On the Roman domination numbers of generalized Petersen graphs

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Abstract For natural numbers n and k, where $n > 2k$, a generalized Petersen graph $P(n,k)$ is obtained by letting its vertex set be $\{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ and its edge set be the union of $u_iu_{i+1}, u_iv_i, v_iv_{i+k}$ over $1 \leq i \leq n$, where subscripts are reduced modulo n. In this paper, an integer programming formulation for Roman domination is established, which is used to give upper bounds for the Roman domination numbers of the generalized Petersen graphs $P(n,3)$ and $P(n,4)$. Together with the dynamic algorithm, we determine the Roman domination number of the generalized Petersen graph $P(n,3)$ for $n \geq 5$.

1 Introduction

In this paper, only directed and undirected graphs without multiple edges or loops are considered. For a graph $G = (V,E)$, $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively. For a function $f : V \to \{0,1,2\}$ and a vertex $u \in V$, we say u is *Roman dominated* by f if $f(u) \neq 0$ or u is adjacent to at least one vertex v for which $f(v) = 2$. A *Roman dominating function* on a graph $G = (V,E)$ is a function $f : V \to \{0,1,2\}$ satisfying the condition that every vertex u is *Roman dominated* by f. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* $\gamma_R(G)$ of G.

For natural numbers n and k, where $n > 2k$, a generalized Petersen graph $P(n,k)$ is obtained by letting its vertex set be $\{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ and its edge set be the union of $u_iu_{i+1}, u_iv_i, v_iv_{i+k}$ over $1 \leq i \leq n$, where subscripts are reduced modulo n (see [1, 2]). If $n = 5, k =$

2, the graph $P(5,2)$, shown in Fig. 1, is the Petersen graph which serves as a counterexample of many conjectures.

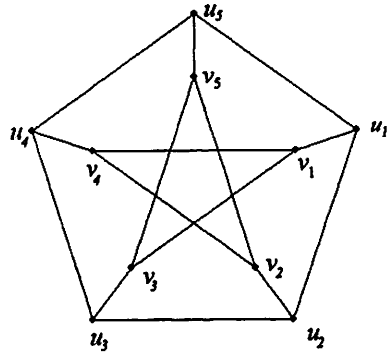

Fig. 1: The Petersen graph $P(5,2)$

In the literature, dominations such as the standard domination, Roman domination and rainbow domination of the generalized Petersen graphs have attracted lots of interests [3, 4, 5, 6, 7]. In particular, Wang et al. [7] obtain the following result of the Roman domination numbers of generalized Petersen graphs.

Theorem 1 [7] *Let $n \geq 5$. Then*

$$\gamma_R(P(n,2)) = \left\lceil \frac{8n}{7} \right\rceil.$$

In this paper, an integer programming formulation for Roman domination is established. By solving the integer programming formulation for Roman domination, we established the upper bound for the Roman domination numbers of the generalized Petersen graphs $P(n,3)$ and $P(n,4)$. Together with the dynamic algorithm, we determine the Roman domination number of the generalized Petersen graph $P(n,3)$ for $n \geq 5$.

2 Constructing Roman dominating function of the generalized Petersen graphs by dynamic algorithm

We first define the graph H_n which will be used in the sequel. For natural numbers $n \geq 5$, the graph H_n is obtained by letting its vertex set be

$\{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ and its edge set be the union of E_1, E_2 and E_3, where $E_1 = \{u_i u_{i+1} | 1 \leq i \leq n-1\}$, $E_2 = \{u_i v_i | 1 \leq i \leq n\}$, $E_3 = \{v_i v_{i+2} | 1 \leq i \leq n-2\}$. Then the graph H_n is the subgraph of $P(m, 2)$ for $m \geq n \geq 5$. The right side of Fig. 2 shows the graph H_5.

The dynamic algorithm was proposed to solve these invariants for rotagraphs and fasciagraphs (see [8]) in a general framework and later used several times [9]. In this paper, we use the dynamic algorithm on the generalized Petersen graphs. We here focus on the generalized Petersen graphs $P(n, 2)$. For fixed k, the graphs $P(n, k)$ can be processed similarly.

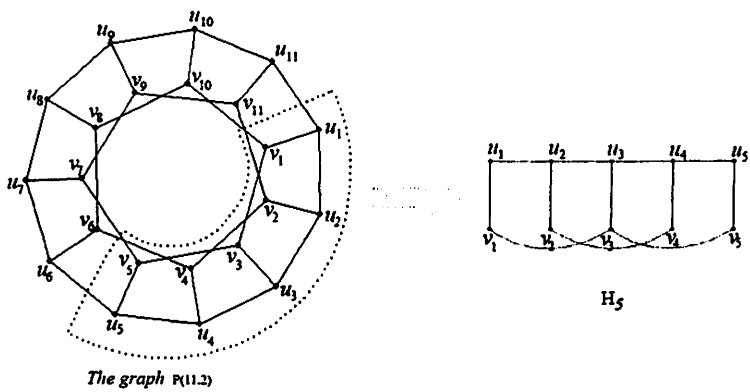

Fig. 2: The subgraph H_5 in the Petersen graphs $P(11, 2)$

We consider functions $f : V(H_5) \to \{0, 1, 2\}$ on the vertex set of the graph H_5. We say f is *locally feasible* if both u_3 and v_3 are *Roman dominated by* f. We define the local weight $lw(f)$ of f as the sum of $f(u_3)$ and $f(v_3)$.

We construct a weighted auxiliary digraph W in the following way: the vertices of W are all possible locally feasible Roman dominating functions of H_5. For convenience we denote the vertices of W in the following way: for a locally feasible function f of H_5, let $s = s_1 s_2 s_3 s_4 s_5 \in V(W)$, where $s_i = (f(u_i), f(v_i))$. There is an arc between two vertices s and t (which represent locally feasible functions of H_5) if and only if $s_i = t_{i-1}$ for all $i = 2, 3, 4, 5$ where $s = s_1 s_2 s_3 s_4 s_5$ and $t = t_1 t_2 t_3 t_4 t_5$. There is an example of two locally feasible functions that are connected by an arc in W in Fig. 3.

Moreover we define the weight of an arc st as the sum of $f(u_3)$ and $f(v_3)$, where f is the locally feasible function represented in W with s.

Fig. 3 shows how two locally feasible functions s and t of H_5, such that $st \in E(W)$, can be used to form a function $h : V(H_6) \to \{0,1,2\}$. In this example, $s_1 = (0,0), s_2 = (1,0), s_3 = (0,1), s_4 = (2,1), s_5 = (0,0)$ and $t_1 = (1,0), t_2 = (0,1), t_3 = (2,1), t_4 = (0,0), t_5 = (0,1)$. Moreover, s is *locally feasible* because u_3 and v_3 are *Roman dominated* by s, and t is locally feasible because also there u_3 and v_3 are Roman dominated by t.

By "gluing" consecutive locally feasible functions of H_5, we can obtain functions on H_n with all its inner vertices Roman dominated. We call such a function to be a partial Roman dominating function. As we can see, a partial Roman dominating function on H_6 can be obtained by a path of two vertices of W. Similarly, a partial Roman dominating function on H_n can be obtained by a path of $n-4$ vertices of W. When considering a cycle instead of a path in W, we therefore construct a Roman dominating function of $P(n,2)$ and we have the following result:

Theorem 2 *Let $n \geq 5$. A Roman dominating function f on $P(n,2)$ corresponds to a closed directed walk of length n in W. The minimum weight over all closed directed walks of length n is the Roman domination number.*

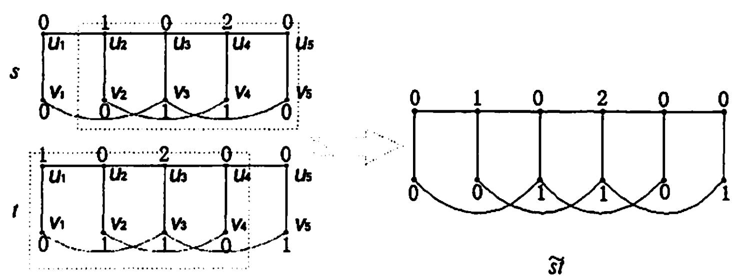

Fig. 3: An example of how an edge is formed

In order to obtain the Roman domination number of $P(n,2)$, we need to apply Theorem 2. The algorithms with path algebra approach, introduced in [10] and used in [11, 12], are implemented and carried out. We rediscovered Theorem 1 from the result of computation. For more details, we refer the reader to the reference [11] and here we omit it.

3 Integer linear programming formulation for Roman domination number

Since the Roman domination number of $P(n,2)$ is known, we here present values of the Roman domination number of $P(n,3)$ which is more complicated. By using dynamic algorithm, we obtained the following result:

Theorem 3 *Let $n \geq 5$. Then*

$$\gamma_R(P(n,3)) = \begin{cases} n, & n \equiv 0 \ (mod\ 4) \\ n+1, & n \equiv 1,3 \ (mod\ 4) \\ n+2, & n \equiv 2 \ (mod\ 4) \end{cases}$$

Although Theorem 3 can be proved by using the dynamic algorithm, it is not easy to explicitly construct a minimum Roman dominating function. We usually call minimum Roman dominating function a γ_R-function. In this section, we give an integer programming formulation for Roman domination problem of a graph which is used to construct a minimum Roman dominating function. Let $G = (V, E)$, the vertices of G be labeled with $\{1, 2, \cdots, n\}$, and let f be a Roman dominating function of G. For a vertex $i \in V$ and an integer $j \in \{0, 1, 2\}$, let $x_{i,j}$ equal 1 if i is labeled with j (namely, $f(i) = j$), 0 otherwise.

We use the following integer programming formulation for Roman domination problem of a graph G:

LP Roman Domination:

$$\text{minimize} \sum_{1 \leq i \leq n, j \in \{1,2\}} j x_{i,j} \tag{1}$$

subject to:

$$x_{i,0} + x_{i,1} + x_{i,2} = 1, \qquad i \in V; \tag{2}$$

$$\sum_{\ell \in N(i)} x_{\ell,2} \geq x_{i,0}, \qquad i \in V; \tag{3}$$

$$x_{i,j} \in \{0,1\}, \qquad 1 \leq i \leq n, j \in \{0,1,2\}. \tag{4}$$

Constrains (2) guarantees each vertex is labeled with only one number. Constrains (3) guarantees that every vertex i for which $f(i) = 0$ is adjacent to at least one vertex ℓ for which $f(\ell) = 2$. Binary requirements on the $x_{i,j}$ variables are given by (4).

The lower bounds are detected by searching closed walks of given lengths in the auxiliary graph W (applying Theorem 2), and the upper bounds are established by solving this integer programming formulation for Roman domination. We use two lines to denote a Roman dominating function, where in the first line there are the function values of vertices $\{u_1, u_2, \cdots, u_n\}$, and in the second line of the vertices of $\{v_1, v_2, \cdots, v_n\}$, such that u_i lies above v_i for all i. We take the result of Case 2 for example. Let $n = 5$, then the Roman dominating function is shown in Fig. 4 (a). Let $n = 9$, then the Roman dominating function is shown in Fig. 4 (b).

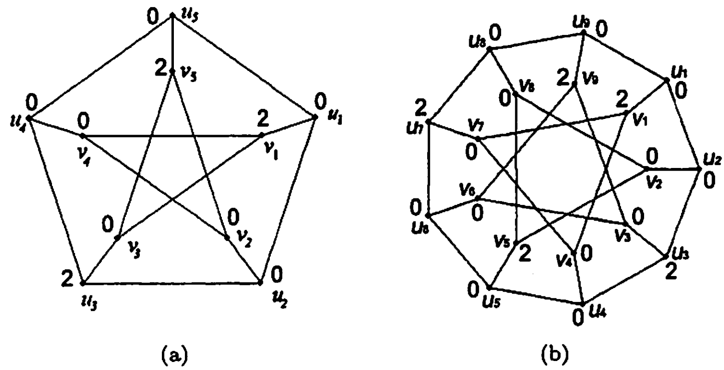

Fig. 4: (a) Roman dominating function of $P(5,3)$ (b) Roman dominating function of $P(9,3)$

We show the desired Roman dominating function for other cases by distinguished n into the following 4 cases:

Case 1. $n = 4k$ with $k \geq 2$:
By repeating the following pattern, we have $\gamma_R(P(n,3)) \leq n$.

0	0	2	0
2	0	0	0

Case 2. $n = 4k + 1$ with $k \geq 1$:
By repeating the leftmost 4 columns of the following pattern, we have $\gamma_R(P(n,3)) \leq n + 1$.

0	0	2	0	|	0
2	0	0	0	|	2

Case 3. $n = 4k + 2$ with $k \geq 1$:
By repeating the leftmost 4 columns of the following pattern, we have $\gamma_R(P(n,3)) \leq n + 2$.

0	0	2	0	|	0	2
2	0	0	0	|	2	0

Case 4. $n = 4k + 3$ with $k \geq 1$:
$\gamma_R(P(7,3)) \leq 8$.

```
0 2 0 0 2 0 0
1 0 0 0 0 1 2
```

By repeating the leftmost 4 columns of the following pattern, we have $\gamma_R(P(n,3)) \leq n + 1$ for $n \geq 11$.

```
0 0 2 0 | 0 0 2 0 0 2 0
2 0 0 0 | 2 1 0 0 0 0 1
```

By giving the constructions, we have the following result:

Theorem 4 *Let* $n \geq 13$. *Then*

$$\gamma_R(P(n,4)) \leq \begin{cases} \lceil \frac{14n}{13} \rceil + 1, & n \equiv 6, 8, 12 \text{ (mod 13)} \\ \lceil \frac{14n}{13} \rceil, & n \equiv 0, 1, 2, 3, 4, 5, 7, 9, 10, 11 \text{ (mod 13)} \end{cases}$$

Proof. We show the desired Roman dominating function by distinguished n into the following 13 cases with the exception of $n = 14$:
Case 1. $n = 13k$ with $k \geq 1$:
By repeating the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil$.

```
0 0 2 0 0 2 0 0 2 0 0 2 0
2 0 0 2 0 0 0 0 0 0 2 0 0
```

Case 2. $n = 13k + 1$ with $k \geq 2$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil$.

```
2 0 0 2 0 0 2 0 0 0 0 0 0 |2 0 0 2 0 1 0 2 0 0 0 0 0
0 0 0 0 0 0 0 0 2 2 2 2 0 |0 0 0 0 1 0 0 0 0 2 2 2 2 0
```

Case 3. $n = 13k + 2$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil$.

```
2 0 0 2 0 0 0 0 0 0 2 0 0 | 2 0
0 0 0 0 0 2 2 2 2 0 0 0 0 | 0 1
```

Case 4. $n = 13k + 3$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil$.

```
0 2 0 0 0 2 0 0 2 0 0 2 0 | 0 2 0
2 0 0 2 0 0 2 0 0 0 0 0 0 | 2 0 0
```

Case 5. $n = 13k + 4$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil$.

2 0 0 2 0 0 0 0 0 2 0 0 | 2 0 1 0
0 0 0 0 0 2 2 2 2 0 0 0 0 | 0 1 1 1

Case 6. $n = 13k + 5$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil$.

0 2 0 0 0 2 0 0 2 0 0 2 0 | 0 0 2 0 1
2 0 0 2 0 0 2 0 0 0 0 0 0 | 2 0 0 2 0

Case 7. $n = 13k + 6$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil + 1$.

0 0 2 0 0 2 0 0 2 0 0 2 0 | 0 0 2 0 2 0
2 0 0 2 0 0 0 0 0 0 2 0 0 | 2 0 0 2 0 0

Case 8. $n = 13k + 7$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil$.

2 0 0 2 0 0 2 0 0 0 2 0 0 | 2 0 0 2 0 0 0
0 0 0 0 0 2 0 0 2 0 0 2 0 | 0 0 0 0 1 2 1

Case 9. $n = 13k + 8$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil + 1$.

2 0 0 0 0 0 0 2 0 0 2 0 0 | 2 0 0 0 0 2 0 0
0 0 2 2 2 2 0 0 0 0 0 0 0 | 0 1 2 2 1 0 0 0

Case 10. $n = 13k + 9$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil$.

0 2 0 0 0 2 0 0 2 0 0 2 0 | 0 2 0 0 1 0 2 0 1
2 0 0 2 0 0 2 0 0 0 0 0 0 | 2 0 1 2 0 0 0 0 0

Case 11. $n = 13k + 10$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n,4)) \leq \lceil \frac{14n}{13} \rceil$.

0 2 0 0 2 0 0 2 0 0 0 2 0 | 0 2 0 0 2 0 0 0 2 0
0 0 0 0 0 0 2 0 0 2 0 0 2 | 0 0 0 0 0 0 2 1 0 2

Case 12. $n = 13k + 11$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n, 4)) \leq \lceil \frac{14n}{13} \rceil$.

2 0 0 2 0 0 0 0 0 0 2 0 0 | 2 0 0 2 0 0 0 0 2 0 0
0 0 0 0 0 2 2 2 2 0 0 0 0 | 0 0 0 0 1 2 2 1 0 0 0

Case 13. $n = 13k + 12$ with $k \geq 1$:
By repeating the leftmost 13 columns of the following pattern, we have $\gamma_R(P(n, 4)) \leq \lceil \frac{14n}{13} \rceil + 1$.

2 0 0 2 0 0 0 0 0 0 2 0 0 | 2 0 1 0 2 0 0 0 0 2 0 0
0 0 0 0 0 2 2 2 2 0 0 0 0 | 0 2 0 0 0 0 2 2 1 0 0 0

If $n = 14$, we use the following pattern, we have $\gamma_R(P(14, 4)) \leq 16$ which is the desired upper bound.

0 2 0 0 2 0 1 0 1 0 0 0 2 0
0 0 1 0 0 0 0 2 1 2 2 0 0 0

□

By solving the integer programming formulation for Roman domination problem, we also obtain some exact values for the Roman domination number of $P(n, 4)$ (see Table 1). From the results of Table 1, we guess the upper bounds in Theorem 4 match the exact values.

Table 1: Exact values of $\gamma_R(P(n, 4))$ for $10 \leq n \leq 51$

n	10	11	12	13	14	15	16	17	18	19	20	21
$\gamma_R(P(n, 4))$	11	12	14	14	16	17	18	19	20	22	22	24
n	22	23	24	25	26	27	28	29	30	31	32	33
$\gamma_R(P(n, 4))$	24	25	26	28	28	30	31	32	33	34	36	36
n	34	35	36	37	38	39	40	41	42	43	44	45
$\gamma_R(P(n, 4))$	38	38	39	40	42	42	44	45	46	47	48	50
n	46	47	48	49	50	51						
$\gamma_R(P(n, 4))$	50	52	52	53	54	56						

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