THE ZERO-DIVISOR GRAPH OF A MEET-SEMILATTICE

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ABSTRACT. In this paper, we introduce the zero-divisor graph $\Gamma(L)$ of a meet-semilattice L with 0. It is shown that $\Gamma(L)$ is connected with $diam\Gamma(L) \leq 3$ and if $\Gamma(L)$ contains a cycle, then the core K of $\Gamma(L)$ is a union of 3-cycles and 4-cycles.

KEY WORDS AND PHRASES: Zero-divisor graph, girth, diameter, star graph, path, atom, core, integral meet-semilattice.

1. Introduction

Beck [3] introduced the notion of coloring in a commutative ring R as follows. Let G be a simple graph whose vertices are the elements of R and two distinct vertices x and y are adjacent in G if $xy = 0$ in R. The graph G is known as the zero-divisor graph of R.

Anderson et al. [1] and Anderson and Livingston [2] studied graphs on commutative rings. Let R be a commutative ring with 1 and let $Z(R)$ be its set of all zero-divisors. They associated a (simple) graph $\Gamma(R)$ to R with vertex set $Z^*(R) = Z(R) - \{0\}$, the set of nonzero zero-divisors of R and distinct $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$ and called this graph as the zero-divisor graph of R. They gave relationship between ring-theoretic properties of R and graph-theoretic properties of $\Gamma(R)$. Later, Demeyer et al. [4] studied graphs on commutative semigroups with 0 in a similar manner.

Nimbhorkar et al. [6] introduced the zero-divisor graph for a meet-semilattice L with 0 and proved a form of Beck's conjecture. They associated the zero-divisor graph to L with 0, whose vertices are the elements of L and two distinct elements $x, y \in L$ are adjacent if and only if $x \wedge y = 0$. They correlated properties of semilattices with coloring of the associated graph. Also Nimbhorkar et al. [7] studied graphs on lattices with 0 by defining adjacency in a similar way. They have shown that when the lattice is atomic or distributive then the chromatic number of the associated graph is equal to the clique number of the graph.

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In this paper, we define a graph on a meet-semilattice L with 0. An element $a \in L$ is called a *zero-divisor* if there exists a nonzero $b \in L$ such that $a \wedge b = 0$. We denote by $Z(L)$ the set of all zero-divisors of L. We associate a graph $\Gamma(L)$ to L with vertex set $Z^*(L) = Z(L) - \{0\}$, the set of nonzero zero-divisors of L and distinct $x, y \in Z^*(L)$ are adjacent if and only if $x \wedge y = 0$ and call this graph as the *zero-divisor graph* of L. In a meet-semilattice L with 0, a nonzero element $a \in L$ is called an *atom* if there is no $x \in L$ such that $0 < x < a$.

In section 2, we show that $\Gamma(L)$ is connected with $diam\Gamma(L) \leq 3$. We show that, if $\Gamma(L)$ contains a cycle, then the core K of $\Gamma(L)$ is a union of 3-cycles and 4-cycles. Moreover, any vertex in $\Gamma(L)$ is either a vertex of the core K of $\Gamma(L)$ or else is a pendant vertex of $\Gamma(L)$. In section 3, we study graphs of product of meet-semilattices and obtain some properties of such graphs. The undefined terms are from West [8] and Grätzer [5].

A graph Γ is called a *star graph* if it has a vertex adjacent to every other vertex and these are the only adjacency relations. A star graph with n vertices is denoted by $K_{1,n}$. A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if they are consecutive in the list. A path with n vertices is denoted by P_n. A graph Γ is called *connected* if there is a path between any two distinct vertices. A graph Γ is *complete* if any two vertices are adjacent. We denote the complete graph with n vertices by K_n. A vertex of a graph Γ is called a *pendant vertex* if its degree is 1. For distinct vertices x and y of a graph Γ, let $d(x, y)$ be the length of the shortest path from x to y; ($d(x, y) = \infty$ if there is no such path). The diameter of Γ is $diam\Gamma = sup\{d(x, y) \mid x$ and y are distinct vertices of $\Gamma\}$. The girth of Γ, denoted by $gr(\Gamma)$, is defined as the length of the shortest cycle in Γ. ($gr(\Gamma) = \infty$ if Γ contains no cycles). The core of Γ is defined as the union of cycles in Γ.

2. Properties of $\Gamma(L)$

In this section, we show that $\Gamma(L)$ is always connected and has small diameter and girth.

We begin this section with some examples.

The zero-divisor graph of a finite meet-semilattice with only one atom is the empty graph. The zero-divisor graph of the meet-semilattice in Figure 1 is the empty graph.

However, this does not hold for infinite meet-semilattices with one atom. For, consider the infinite meet-semilattice given in Figure 2, where the descending dots represent infinite descending chain. It has only one atom c but its graph $\Gamma(L)$ is an infinite star graph.

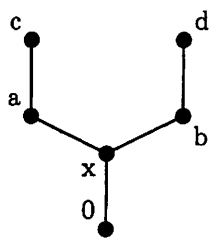

Figure 1

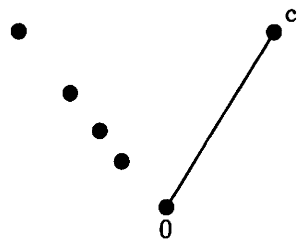

Figure 2

The following example shows that nonisomorphic meet-semilattices may have the same zero-divisor graph.

Example 1. The meet-semilattices in Figures 3 and 4 are not isomorphic but their zero-divisor graph is the path P_2.

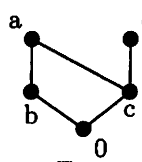

Figure 3

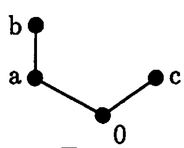

Figure 4

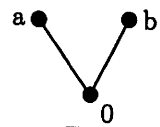

Figure 5

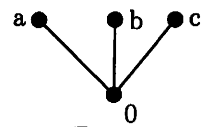

Figure 6

Example 2. All connected graphs with less than four vertices may be realized as $\Gamma(L)$ for some meet-semilattice L with 0.

There is only one connected graph K_2 with two vertices and one can see that $K_2 = \Gamma(L)$, where L is the meet-semilattice given in Figure 5.

Note that P_3 and K_3 are the only connected graphs on three vertices. Both are realizable as the zero-divisor graph of a meet-semilattice L with 0. The corresponding meet-semilattices are given in Figures 4 and 6 respectively.

There are eleven graphs with four vertices of which only six are connected. Of these six, the five graphs shown in Figure 7 to Figure 11 can be realized as $\Gamma(L)$. The corresponding meet-semilattices of these graphs are shown in Figure 7a to Figure 11a respectively.

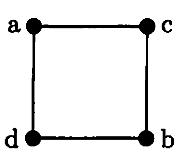

Figure 7

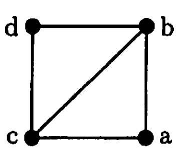

Figure 8

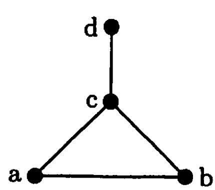

Figure 9

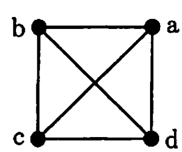

Figure 10

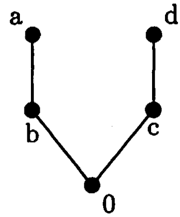

Figure 7a

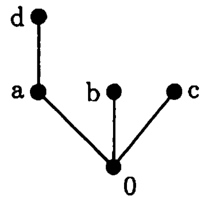

Figure 8a

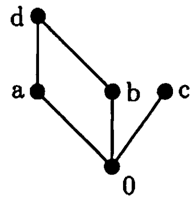

Figure 9a

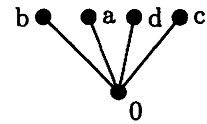

Figure 10a

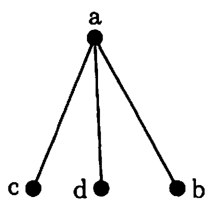

Figure 11

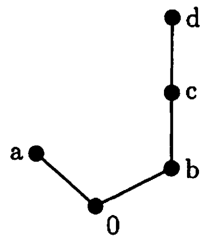

Figure 11a

We next observe that the graph P_4 cannot be realized as $\Gamma(L)$ for any meet-semilattice L with 0. Let $V(P_4) = \{a, b, c, d\}$ with $a \wedge b = b \wedge c = c \wedge d = 0$ and no other meet is zero. If $a \wedge d = x$ for some x then $x \leq a$, $x \leq d$ hence $x \wedge b = 0$, $x \wedge c = 0$ that is x is a common neighbour of b and c, a contradiction. Hence P_4 cannot be realized as $\Gamma(L)$.

Remark 2.1. We have seen above that $\Gamma(L)$ can be a 3-cycle or a 4-cycle. But, $\Gamma(L)$ cannot be an n-cycle for any $n \geq 5$. For, consider an n-cycle G given by $a_1 - a_2 - a_3 \cdots a_n - a_1$ with $n \geq 5$. Suppose that $G = \Gamma(L)$ for some meet-semilattice L with 0. Let $a_2 \wedge a_4 = x$ then $x \leq a_2$, $x \leq a_4$ gives $x \wedge a_3 = 0$, $x \wedge a_5 = 0$ that is x is a common neighbour of a_3 and a_5. We note that if $x = a_4$, then $a_4 \leq a_2$ and hence $a_4 \wedge a_1 = 0$, a contradiction. This shows that $a_2 \wedge a_4$ does not exist.

Let L be a meet-semilattice with 0. Then $\Gamma(L)$ is complete if and only if $x \wedge y = 0$ for all $x, y \in Z^*(L)$. For the meet-semilattice L $= M_n = \{0, a_1, \cdots, a_n\}$, where $a_i \wedge a_j = 0$, for all $i \neq j$, $\Gamma(L)$ is the complete graph K_n.

Theorem 2.1. *Let L be a meet-semilattice with 0, then $\Gamma(L)$ is connected and $diam(\Gamma(L)) \leq 3$. Moreover, if $\Gamma(L)$ contains a cycle, then $gr\Gamma(L) \leq 4$.*

Proof. Let $x, y \in Z^*(L)$ be distinct. If $x \wedge y = 0$, then $d(x, y) = 1$. Suppose that $x \wedge y \neq 0$, then there are $a, b \in Z^*(L) - \{x, y\}$ with $a \wedge x = b \wedge y = 0$.

If $a = b$ then $x - a - y$ is a path of length 2; thus $d(x,y) = 2$. Thus we may assume that $a \neq b$. If $a \wedge b = 0$, then $x - a - b - y$ is a path of length 3 and hence $d(x,y) \leq 3$. If $a \wedge b \neq 0$, then $x - a \wedge b - y$ is a path of length 2; thus $d(x,y) = 2$. Hence $d(x,y) \leq 3$ and thus $diam(\Gamma(L)) \leq 3$.

As seen above, there exists a path between any two distinct elements in $Z^*(L)$ and so $\Gamma(L)$ is connected.

Now, suppose that $\Gamma(L)$ contains a cycle. If $gr\Gamma(L) \geq 5$, then $\Gamma(L)$ contains an n-cycle say $a_1 - a_2 - a_3 \cdots a_n - a_1$ with $n \geq 5$.

Let $a_2 \wedge a_4 = x$, $a_3 \wedge a_5 = y$, $a_5 \wedge a_2 = z$ in L. Then $x \wedge y = 0$, $y \wedge z = 0$, $z \wedge x = 0$, a contradiction to the assumption that $gr\Gamma(L) \geq 5$. Hence $gr\Gamma(L) \leq 4$. □

Remark 2.2. There exists a lattice L such that $gr\Gamma(L) = 4$, see Figure 7a and Figure 7.

Theorem 2.2. *If $a - x - b$ is a path in $\Gamma(L)$, then either x is an atom in L or $a - x - b$ is contained in a cycle of length ≤ 4.*

Proof. Suppose $a - x - b$ is a path in $\Gamma(L)$ and x is not an atom in L then there is a nonzero $c < x$. Then $a \wedge c = b \wedge c = 0$. Hence $a - x - b - c - a$ is a cycle of length equal to 4. □

Theorem 2.3. *If L does not contain any atom, then any edge in $\Gamma(L)$ is contained in a cycle of length ≤ 4, and therefore $\Gamma(L)$ is a union of 3-cycles and 4-cycles.*

Proof. Let $a - x$ be an edge in $\Gamma(L)$. Since $\Gamma(L)$ is connected and $|\Gamma(L)| \geq 3$, there exists a vertex b in $\Gamma(L)$ with $a - x - b$ or $x - a - b$ is a path in $\Gamma(L)$. In the first case, if $b \wedge a = 0$ then $a - x - b - a$ is a 3-cycle. If $b \wedge a \neq 0$, since x is not an atom then there exists a $c < x$. Then $a \wedge c = 0$, $b \wedge c = 0$. Hence $a - x - b - c - a$ is a cycle of length 4. Thus x is contained in a cycle of length ≤ 4, so $a - x$ is an edge of either a 3-cycle or a 4-cycle. In the second case, if $x \wedge b = 0$ then $x - a - b - x$ is a 3-cycle. If $x \wedge b \neq 0$, since a is not an atom then there exists a $d < a$. Then $d \wedge x = 0$, $d \wedge b = 0$. Hence $d - x - a - b - d$ is a cycle of length 4. Thus a is contained in a cycle of length ≤ 4, so $a - x$ is an edge of a 4-cycle. Hence $a - x$ is an edge of a 3-cycle or a 4-cycle. □

Theorem 2.4. *Let L be a meet-semilattice with 0. If $\Gamma(L)$ contains a cycle, then the core K of $\Gamma(L)$ is a union of 3-cycles and 4-cycles and any vertex in $\Gamma(L)$ is either a vertex of the core K of $\Gamma(L)$ or is an end of $\Gamma(L)$.*

Proof. Let $a_1 \in K$ and suppose that a_1 does not belong to any 3-cycle or a 4-cycle in $\Gamma(L)$. Then a_1 is in some n-cycle $a_1 - a_2 - a_3 \cdots a_n - a_1$ with $n \geq 5$. By Theorem 2.2, a_1 is an atom in L. Then $a_1 \leq a_4$ and implies that $a_1 \wedge a_3 = 0$, a contradiction. Hence a_1 is in a 3-cycle or a 4-cycle.

Now suppose that a is any vertex in $\Gamma(L)$. If $a \notin K$ and a is not a pendant vertex then the following possibilities hold. (i) a is contained in a path of the form $x - y - a - b$ with $b \in K$ or (ii) a is contained in a path of the form $x - a - b$ with $b \in K$.

Since $b \in K$, b is contained in a 3-cycle or a 4-cycle, say $b - c - d - b$ or $b - c - d - e - b$.

In (i), we get $d(x, c) = 4$, contradicts $diam(\Gamma(L)) \leq 3$. Hence (i) cannot hold.

In (ii), we get $x - a - b - c - d - b$ or $x - a - b - c - d - e - b$. Hence by Theorem 2.2, a must be an atom. Therefore, $a \wedge c = a$. This gives $a \wedge d = 0$, a contradiction as $a \notin K$. Thus (ii) cannot hold. □

Theorem 2.5. *Let L be a meet-semilattice with 0 and $a \in Z^*(L)$ be a pendant vertex in $\Gamma(L)$. Let x be the vertex adjacent to a. Then x is an atom in L.*

Proof. If $\Gamma(L)$ has only two vertices, then the result is trivial. Otherwise suppose $\Gamma(L)$ has more than two vertices. Since a is an end vertex and $\Gamma(L)$ is connected, there exists a vertex b in $\Gamma(L)$ such that $x - b$ is an edge in $\Gamma(L)$. Then $a - x - b$ is a path in $\Gamma(L)$ not contained in a cycle, and by Theorem 2.2, x is an atom in L. □

Theorem 2.6. *If L does not contain any atom then every pair of vertices in $\Gamma(L)$ is contained in a cycle of length ≤ 6.*

Proof. Let a, b be vertices of $\Gamma(L)$. If $a - b$ is an edge in $\Gamma(L)$, then by the Theorem 2.3, $a - b$ is an edge of a 3-cycle or a 4-cycle.

If $a-b$ is not an edge, then $d(a, b) = 2$ or $d(a, b) = 3$. Suppose $d(a, b) = 2$, then there is a path $a - x - b$ and since x is not an atom, by Theorem 2.2, $a-x-b$ is contained in a cycle of length ≤ 4. If $d(a, b) = 3$, then there exists a path $a - x - y - b$. Since x, y are not atoms, there exist nonzero $c, d \in L$ such that $c < x$ and $d < y$. Then $c \wedge a = c \wedge y = 0$ and $d \wedge b = d \wedge x = 0$. Thus we get two cycles $a - x - y - c - a$ and $b - y - x - d - b$. Thus there exists a cycle $a - x - d - b - y - c - a$ of length less than or equal to 6 containing the vertices a, b. □

3. Integral Meet-semilattices and Graphs of Product of Meet-semilattices

We say that a meet-semilattice L with 0 is an integral meet-semilattice if for $a, b \in L$, $a \wedge b = 0$ implies $a = 0$ or $b = 0$, for example, the meet-semilattice in Figure 1.

Theorem 3.1. *If L_1 and L_2 are integral meet-semilattices with 0 such that $|L_1| = m + 1$, $|L_2| = n + 1$ and $L \cong L_1 \times L_2$ then $\Gamma(L)$ is the complete bipartite graph $K_{m,n}$.*

Proof. If $L_1 = \{0, a_1, \cdots, a_m\}$ and $L_2 = \{0, b_1, \cdots, b_n\}$ then the pairs of the form $(a_i, 0)$ and $(0, b_j)$ are all adjacent.

Moreover, no pairs of the form $(a_i, 0)$, $(a_k, 0)$ are adjacent, since $a_i \wedge a_k \neq 0$ in L_1. Similarly, no pairs of the form $(0, b_i)$, $(0, b_j)$ are adjacent. The resulting graph is a complete bipartite graph with partitions $A = \{(a_1, 0), \cdots, (a_m, 0)\}$, and $B = \{(0, b_1), \cdots, (0, b_n)\}$. □

Theorem 3.2. *Let L_1 and L_2 be two meet-semilattices with* 0 *and* $L = L_1 \times L_2$. *Then* $gr(\Gamma(L)) = \infty$ *if and only if either*

(1) $|\Gamma(L)| \leq 2$ *or*
(2) $|\Gamma(L)| = 3$ *and* $\Gamma(L)$ *is not complete or*
(3) $L \cong C_2 \times L_2$, *where L_2 is an integral meet-semilattice and C_2 is the two element chain. In this case, $\Gamma(L)$ is a star graph.*

Proof. Suppose $gr\Gamma(L) = \infty$, then either $|\Gamma(L)| \leq 2$ or $|\Gamma(L)| = 3$ and $\Gamma(L)$ is not complete. Suppose both these fail.

Case 1: Let L_1, L_2 be two meet-semilattices but one of these say L_1 is not an integral meet-semilattice. There exist nonzero elements $a, b \in L_1$ with $a \wedge b = 0$. Let $c \in L_2$, $c \neq 0$. Then $(a,0), (b,0), (0,c) \in L_1 \times L_2$ form a 3-cycle in $\Gamma(L)$, which is a contradiction to $gr\Gamma(L) = \infty$. Hence both L_1, L_2 must be integral meet-semilattices.

Case 2: Suppose that both L_1 and L_2 are integral meet-semilattices with $|L_1| > 2$, $|L_2| > 2$. Then choose nonzero $a, b \in L_1$ and $c, d \in L_2$. The elements $(a,0), (0,c), (b,0)$, $(0,d)$ form a $4-cycle$ in $\Gamma(L)$, a contradiction. Thus either L_1 or L_2 is C_2 and the other is an integral meet-semilattice.

Conversely, suppose either $|\Gamma(L)| \leq 2$ or $|\Gamma(L)| = 3$ and $\Gamma(L)$ is not complete or $L \cong L_1 \times L_2$ where $L_1 = C_2$ and L_2 is an integral meet-semilattice. We have to prove that $gr\Gamma(L) = \infty$.

If $|\Gamma(L)| = 2$ then the zero-divisor graph is P_2. Hence $gr\Gamma(L) = \infty$.

If $|\Gamma(L)| = 3$ and $\Gamma(L)$ is not complete then the zero-divisor graph is P_3. Hence $gr\Gamma(L) = \infty$.

Suppose $L \cong L_1 \times L_2$ where $L_1 = C_2$ and L_2 is an integral meet-semilattice. Let $(1,0), (0,a_1), \cdots, (0,a_n)$ form a cycle in $\Gamma(L)$. Then $(0,a_1), (0,a_2)$ are adjacent. Hence $a_1 \wedge a_2 = 0$ for nonzero $a_1, a_2 \in L_2$, a contradiction since L_2 is an integral meet-semilattice. Hence $gr\Gamma(L) = \infty$. □

Corollary 3.1. *Let L_1 and L_2 be two meet-semilattices with* 0 *and* $L = L_1 \times L_2$. *Then $\Gamma(L)$ does not contain a 3-cycle if and only if L_1, L_2 are integral meet-semilattices.*

Remark 3.1. For any star graph with n elements there corresponds a meet-semilattice with 0.

Theorem 3.3. *Let L_1 and L_2 be two meet-semilattices with* 0 *and* $L = L_1 \times L_2$. *Then exactly one of the following holds:*

(1) $\Gamma(L)$ *has a cycle of length* 3 *or* 4 *(that is* $gr\Gamma(L) \leq 4$*),*
(2) $\Gamma(L)$ *is a star graph.*

Proof. Case (1): Suppose $L \cong L_1 \times L_2$, where at least one of L_1 and L_2 is not an integral meet-semilattice, say L_1 is not an integral meet-semilattice. Then there exist nonzero $a, b \in L_1$, with $a \wedge b = 0$ and choose nonzero $c \in L_2$. Then $(a, 0), (b, 0), (0, c)$ form a cycle of length 3 in $\Gamma(L)$.

Case (2): Let $L \cong L_1 \times L_2$ where L_1, L_2 both are integral meet-semilattices with $|L_1| > 2$, $|L_2| > 2$. Let $a, b \in L_1$, and $c, d \in L_2$ be nonzero elements. Then $(a, 0), (0, c), (b, 0), (0, d)$ form a cycle of length 4 in $\Gamma(L)$.

Case (3): Let $L \cong L_1 \times L_2$, where either $|L_1| = 2$ or $|L_2| = 2$. Let $|L_1| = 2$ and L_2 be an integral meet-semilattice then by Theorem 3.2, $\Gamma(L)$ is a star graph. □

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