

# The number of spanning trees of composite graphs\*

Jianxi Li<sup>a</sup>

<sup>a</sup>Department of Mathematics & Information Science,  
Zhangzhou Normal University, Zhangzhou, Fujian, P.R. China  
ptjxli@hotmail.com(J. Li, Corresponding author)

Wai Chee Shiu<sup>b</sup>

<sup>b</sup>Department of Mathematics, Hong Kong Baptist University,  
Kowloon Tong, Hong Kong, P. R. China.  
wcshiu@hkbu.edu.hk(W.C. Shiu)

## Abstract

In this paper, some formulae for computing the numbers of spanning trees of the corona and the join of graphs are deduced.

**Key words:** Spanning trees, Laplacian spectrum, Matrix-Tree Theorem.

**AMS Subject Classifications:** 05C50.

## 1 Introduction

Let  $G$  be a simple connected graph with edge set  $E(G)$  and vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ . The number of spanning trees of  $G$ , denoted by  $t(G)$ , is the total number of distinct spanning subgraphs of  $G$  that are trees. Let  $A(G)$  and  $D(G)$  be the adjacency matrix and the diagonal matrix of vertex degrees of  $G$ , respectively. The Laplacian matrix of  $G$  is defined as  $L(G) = D(G) - A(G)$ , and the Laplacian characteristic polynomial  $\Phi(G, x)$  of  $G$  is defined as  $\Phi(G, x) = \det(xI - L(G))$ . It is easy to see that  $L(G)$

---

\*Partially supported by National Science Foundation (NSF) of China(No.11101358); NSF of Fujian(Nos.2011J05014, 2011J01026); Project of Fujian Education Department(Nos.JA11165, JA12208, JA12209); FRG, Hong Kong Baptist University.

is a symmetric positive semidefinite matrix having 0 as an eigenvalue. The Laplacian spectrum of  $G$  is

$$S(G) = (\mu_1(G), \mu_2(G), \dots, \mu_n(G)),$$

where  $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G) = 0$ , are the eigenvalues of  $L(G)$  (or the Laplacian eigenvalues of  $G$ ) arranged in non-increasing order. When one graph is under discussion, we may write  $\mu_i$  instead of  $\mu_i(G)$ . For a connected graph  $G$  of order  $n$ , it has been proven [1, p.284] that:

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i. \quad (1.1)$$

This formula can be used to obtain some sharp upper bounds for  $t(G)$  in terms of graph structural parameters such as the number of vertices, the number of edges, maximum vertex degree, minimum vertex degree, connectivity, chromatic number and matching number in [2]. In this paper, we mainly use this formula to compute the number of spanning trees of the corona and the join of graphs, respectively.

## 2 Preliminaries

Let  $G$  and  $H$  be two graphs. The corona  $G \circ H$  is obtained by taking one copy of  $G$  and  $|V(G)|$  copies of  $H$ , and by joining each vertex of the  $i$ th copy of  $H$  to the  $i$ th vertex of  $G$ ,  $i = 1, 2, \dots, |V(G)|$ . The vertex-disjoint union of the graphs  $G$  and  $H$  is denoted by  $G \cup H$ . The join  $G \vee H$  is obtained from  $G \cup H$  by adding all possible edges from vertices of  $G$  to vertices of  $H$ , i.e.,  $G \vee H = \overline{G} \cup \overline{H}$ , where  $\overline{G}$  is the complement of a graph  $G$ .

**Lemma 2.1** ([4]) *Let  $G$  and  $H$  be two graphs of order  $r$  and  $s$ , respectively. If  $S(G) = (\mu_1(G), \mu_2(G), \dots, \mu_r(G))$  and  $S(H) = (\mu_1(H), \mu_2(H), \dots, \mu_s(H))$ , then the Laplacian eigenvalues of  $G \circ H$  are:*

- (a)  $\frac{\mu_i(G) + s + 1 \pm \sqrt{(s+1)^2 - 4\mu_i(G)}}{2}$  with multiplicity 1 for  $i = 1, 2, \dots, r$ , and
- (b)  $\mu_j(H) + 1$  with multiplicity  $r$  for  $j = 1, 2, \dots, s - 1$ .

**Lemma 2.2** ([5]) *Let  $G$  and  $H$  be two connected graphs of order  $r$  and  $s$ , respectively. If*

*$S(H) = (\mu_1(H), \mu_2(H), \dots, \mu_s(H))$ , then the Laplacian polynomial of  $G \circ H$  can be expressed as follows*

$$\Phi(G \circ H, x) = \left( \prod_{i=1}^{s-1} (x-1 - \mu_i(H))^r \right) \begin{vmatrix} -L(G) & -(x-s-1)I_r \\ xI_r & (x-1)I_r \end{vmatrix}.$$

**Lemma 2.3** ([3]) *Let  $G$  and  $H$  be two graphs of order  $r$  and  $s$ , respectively. If  $S(G) = (\mu_1(G), \mu_2(G), \dots, \mu_r(G))$  and*

*$S(H) = (\mu_1(H), \mu_2(H), \dots, \mu_s(H))$ , then the Laplacian spectrum of  $G \vee H$  is  $S(G \vee H) = (r+s, \mu_1(G)+s, \mu_2(G)+s, \dots, \mu_{r-1}(G)+s, \mu_1(H)+r, \mu_2(H)+r, \dots, \mu_{s-1}(H)+r, 0)$ .*

### 3 Main results

Let  $G$  and  $H$  be two graphs of orders  $r$  and  $s$ , respectively. In this section, the number of spanning trees of  $G \circ H$  and  $G \vee H$  are computed, respectively.

#### 3.1 $G \circ H$

If  $G$  is connected, then  $G \circ H$  is also connected.

$\frac{\mu_r(G)+s+1-\sqrt{(s+1)^2-4\mu_r(G)}}{2} = 0$  since  $\mu_r(G) = 0$ . Combining with Lemma 2.1, we have

**Theorem 3.1** *Let  $G$  be a connected graph of order  $r$ , and  $H$  be a graph of order  $s$ . If*

*$S(G) = (\mu_1(G), \dots, \mu_{r-1}(G), 0)$  and  $S(H) = (\mu_1(H), \dots, \mu_{s-1}(H), 0)$ , then*

$$t(G \circ H) = \frac{\prod_{i=1}^{r-1} [\mu_i^2(G) + 2(s+3)\mu_i(G)] \prod_{i=1}^{s-1} (\mu_i(H) + 1)^r}{r 4^{r-1}}. \quad (3.2)$$

**Proof.** By Lemma 2.1 and Eq.(1.1), we have

$$\begin{aligned} t(G \circ H) &= \frac{(s+1) \prod_{i=1}^{r-1} \frac{(\mu_i(G)+s+1)^2 - (s+1)^2 + 4\mu_i(G)}{4} \prod_{i=1}^{s-1} (\mu_i(H) + 1)^r}{(s+1)r} \\ &= \frac{\prod_{i=1}^{r-1} [\mu_i^2(G) + 2(s+3)\mu_i(G)] \prod_{i=1}^{s-1} (\mu_i(H) + 1)^r}{r 4^{r-1}}. \end{aligned}$$

This completes the proof of (3.2).  $\square$

The expression (3.2) is somewhat complicated. In what follows, we will give a simpler expression for  $t(G \circ H)$  in terms of  $t(G)$  and  $t(H \vee K_1)$ .

**Theorem 3.2** *Let  $G$  be a connected graph of order  $r$ , and  $H$  be a graph of order  $s$  with*

$S(H) = (\mu_1(H), \dots, \mu_{s-1}(H), 0)$ . Then

$$t(G \circ H) = t(G) \prod_{i=1}^{s-1} (\mu_i(H) + 1)^r. \quad (3.3)$$

**Proof.** Let

$$f(x) := \begin{vmatrix} -L(G) & -(x-s-1)I_r \\ xI_r & (x-1)I_r \end{vmatrix} = \sum_{i=0}^{2r} a_i x^i.$$

Then, by Lemma 2.2, we have

$$\Phi(G \circ H, x) = f(x) \left( \prod_{i=1}^{s-1} (x-1-\mu_i(H))^r \right).$$

Since  $1 + \mu_i(H) > 0$  for  $1 \leq i \leq s-1$ ,  $f(x)$  can be written as

$$f(x) = x(x-b_1)(x-b_2) \cdots (x-b_{2r-1}),$$

where  $b_i > 0$  is a root of equation  $f(x) = 0$ ,  $1 \leq i \leq 2r-1$ .

Hence, (1.1) implies that

$$t(G \circ H) = \frac{\prod_{i=1}^{2r-1} b_i \prod_{i=1}^{s-1} (\mu_i(H) + 1)^r}{(s+1)r} = \frac{(-a_1) \prod_{i=1}^{s-1} (\mu_i(H) + 1)^r}{(s+1)r}.$$

In what follows, we will prove that  $a_1 = -r(s+1)t(G)$ .

For a matrix  $C$ , let  $C(i)$  and  $C(i, j)$  denote the submatrices obtained from  $C$  by deleting the  $i$ th row and column, and by deleting the  $i$ th row and  $j$ th column, respectively. Let  $C^*(i)$  and  $C^\dagger(i)$  denote the submatrices obtained from  $C$  by deleting the  $i$ th row and  $i$ th column, respectively. Let  $I_r$  and  $0_r$  be the identity matrix and the zero matrix of order  $r$ , respectively.

Since

$$\begin{aligned} f(x) &= \begin{vmatrix} -L(G) & -(x-s-1)I_r \\ xI_r & (x-1)I_r \end{vmatrix} \\ &= \left[ \begin{array}{cc} -L(G) & -(x-s-1)I_r \\ xI_r & (x-1)I_r \end{array} \right] \begin{bmatrix} I_r & 0_r \\ -I_r & I_r \end{bmatrix} \left[ \begin{array}{cc} I_r & I_r \\ 0_r & I_r \end{array} \right] \\ &= \begin{vmatrix} (x-s-1)I_r - L(G) & -L(G) \\ I_r & xI_r \end{vmatrix}. \end{aligned}$$

Let  $M := \begin{bmatrix} -(s+1)I_r - L(G) & -L(G) \\ I_r & 0_r \end{bmatrix}$ . Note that  $f(0) = \det M$ .

Since the  $(r+i)$ th row of  $M(i)$  has all zero entries when  $1 \leq i \leq r$ ,  $\det M(i) = 0$  for  $1 \leq i \leq r$ .

$$\begin{aligned} a_1 &= \sum_{i=1}^{2r} \det M(i) = \sum_{i=r+1}^{2r} \det M(i) \\ &= \sum_{i=1}^r \begin{vmatrix} -(s+1)I_r - L(G) & -L^\dagger(G)(i) \\ I_r^*(i) & 0_{r-1} \end{vmatrix} \end{aligned}$$

Let  $A_i$  be the  $r \times r$  matrix whose  $(i, i)$ -entry is  $s+1$  and other entries are all zero. By consecutively interchanging the  $i$ th column with the  $(i+1)$ th,  $(i+2)$ th,  $\dots$  and  $(r+i-1)$ th columns in the last determinant. We have

$$\begin{aligned} a_1 &= (-1)^r (-1)^{r-1} \sum_{i=1}^r \begin{vmatrix} (s+1)I_r^\dagger(i) + L^\dagger(G)(i) & L(G) + A_i \\ I_{r-1} & 0_r^*(i) \end{vmatrix} \\ &= (-1)(-1)^{2r-2} \sum_{i=1}^r \begin{vmatrix} L(G) + A_i & (s+1)I_r^\dagger(i) + L^\dagger(G)(i) \\ 0_r^*(i) & I_{r-1} \end{vmatrix} \\ &= - \sum_{i=1}^r \det(L(G) + A_i) = - \sum_{i=1}^r (\det L(G) + (s+1) \det L(G)(i)) \\ &= -(s+1) \sum_{i=1}^r t(G) = -r(s+1)t(G). \end{aligned}$$

Therefore,  $t(G \circ H) = \frac{(-a_1) \prod_{i=1}^{s-1} (\mu_i(H)+1)^r}{(s+1)^r} = t(G) \prod_{i=1}^{s-1} (\mu_i(H) + 1)^r$ , which completes the proof.  $\square$

Note that  $S(H \vee K_1) = (s+1, \mu_1(H)+1, \dots, \mu_{s-1}(H)+1, 0)$  and  $t(H \vee K_1) = \prod_{i=1}^{s-1} (\mu_i(H) + 1)$ . Then (3.3) can be rewritten as follows.

**Theorem 3.3** *Let  $G$  be a connected graph of order  $r$ . Then  $t(G \circ H) = t(G)t^r(H \vee K_1)$ .*

### 3.2 $G \vee H$

**Theorem 3.4** *Let  $G$  and  $H$  be two graphs of order  $r$  and  $s$ , respectively. If*

*$S(G) = (\mu_1(G), \dots, \mu_{r-1}(G), 0)$  and  $S(H) = (\mu_1(H), \dots, \mu_{s-1}(H), 0)$ , then*

$$t(G \vee H) = \prod_{i=1}^{r-1} (s + \mu_i(G)) \prod_{i=1}^{s-1} (r + \mu_i(H)).$$

**Proof.** By Lemma 2.3 and Eq.(1.1), the result follows.  $\square$

Let  $K_n$  and  $\overline{K}_n$  denote the complete graph and empty graph of order  $n$ , respectively,  $K_{m,n}$  denote the complete bipartite graph such that one part has  $n$  vertices and the other has  $m$  vertices. It is interesting that  $t(G \vee H)$  not only can be determined by the Laplacian spectra of  $G$  and  $H$ , but also can be expressed as the following form.

**Theorem 3.5** *Let  $G$  and  $H$  be two graphs of order  $r$  and  $s$ , respectively. Then*

$$t(G \vee H) = \frac{t(G \vee \overline{K}_s)t(H \vee \overline{K}_r)}{t(K_{r,s})}.$$

**Proof.** By Lemma 2.3, the Laplacian spectra of  $G \vee \overline{K}_s$  and  $H \vee \overline{K}_r$  are

$$S(G \vee \overline{K}_s) = (r + s, \underbrace{r, \dots, r}_{s-1}, \mu_1(G) + s, \dots, \mu_{r-1}(G) + s, 0),$$

$$S(H \vee \overline{K}_r) = (r + s, \underbrace{s, \dots, s}_{r-1}, \mu_1(H) + r, \dots, \mu_{s-1}(H) + r, 0).$$

Note that the Laplacian spectrum of  $K_{r,s}$  is

$$S(K_{r,s}) = (r + s, \underbrace{r, \dots, r}_{s-1}, \underbrace{s, \dots, s}_{r-1}, 0).$$

Therefore we have  $t(G \vee \overline{K_s}) = r^{s-1} \prod_{i=1}^{r-1} (\mu_i(G) + s)$ ,  $t(H \vee \overline{K_r}) = s^{r-1} \prod_{i=1}^{s-1} (\mu_i(H) + r)$  and  $t(K_{r,s}) = r^{s-1} s^{r-1}$ .

Hence, by Theorem 3.4, the result follows.  $\square$

**Theorem 3.6** *Let  $G$  and  $H$  be two connected graphs of order  $r$  and  $s$ , respectively. Then*

$$\frac{t(G \vee H)}{t(G)t(H)} \geq (r + s)^2 \left(1 + \frac{r}{s}\right)^{s-2} \left(1 + \frac{s}{r}\right)^{r-2},$$

and equality holds if and only if  $G \vee H$  is complete.

**Proof.** Let  $S(G) = (\mu_1(G), \dots, \mu_{r-1}(G), 0)$  and

$S(H) = (\mu_1(H), \dots, \mu_{s-1}(H), 0)$ . Then by Theorem 3.4, we have

$$\frac{t(G \vee H)}{t(G)t(H)} = rs \prod_{i=1}^{r-1} \left(1 + \frac{s}{\mu_i(G)}\right) \prod_{i=1}^{s-1} \left(1 + \frac{r}{\mu_i(H)}\right).$$

And

$$(r + s)^2 \left(1 + \frac{s}{r}\right)^{r-2} \left(1 + \frac{r}{s}\right)^{s-2} = rs \left(1 + \frac{s}{r}\right)^{r-1} \left(1 + \frac{r}{s}\right)^{s-1}.$$

Hence,

$$\frac{\frac{t(G \vee H)}{t(G)t(H)}}{(r + s)^2 \left(1 + \frac{s}{r}\right)^{r-2} \left(1 + \frac{r}{s}\right)^{s-2}} = \prod_{i=1}^{r-1} \frac{1 + \frac{s}{\mu_i(G)}}{1 + \frac{s}{r}} \prod_{i=1}^{s-1} \frac{1 + \frac{r}{\mu_i(H)}}{1 + \frac{r}{s}}.$$

Note that  $\mu_i(G) \leq r$  for all  $1 \leq i \leq r - 1$  and  $\mu_i(H) \leq s$  for all  $1 \leq i \leq s - 1$ . Hence the desired inequality holds, and the equality holds if and only if  $\mu_i(G) = r$  for all  $1 \leq i \leq r - 1$  and  $\mu_i(H) = s$  for all  $1 \leq i \leq s - 1$ . Therefore  $G$  and  $H$  are complete graphs. Thus  $G \vee H$  is also complete.  $\square$

### 3.3 Examples

The Laplacian spectra of  $P_n$ ,  $C_n$  and  $K_n$  [1] are

$$S(P_n) = \left( 4 \sin^2 \frac{(n-1)\pi}{2n}, 4 \sin^2 \frac{(n-2)\pi}{2n}, \dots, 4 \sin^2 \frac{\pi}{2n}, 0 \right),$$

$$S(C_n) = \left( 4 \sin^2 \frac{(n-1)\pi}{n}, 4 \sin^2 \frac{(n-2)\pi}{n}, \dots, 4 \sin^2 \frac{\pi}{n}, 0 \right) \quad \text{and}$$

$$S(K_n) = (n, n, \dots, n, 0).$$

And it is well known that  $t(P_n) = 1$ ,  $t(C_n) = n$  and  $t(K_n) = n^{n-2}$ .

The fan graph  $F_{r,s}$  and cone graph  $C_{r,s}$  are defined as  $P_r \vee \overline{K_s}$  and  $C_r \vee \overline{K_s}$ , respectively. Hence by Theorem 3.4, we have

$$t(F_{r,s}) = r^{s-1} \prod_{i=1}^{r-1} \left( s + 4 \sin^2 \frac{i\pi}{2r} \right) \quad \text{and} \quad t(C_{r,s}) = r^{s-1} \prod_{i=1}^{r-1} \left( s + 4 \sin^2 \frac{i\pi}{r} \right).$$

In particular, for the fan  $F_{r+1}$  and wheel graph  $W_{r+1}$  which are defined as  $F_{r,1}$  and  $C_{r,1}$ , we have

$$t(F_{r+1}) = \prod_{i=1}^{r-1} \left( 1 + 4 \sin^2 \frac{i\pi}{2r} \right) \quad \text{and} \quad t(C_{r+1}) = \prod_{i=1}^{r-1} \left( 1 + 4 \sin^2 \frac{i\pi}{r} \right).$$

The  $r$ -corona graph of a graph  $G$ , denoted by  $I_r(G)$ , is defined as  $G \circ \overline{K_r}$ . Since  $\overline{K_r} \vee K_1 = S_{r+1}$ , and  $t(S_{r+1}) = 1$ , by Theorem 3.3, we have

$$t(I_r(G)) = t(G).$$

## Acknowledgements

The authors sincerely thank the referee for careful reading of the manuscript and his/her suggestions which improved the presentation of the manuscript.

## References

- [1] C. Godsil, G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, **207**, Springer-Verlag, New York, 2001.
- [2] J. Li, W.C. Shiu, A. Chang, The number of spanning trees of a graph, *Appl. Math. Letters*, **23**(2010), 286-290.
- [3] R. Merris, Laplacian graph eigenvectors, *Linear Algebra Appl.*, **278**(1998), 221-236.
- [4] S. Barik, S. Pati, B.K. Sarma, The spectrum of the corona of two graphs, *SIAM J. Discrete Math.*, **21**(2007), 47-56.
- [5] H. Xu, The Laplacian spectrum of coronas, *J. Xiamen Univ.(Nat. Sci.)*, **44**(2005), 745-748 (in Chinese).