices of z_0 for which z_0 is in the interior of the triangle $z_1z_2z_3$. (Indeed, the convex hull of all the vertices is a hexagon, with two vertices in its interior. The vertex z_0 may be chosen to be either of the two vertices in the interior of the hexagon.) Make such a choice of z_0. Let θ_1, θ_2 and θ_3 be the angles (in radian) subtended at z_0 by z_2 and z_3, by z_1 and z_3, and by z_1 and z_2, respectively. By our choice of z_0, we have $0 \leq \theta_i \leq \pi$ for $i = 1, 2, 3$ and $\theta_1 + \theta_2 + \theta_3 = 2\pi$. Also, we may arrange the notation so that $\theta_1 \leq \theta_2 \leq \theta_3$. Therefore, $\theta_3 \geq \frac{2\pi}{3}$. Since cosine is strictly decreasing on $[0, \pi]$, it follows that $cos\theta_1 \geq cos\theta_2$ and $cos\theta_3 \leq cos(\frac{2\pi}{3}) = -\frac{1}{2}$. Therefore we get $cos\theta_1 - cos\theta_2 - cos\theta_3 \geq -cos\theta_3 \geq \frac{1}{2}$.

Hence, if u is the common neighbour of z_2 and z_3 other than z_0, then the squared euclidean distance between u and z_1 is $3+2(cos\theta_1 - cos\theta_2 - cos\theta_3) \geq 4$. Thus the euclidean distance between u and z_1 is at least 2. If equality holds here, then, by the above argument, we must have $\theta_1 = \theta_2 = \theta_3 = \frac{2\pi}{3}$, so that two of the points representing the vertices of Q_3 coincide (viz. z_0 and its "antipode" in Q_3). Contradiction. ∎

Proof of Theorem 2: Let H be a primitive s.r.g with a given unit distance representation. By Lemma 1, $(\lambda, \mu)=(0,1)$, $(0,2)$ or $(1,2)$.

First suppose $(\lambda, \mu)=(0,1)$, i.e., H is a Moore graph. Thus H is the pentagon, the Petersen or the Hoffman-Singleton graph or a Moore graph of degree 57. We need to rule out the last case. Suppose H has parameters $(3250, 57, 0, 1)$. Then, by the Hoffman bound on the coclique number of a regular graph (see Proposition 3.7.2, page 92 in [2]), the coclique number of H is at most 400 and hence $\chi(H) \geq \frac{3250}{400} > 8$. This contradicts Proposition 1.

Now suppose $(\lambda, \mu)=(0,2)$ or $(1,2)$. Suppose the degree k of H satisfies $k > 2$ if $(\lambda, \mu)=(0,2)$ and $k > 4$ if $(\lambda, \mu)=(1,2)$. In the first case, the neighbours of any vertex are mutually non-adjacent. In the second case, H induces the disjoint union of $\frac{k}{2}$ copies of K_2 on the neighbours of any vertex. Because of our assumption on k, in either case, we can take a vertex z_0 and three mutually non-adjacent neighbours z_1, z_2 and z_3 of z_0. For $1 \leq i \neq j \leq 3$, let z_{ij} be the unique common neighbour, other than z_0, of z_i and z_j. Since K_4 is not a unit distance graph, the points z_1, z_2, z_3 can not be at mutual euclidean distance 1. Say, without loss of generality, that the distance between z_1 and z_2 is not equal to 1. Notice that if a, b, c, d are points in $\mathbb{R}^2$ such that the line segments ab, bc, cd and da have euclidean length 1, then we must have $a + c = b + d$ (the parallelogram law for vector addition), which determines d in terms of a, b, c. Using this observation repeatedly, we see that if we assume (without loss of generality) $z_0 = 0$ (the origin), then $z_{12} = z_1 + z_2$, $z_{13} = z_1 + z_3$, and $z_{23} = z_2 + z_3$. Since the distance between z_1 and z_2 is not 1, it follows that the distance between z_{13} and z_{23} is not 1. Therefore, z_{13} and z_{23} are non-adjacent in H. Hence there is a vertex $u \neq z_3$ which is adjacent to both z_{13} and z_{23} (as $\mu = 2$). By the above observation, we have $u = z_{13} + z_{23} - z_3 = z_1 + z_2 + z_3$. Thus, eight of the points representing the vertices of H are $0, z_1, z_2, z_3, z_1 + z_2, z_2 + z_3, z_1 + z_3, z_1 + z_2 + z_3$. Now, z_1, z_2 and z_3 are distinct points of norm 1, and $z_1 + z_2 \neq 0$, $z_2 + z_3 \neq 0$ and $z_3 + z_1 \neq 0$. Also, if 0 and u represented the same vertex of H, then the adjacent vertices 0 and z_{13} would have two common neighbours (namely, z_1 and z_3); this is not possible since $\lambda \leq 1$. Thus $z_1 + z_2 + z_3 \neq 0$. It follows that these eight points provide a unit distance representation of the hypercube Q_3 (even though Q_3 may not be a subgraph of H). Therefore, by Lemma 2, the euclidean diameter of this set of eight points is greater than 2. Therefore, the euclidean diameter of the given unit distance representation is greater than 2. This is a contradiction since the euclidean diameter of any unit distance representation of H is at most the graphical diameter of H, and the graphical diameter of any primitive s.r.g is clearly equal to 2. This contradiction proves that if $(\lambda, \mu)=(0, 2)$ then $k \leq 2$, and if $(\lambda, \mu)=(1, 2)$ then $k \leq 4$.

Now, to complete the proof, it suffices to note that there is no primitive s.r.g with $k \leq 2, \lambda = 0, \mu = 2$, and $K_3 \times K_3$ is the only primitive s.r.g with $k \leq 4, \lambda = 1, \mu = 2$. ∎

Remark 4: (Chromatic number of Hoffman-Singleton) The Hoffman-Singleton graph Ho-Si is the unique s.r.g with parameters $(50, 7, 0, 1)$. By the Hoffman bound, its coclique number is at most 15, and hence its chromatic number is $\geq \frac{50}{15} > 3$. We show by explicit construction that it has a proper vertex colouring in four colours, so that $\chi(Ho - Si) = 4$. By a well known construction of Ho-Si (see [3]), this graph has a coclique of size 15 (in fact, it has exactly hundred such cocliques, and the automorphism group of the graph acts transitively on them); fix such a coclique C_0. Then the induced subgraph H_0 of Ho-Si on the complement of C_0 has the following description. Take a set S of seven symbols. Then the vertex set V_0 of H_0 consists of the $\binom{7}{3} = 35$ subsets of size 3 of the set S. Two vertices $\alpha, \beta \in V_0$ are adjacent in H_0 if and only if $\alpha \bigcap \beta = \emptyset$. Fix two elements $x \neq y$ of S. Let $C_1 = \{\alpha \in V_0 : x \in \alpha\}$, $C_2 = \{\alpha \in V_0 : x \notin \alpha, y \in \alpha\}$ and $C_3 = \{\alpha \in V_0 : x \notin \alpha, y \notin \alpha\}$. Then C_i, $i = 1, 2, 3$, are three pairwise disjoint cocliques of H_0, covering the vertex set of H_0. Therefore, if we assign the "colour" i to the vertices of C_i $(i = 1, 2, 3)$, we get a proper vertex colouring of H_0 in three colours. Extend this to a proper vertex colouring of the Hoffman-Singleton graph in four colours by assigning the "colour" 0 to the vertices in C_0. This shows that $\chi(Ho - Si) = 4$.

It is easy to see that all the graphs in Figure 2 have chromatic number ≤ 3. Therefore, in view of the remark above, Theorem 2 implies:

Corollary 2 *All the strongly regular unit distance graphs have chromatic number* ≤ 4.

Thus, there is no counter example to the four colour conjecture among strongly regular graphs.

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