Roman reinforcement numbers of digraphs

N. Dehgardi and S.M. Sheikholeslami[1]
Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz, I.R. Iran
[1]s.m.sheikholeslami@azaruniv.edu

D. Meierling and L. Volkmann
Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany
meierling,volkm@math2.rwth-aachen.de

Abstract

Let $D = (V, A)$ be a finite and simple digraph. A *Roman dominating function* on D is a labeling $f\colon V(D) \to \{0, 1, 2\}$ such that every vertex v with label 0 there is a vertex w with label 2 such that wv is an arc in D. The *weight* of an RDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, equals the minimum weight of an RDF on D. The Roman reinforcement number $r_R(D)$ of a digraph D is the minimum number of arcs that must be added to D in order to decrease the Roman domination number. In this paper, we initiate the study of Roman reinforcement number in digraphs and we present some sharp bounds for $r_R(D)$. In particular, we determine the Roman reinforcement number of some classes of digraphs.

Key Words: Domination, Reinforcement, Roman domination, Roman Reinforcement, digraph.
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1 Introduction

Let D be a finite simple digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. A digraph without directed cycles of length 2 is an *oriented graph*. The order $n = n(D)$ of a digraph D is the number of its vertices.

We write $d_D^+(v) = d^+(v)$ for the outdegree of a vertex v and $d_D^-(v) = d^-(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N_D^-(v)$ and $N^+(v) = N_D^+(v)$, respectively. Let $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$. For a set S of vertices, we define $N^-(S) = \bigcup_{v\in S} N^-(v)$, $N^-[S] = \cup_{v\in S} N^-[v]$, $N^+(S) = \bigcup_{v\in S} N^+(v)$ and $N^+[S] = \bigcup_{v\in S} N^+[v]$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v. Consult [13] for the notation and terminology which are not defined here. In this note, we consider only finite simple digraphs.

A subset S of vertices of D is a *dominating set* if $N^+[S] = V$. The *domination number* $\gamma(D)$ is the minimum cardinality of a dominating set of D. The domination number was introduced by Lee [6]. The *reinforcement number* $r(D)$ of a digraph D is the minimum number of arcs that must be added to D in order to decrease the domination number [4].

A *Roman dominating function* (RDF) on a digraph $D = (V, A)$ is a function $f\colon V \to \{0, 1, 2\}$ satisfying the condition that every vertex v with $f(v) = 0$ has an in-neighbor u with $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, equals the minimum weight of an RDF on D. A *$\gamma_R(D)$-function* is a Roman dominating function of D with weight $\gamma_R(D)$. Roman domination for digraphs was introduced by Kamaraj and Jakkammal [5] and has been studied by several authors (see for example [11]). A Roman dominating function $f\colon V \to \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) of V (or (V_0^f, V_1^f, V_2^f) to refer to f), where $V_i = \{v \in V\colon f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. Since $V_1^f \cup V_2^f$ is a dominating set when f is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, we have

$$\gamma(D) \le \gamma_R(D) \le 2\gamma(D). \tag{1}$$

For a digraph D, a subset S of $V(D)$ and $x \in S$, the *private neighborhood* of x with respect to S is the set $PN(x, S) = N^+[x] - N^+[S - \{x\}]$. It is clear that every vertex of V_2 of a $\gamma_R(D)$-function has at least two V_2-private neighbors.

Let (V_0, V_1, V_2) be a $\gamma_R(D)$-function on a digraph D. If there exists an arc uv in D such that $u, v \in V_1$, then $(V_0 \cup \{v\}, V_1 \setminus \{u, v\}, V_2 \cup \{u\})$ is also a $\gamma_R(D)$-function of D. We call a $\gamma_R(D)$-function with the property that $|V_2|$ is maximum a *nice γ_R-function* of D.

The definition of Roman dominating functions for undirected graphs was given by Steward [12] and ReVelle and Rosing [10]. Cockayne, Dreyer Jr., Hedetniemi and Hedetniemi [2] as well as Chambers, Kinnersley, Prince and West [1] continued the investigation and have given a lot of results on Roman domination.

The *Roman reinforcement number* $r_R(D)$ of a digraph D is the minimum number of arcs that must be added to D in order to decrease the Roman domination number. It is obvious that if $\gamma_R(D) \in \{1, 2\}$, then addition of arcs edges does not reduce the Roman domination number. In these cases we define $r_R(D) = 0$.

Our purpose in this paper is to initiate the study of the Roman reinforcement number of digraphs. We present some sharp bounds on the Roman reinforcement number and we also determine exact values for the Roman reinforcement number of some classes of digraphs.

We make use of the following observations and properties. The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G by replacing each edge e of G by two oppositely oriented arcs. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 1. For any graph G, $\gamma_R(G) = \gamma_R(D(G))$, $r(G) = r(D(G))$ and $r_R(G) = r_R(D(G))$.

Theorem A ([4]). For any digraph D of order n and $\gamma(D) = 2$, $r(D) = n - \Delta^+(D) - 1$.

Observation 2. If D is a digraph of order n and E_1 is a minimum subset of arcs of $D(K_n) - A(D)$ such that $\gamma_R(D+E_1) < \gamma_R(D)$, then $\gamma_R(D+E_1) = \gamma_R(D) - 1$.

Observation 3. If D is an empty digraph of order $n \geq 3$, then $r_R(D) = 2$.

Proposition 4. Let D be a digraph. Then $\gamma_R(D) = 2\gamma(D)$ if and only if D has a $\gamma_R(D)$-function $f = (V_0, V_1, V_2)$ with $|V_1| = 0$.

Proof. Let $\gamma_R(D) = 2\gamma(D)$, and let S be a dominating set of D with $|S| = \gamma(D)$. Then $f = (V(D) - S, \emptyset, S) = (V_0, V_1, V_2)$ is an RDF on D such that

$$\omega(f) = 2|S| = 2\gamma(D) = \gamma_R(D)$$

and therefore f is a $\gamma_R(D)$-function with $V_1 = \emptyset$.

Conversely, let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function with $|V_1| = 0$ and, consequently, $\gamma_R(D) = 2|V_2|$. Then V_2 is also a dominating set of D, and hence we deduce that $2\gamma(D) \leq 2|V_2| = \gamma_R(D)$. Applying the second inequality in (2), we obtain the identity $\gamma_R(D) = 2\gamma(D)$, and the proof is complete. □

2 Roman reinforcement number and reinforcement number of digraphs

In this section we show that $r(D) = r_R(D) + 1$ for any digraph D with $\gamma_R(D) = 2\gamma(D) \geq 4$. We start with the following lemmas.

Lemma 5. For a digraph D of order n with $\gamma_R(D) \geq 4$ and $\gamma_R(D) = 2\gamma(D)$,

$$r_R(D) \leq r(D) - 1.$$

Proof. Let E be a set of augmenting arcs such that $|E| = r(D)$ and $\gamma(D + E) \leq \gamma(D) - 1$. Then the second inequality in (1) implies that

$$\gamma_R(D + E) \leq 2\gamma(D + E) \leq 2\gamma(D) - 2 = \gamma_R(D) - 2.$$

If e is an arbitrary arc of E, then it follows that

$$\gamma_R(D + (E - \{e\})) \leq (\gamma_R(D) - 2) + 1 = \gamma_R(D) - 1.$$

Hence, $|E - \{e\}| \geq 1$ and $r_R(D) \leq |E| - 1 = r(D) - 1$. This completes the proof. □

Lemma 6. Let D be a digraph with $\gamma_R(D) = 2\gamma(D) \geq 4$. If there exists an arc $e \in A(\overline{D})$ such that $\gamma_R(D + e) = \gamma_R(D) - 1$, then $D + e$ has a γ_R-function $f = (V_0, V_1, V_2)$ with $|V_1| = 1$.

Proof. The proof is by induction on $\gamma(D)$.

Let $\gamma_R(D) = 2\gamma(D) = 4$. Note that by Observation 2, every γ_R-function of $D + e$ has weight 3. Let $f = (V_0, V_1, V_2)$ be a nice γ_R-function of $D + e$ and assume that the proposition is not correct. It follows that $f = (\emptyset, V_1, \emptyset)$ with $|V_1| = 3$. Because f is nice, we conclude that V_1 is an independent set. Therefore, $D + e$ consists of three independent vertices, a contradiction.

Let $\gamma_R(D) = 2\gamma(D) \geq 6$ and suppose that the proposition is true for every digraph D' with $\gamma_R(D) > \gamma_R(D') = 2\gamma(D')$. According to Proposition 4, there exists a γ_R-function $f = (V_0^f, \emptyset, V_2^f)$ of D. Moreover, since $r_R(D) = 1$, there exists a vertex $x \in V_2^f$ such that no private neighbor of x is incident to e. Let D^* be obtained from D by deleting the set X of all private neighbors of x. It is easy to see that $(V_0^f \setminus X, \emptyset, V_2^f \setminus \{x\})$ is a γ_R-function of D^* and $V_2^f \setminus \{x\}$ is a dominating set of D^*. Hence, $\gamma_R(D^*) = \gamma_R(D) - 2$ and $\gamma(D^*) = \gamma(D) - 1$. Furthermore, $\gamma_R(D^* + e) = \gamma_R(D^* + e) - 1$ and thus, by induction, $D^* + e$ has a γ_R-function $g = (V_0^g, \{v\}, V_2^g)$. It follows that $h = (V_0^g \cup (X \setminus \{x\}), \{v\}, V_2^g \cup \{x\})$ is a γ_R-function of $D + e$ with $|V_1^h| = 1$. This completes the proof of this lemma. □

Theorem 7. If D is a digraph with $\gamma_R(D) = 2\gamma(D) \geq 4$, then $r(D) = r_R(D) + 1$.

Proof. Note that $r(D) \geq r_R(D) + 1$ by Lemma 5. Let E be a minimum augemtning set such that $|E| = r_R(D)$ and $\gamma_R(D + E) = \gamma_R(D) - 1$. If $e \in E$, then $D + E \setminus \{e\}$ fulfills the assumptions of Lemma 6. It follows that $D + E$ has a γ_R-function $f = (V_0, V_1, V_2)$ with $|V_1| = 1$. Let v be an arbitrary vertex of V_2 and $V_1 = \{w\}$. It follows that V_2 is a dominating set of $D + (E \cup \{vw\})$ of order $\gamma(D) - 1$. Hence, $r(D) \leq r_R(D) + 1$ and the proof is complete. □

As an application of Observation 1, Theorem A and Theorem 7, we have the next result.

Corollary 8 (Jafari Rad et al. [9] 2011). If G is a graph with $\gamma_R(G) = 2\gamma(G) \geq 4$, then $r_R(G) = r(G) - 1$.

3 Bounds on Roman reinforcement number

Proposition 9. If D is a digraph of order n with $\gamma_R(D) \geq 3$, then

$$r_R(D) \leq n - \Delta^+(D) - \gamma_R(D) + 2.$$

Proof. Since $\gamma_R(D) \geq 3$, $\Delta^+(D) \leq n - 2$. Let u be a vertex of maximum out-degree and let $E' = \{(u, v) : v \in V(D) - N^+[u]\}$. Then $(V(D) - \{u\}, \emptyset, \{u\})$ is a Roman dominating function of $D + E'$ and thus, $\gamma_R(D + E') = 2$. Hence, $r_R(D) \leq |E'| = n - \Delta^+(D) - 1$. Therefore there are $r_R(D) - 1$ vertices $v_1, v_2, \ldots, v_{r_R(D)-1}$ in $V(D) - N^+[u]$.

Let D' be a digraph obtained from D by adding $r_R(D) - 1$ arcs (u, v_i) for each $i = 1, 2, \ldots, r_R(D) - 1$. By the definition of $r_R(D)$,

$$\gamma_R(D) = \gamma_R(D') \leq n - \Delta^+(D') + 1 = n - (\Delta^+(D) + r_R(D) - 1) + 1,$$

which yields the result. □

The next example gives a class of digraphs satisfying the conditions of Proposition 9.

Example 10. Let $p \geq 1$ be an integer, and let F be a digraph with vertex set $\{c, x_1, x_2, \ldots, x_p\}$ and arc set $\{cx_1, cx_2, \ldots, cx_p\}$. Let D be the digraph obtained as the disjoint union of two copies of F. Then $n(D) = 2p + 2$, $\gamma_R(D) = 4$, $\Delta^+(D) = p$, and it is easy to see that $r_R(D) = p$. It follows that

$$r_R(D) = n(D) - \Delta^+(D) - \gamma_R(D) + 2 = p,$$

and therefore the upper bound in Proposition 9 is sharp.

Theorem 11. Let D be a digraph of order $n \geq 3$ and $\Delta^+(D) \geq 1$. Then $r_R(D) = 1$ if and only if there is a $\gamma_R(D)$-function $f = (V_0, V_1, V_2)$ with $V_1 \neq \emptyset$.

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function with $V_1 \neq \emptyset$. If $V_2 \neq \emptyset$, then let $u \in V_1$ and $v \in V_2$. Obviously $\gamma_R(D + (v,u)) < \gamma_R(D)$ and thus, $r_R(D) = 1$. So assume that $V_2 = \emptyset$. Then $V_1 = V(D)$. Since $\Delta^+(D) \geq 1$, we may assume $vw \in A(D)$. Obviously $(\{w\}, V_1 - \{v,w\}, \{v\})$ is a $\gamma_R(D)$-function and the result follows as above.

Conversely, let $r_R(D) = 1$. Suppose to the contrary that $V_1^f = \emptyset$ for each $\gamma_R(D)$-function $f = (V_0^f, V_1^f, V_2^f)$. Then $\gamma_R(D)$ is even. Assume that xy is an arc such that $\gamma_R(D + xy) < \gamma_R(D)$. By Observation 2, $\gamma_R(D + xy) = \gamma_R(D) - 1$. Let g be a $\gamma_R(D + xy)$-function. If $g(x) \neq 2$, then g is an RDF for D which is a contradiction. So $g(x) = 2$ and hence $g(y) = 0$. Then $f = (V_0^g - \{y\}, V_1^g \cup \{y\}, V_2^g)$ is a $\gamma_R(D)$-function with $V_1^f \neq \emptyset$. This contradiction completes the proof. □

Corollary 12. For a digraph D of order $n \geq 4$ with $\gamma_R(D) = 4$ and $r_R(D) \geq 2$, $r_R(D) = n - \Delta^+(D) - 2$.

Proof. If $n = 4$ and D is an empty digraph, then the result follows from Observation 3. Hence, let $n \geq 5$ or $n = 4$ and $A(D) \neq \emptyset$. Assume that $f = (V_0, V_1, V_2)$ is a $\gamma_R(D)$-function. Since $r_R(D) \geq 2$, we deduce that $V_1 = \emptyset$. This implies that $\gamma(D) = 2$ and so $\gamma_R(D) = 2\gamma(D)$. It follows from Theorem 7 and Theorem A that $r_R(D) = n - \Delta^+(D) - 2$. □

Corollary 12 shows that the bound in Proposition 9 is sharp when $\gamma_R(D) = 4$.

Corollary 13. If D is a digraph of order $n \geq 3$ and $\Delta^+(D) \geq 1$ such that $\gamma_R(D)$ is odd, then $r_R(D) = 1$.

In the following we transfer an idea from [4] to digraphs to present an upper bound for the Roman reinforcement number. Let D be a digraph, and let S be a subset of $V(D)$ with $|S| \geq 2$. Assume that $\eta(S) = \max\{|N^+[X]|\colon X \subseteq S, |X| = |S| - 1\}$ and define

$$\eta(D) = \max\{\eta(V_1 \cup V_2)\colon f = (V_0, V_1, V_2) \text{ is a nice } \gamma_R(D) - \text{function}\}.$$

We call a nice $\gamma_R(D)$-function $f = (V_0, V_1, V_2)$ an *η-function* if $\eta(D) = \eta(V_1 \cup V_2)$. It is clear that

$$\eta(V_1 \cup V_2) \leq n - 1 \tag{2}$$

for any nice $\gamma_R(D)$-function and hence $\eta(D) \leq n - 1$.

Theorem 14. Let D be a nonempty digraph of order $n \geq 3$ with $\gamma_R(D) \geq 3$. Then

$$r_R(D) \leq n - \eta(D).$$

Proof. Let $f = (V_0, V_1, V_2)$ be an η-function. Since D is nonempty and f is a nice $\gamma_R(D)$-function, we have $V_2 \neq \emptyset$. Suppose that X is a subset of $V_1 \cup V_2$ such that $\eta(D) = \eta(V_1 \cup V_2) = |N^+[X]|$. If $|V_2| \geq 2$, then $X \cap V_2 \neq \emptyset$. So let $|V_2| = 1$ with $V_2 = \{x\}$. Since $\gamma_R(D) \geq 3$, we have $|V_1| \geq 1$. Suppose $y \in V_1$ and set $X' = (X - \{y\}) \cup \{x\}$. It is clear that $\eta(D) = \eta(V_1 \cup V_2) = |N^+[X']|$. Hence, we may assume, without loss of generality, that $X \cap V_2 \neq \emptyset$, say $x \in X \cap V_2$. Since $\eta(V_1 \cup V_2) = |N^+[X]|$, by (2) there exists a subset Y of $V(D)$ that is not dominated by X. Let D' be a digraph obtained from D by adding all arcs leading from x to the vertices in Y. Let $y \in V_1 \cup V_2 - X$. Then $(V_0 \cup \{y\}, V_1 - \{y\}, V_2 - \{y\})$ is an RDF of D'. Hence, $\gamma_R(D') \leq |V_1 - \{y\}| + 2|V_2 - \{y\}| < |V_1| + 2|V_2| = \gamma_R(D)$. It follows that $r_R(D) \leq n - \eta(D)$. □

If D is a digraph and S a subset of $V(D)$, then let $\rho(S) = \min\{|PN(x, S)| : x \in S\}$. The *private neighborhood number* of D is defined by

$$\rho(D) = \min\{\rho(V_2) \colon f = (V_0, V_1, V_2) \text{ is a nice } \gamma_R(D) - \text{function}\}.$$

If $A(D) \neq \emptyset$, then it is clear that

$$\rho(D) \geq 2. \tag{3}$$

Theorem 15. If D is a digraph of order $n \geq 3$ with $\Delta^+(D) \geq 1$, then $r_R(D) \leq \rho(D) - 1$.

Proof. If $\gamma_R(D) = 2$, then $r_R(D) = 0 < \rho(D) - 1$. If $\gamma_R(D) = 3$, then Corollary 13 leads to $r_R(D) = 1 \leq \rho(D) - 1$.

Now assume that $\gamma_R(D) \geq 4$. If there is a $\gamma_R(D)$-function (V_0, V_1, V_2) with $V_1 \neq \emptyset$, then it follows from Theorem 11 that $r_R(D) = 1 \leq \rho(D) - 1$. Assume next that $V_1 = \emptyset$ for each $\gamma_R(D)$-function (V_0, V_1, V_2). Let $f = (V_0, \emptyset, V_2)$ be a nice $\gamma_R(D)$-function such that there exists a vertex $x \in V_2$ with $|PN(x, V_2)| = \rho(D)$. Since $\gamma_R(D) \geq 4$, there exists a vertex $u \in V_2 - \{x\}$.

If $x \in PN(x, V_2)$, then let $D' = D + \{(u, w) \colon w \in PN(x, V_2) - \{x\}\}$. We observe that $g = (V_0, \{x\}, V_2 - \{x\})$ is an RDF of D', and thus $r_R(D) \leq |PN(x, V_2)| - 1 = \rho(D) - 1$.

If $x \notin PN(x, V_2)$, then let $PN(x, V_2) = \{y_1, y_2, \ldots, y_\rho\}$. Now let $D' = D + \{(u, y_i) \colon i = 1, 2, \ldots, \rho - 1\}$. We see that $h = ((V_0 \cup \{x\}) - \{y_\rho\}, \{y_\rho\}, V_2 - \{x\})$ is an RDF of D' and thus, $r_R(D) \leq |PN(x, V_2)| - 1 = \rho(D) - 1$. □

Corollary 16. If D is a digraph of order $n \geq 3$ with $\Delta^+(D) \geq 1$, then

$$r_R(D) \leq \frac{2n}{\gamma_R(D)} - 1.$$

Proof. If there is a $\gamma_R(D)$-function (V_0, V_1, V_2) with $V_1 \neq \emptyset$, then $r_R(D) = 1 \leq \frac{2n}{\gamma_R(D)} - 1$. So assume that $V_1 = \emptyset$ for every $\gamma_R(D)$-function (V_0, V_1, V_2). Then $\gamma_R(D) = 2|V_2|$ for any $\gamma_R(D)$-function (V_0, V_1, V_2). Let $f = (V_0, \emptyset, V_2)$ be a $\gamma_R(D)$-function such that there exists a vertex $x \in V_2$ with $|PN(x, V_2)| = \rho(D)$. Then

$$\frac{\gamma_R(D)}{2}\rho(D) = |V_2|\rho(D) \leq \sum_{v \in V_2} |PN(v, V_2)| \leq n.$$

Now the desired result follows from Theorem 15. □

Corollary 17. Let D be a digraph of order $n \geq 3$ with $\Delta^+(D) \geq 1$, and let $f = (V_0, V_1, V_2)$ be a nice $\gamma_R(D)$-function. Then

$$r_R(D) \leq \min\{d^+(u) \colon u \in V_2\}.$$

An immediate consequence of Corollary 17 is the following result.

Corollary 18. For any digraph D of order $n \geq 3$ and $\Delta^+(D) \geq 1$, $r_R(D) \leq \Delta^+(D)$. Moreover, the bound is sharp for any digraph D with $\Delta^+(D) = 1$.

Example 19. Let F be a digraph with vertex set $\{c, x_1, x_2, \ldots, x_p\}$ with $p \geq 1$ and arc set $\{cx_1, cx_2, \ldots, cx_p\}$. Let H be the digraph obtained from the disjoint union of $t \geq 2$ copies of F. Then $\gamma_R(H) = 2t$, $\Delta^+(H) = p$, and it is easy to see that $r_R(H) = p$ and $\rho(H) = p + 1$. It follows that $\rho(H) - 1 = r_R(H) = \Delta^+(H) = p$, and therefore the bounds given in Theorem 15 and Corollary 18 are sharp.

The next result follows from Proposition 9 and Corollary 18.

Corollary 20. If D is a digraph of order $n \geq 3$ with $\Delta^+(D) \geq 1$, then

$$r_R(D) \leq \frac{n - \gamma_R(D) + 2}{2}.$$

Digraphs D with $\Delta^+(D) = 1$ as well as Example 10 show that the bounds in Corollaries 18 and 20 are sharp.

4 Compositions of digraphs

For two undirected graphs G and H, the *join* $G + H$ is defined as the undirected graph consisting of G and H with each vertex of G adjacent to every vertex of H. In the directed case, there are two possibilities to define the join of two digraphs. Let G and H be digraphs. The digraph $G \to H$ is obtained from G and H by adding all possible arcs from vertices of G to vertices of H, and $G \leftrightarrow H$ is be obtained from $G \to H$ by adding all possible arcs from vertices of H to vertices of G.

Proposition 21. Let G and H be two digraphs such that $\Delta^+(G) \geq 1$ and $\Delta^+(H) \geq 1$. Then

1. $r_R(G \to H) = r_R(G)$,
2. $r_R(G \leftrightarrow H) = \min\{n(G) - \Delta^+(G) - 2, n(H) - \Delta^+(H) - 2\}$.

Proof. (1) Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function with $V_2 \neq \emptyset$. Then f is also an RDF of $G \to H$. Hence $\gamma_R(G \to H) \leq \omega(f) = \gamma_R(G)$. On the other hand, if $g = (V_0, V_1, V_2)$ is a $\gamma_R(G \to H)$-function, then obviously $g|_G = (V_0 \cap V(G), V_1 \cap V(G), V_2 \cap V(G))$ is an RDF of G and thus, $\gamma_R(G \to H) = \gamma_R(G)$.

If $\gamma_R(G) = 2$, then obviously $\gamma_R(G \to H) = 2$ and thus, $r_R(G) = r_R(G \to H) = 0$. Let $\gamma_R(G) \geq 3$ and let E' be a minimum set of augmenting arcs with $|E'| = r_R(G)$ and $\gamma_R(G + E') < \gamma_R(G)$. Then $\gamma_R((G \to H) + E') = \gamma_R((G + E') \to H) = \gamma_R(G + E') < \gamma_R(G) = \gamma_R(G \to H)$. Hence, $r_R(G \to H) \leq r_R(G)$.

Let E_1 be a minimum set of augmenting arcs with $|E_1| = r_R(G \to H)$ and $\gamma_R((G \to H) + E_1) < \gamma_R(G \to H)$. Suppose that E_2 is a subset of E_1 such that two ends of arcs in E_2 lie in $V(G)$. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R((G \to H)+E_1)$-function and let $g = (V_0^f \cap V(G), V_1^f \cap V(G), V_2^f \cap V(G))$. If g is an RDF of $G + E_2$, then we have

$$\begin{aligned}\gamma_R((G \to H) + E_1) &= \omega(f)\\ &\geq \omega(g)\\ &\geq \gamma_R(G + E_2)\\ &= \gamma_R((G \to H) + E_2)\\ &\geq \gamma_R((G \to H) + E_1).\end{aligned}$$

By the choice of E_1, we have $E_2 = E_1$ and hence

$$\gamma_R(G + E_2) \leq \omega(g) = \gamma_R((G \to H) + E_2) < \gamma_R(G \to H) = \gamma_R(G).$$

Thus, $r_R(G) \leq |E_2| = |E_1| = r_R(G \to H)$ and so $r_R(G) = r_R(G \to H)$.

Assume that g is not an RDF of $G + E_2$. Then obviously some vertex u in $V_0^f \cap V(G)$ has an in-neighbor $u' \in V_2^f$ in H where $u'u \in E_1$.

Let $v \in V(G)$ and add an arc vu for each $u \in V_0^f \cap V(G)$ which does not have in-neighbor in $V_2^f \cap V(G)$ and let $E_3 = \{(v,u): u \in V_0^f \cap V(G)$ and u does not have in – neighbor in $V_2^f \cap V(G)\}$. Obviously $|E_2 \cup E_3| \leq |E_1|$ and it is easy to see that $h = ((V_0^f \cap V(G)) - \{v\}, (V_1^f \cap V(G)) - \{v\}, (V_2^f \cap V(G)) \cup \{v\})$ is an RDF of $G + (E_2 \cup E_3)$. Note that $\omega(h) \leq \omega(f)$, since $|V_2^f \cap H| \geq 1$. Now we have

$$\begin{aligned} \gamma_R(G + (E_2 \cup E_3)) &\leq \omega(h) \\ &\leq \omega(f) \\ &= \gamma_R((G \to H) + E_1) \\ &< \gamma_R(G \to H) \\ &= \gamma_R(G). \end{aligned}$$

Thus $r_R(G) \leq |E_2 \cup E_3| \leq |E_1| = r_R(G \to H)$ and hence $r_R(G) = r_R(G \to H)$.

(2) If $\min\{\gamma_R(G), \gamma_R(H)\} \leq 2$, then $\gamma_R(G \leftrightarrow H) \leq 2$ and $r_R(G \leftrightarrow H) = 0$. Otherwise $\gamma_R(G \leftrightarrow H) = 4$. By Corollary 12,

$$\begin{aligned} r_R(G \leftrightarrow H) &= n(G \leftrightarrow H) - \Delta^+(G \leftrightarrow H) - 2 \\ &= n(G) + n(H) - \max\{\Delta^+(G) + n(H), \Delta^+(H) + n(G)\} - 2 \\ &= \min\{n(G) - \Delta^+(G) - 2, n(H) - \Delta^+(H) - 2\}, \end{aligned}$$

and the proof is complete. □

The *corona* GoH of two undirected graphs G and H is formed from one copy of G and $n(G)$ copies of H by joining v_i to every vertex in H_i, where v_i is the ith vertex of G and H_i is the ith copy of H. For digraphs G and H, if all the additional edges are from G to H_i, then we denote the resulting digraph by $G\vec{o}H$.

Proposition 22. Let G and H be two digraphs with $n(H) \geq 2$. Then

$$r_R(G\vec{o}H) = \begin{cases} 0 & \text{if } n(G) = 1, \\ n(H) & \text{if } G \text{ is the empty graph and } n(G) \geq 2, \\ n(H) - 1 & \text{otherwise.} \end{cases}$$

Proof. If $n(G) = 1$, then obviously $\gamma_R(G\vec{o}H) = 2$ and, by definition, $r_R(G\vec{o}H) = 0$. Let $n(G) \geq 2$. In [4], it is shown that $\gamma(G\vec{o}H) = n(G)$. Now we show that $\gamma_R(G\vec{o}H) = 2n(G)$. Obviously, $f = (V(G\vec{o}H) - V(G), \emptyset, V(G))$ is an RDF of $G\vec{o}H$ and thus, $\gamma_R(G\vec{o}H) \leq 2n(G)$. Now let f be a $\gamma_R(G\vec{o}H)$-function. To dominate the vertices of the ith copy of H, i.e., H_i, we must have $\sum_{v \in V(H_i) \cup \{v_i\}} f(v) \geq 2$. Since a single vertex of G does not dominate two vertices in different copies of H, we deduce that $\gamma_R(G\vec{\partial}H) \geq 2n(G)$. Thus, $\gamma_R(G\vec{\partial}H) = 2n(G) = 2\gamma(G\vec{\partial}H)$. We consider two cases.

Case 1 Assume that $A(G) = \emptyset$.
Let $A' = \{(v_1, u)\colon u \in V(H_{n(G)})\}$. Then obviously

$$(\cup_{i=1}^{n(G)} V(H_i), \{v_n\}, \{v_1, v_2, \ldots, v_{n-1}\})$$

is an RDF of $(G\vec{\partial}H)+A'$ and thus, $\gamma_R((G\vec{\partial}H)+A') < \gamma_R(G\vec{\partial}H)$. Hence, $r_R(G\vec{\partial}H) \leq n(H)$.

Let E_1 be a minimum set of augmenting arcs with $|E_1| = r_R(G\vec{\partial}H)$ and $\gamma_R((G\vec{\partial}H) + E_1) < \gamma_R(G\vec{\partial}H)$. By Observation 2, $\gamma_R((G\vec{\partial}H) + E_1) = \gamma_R(G\vec{\partial}H) - 1 = 2n(G) - 1$. Let $W_i = \{v_i\} \cup V(H_i)$, $1 \leq i \leq n(G)$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_R((G\vec{\partial}H) + E_1)$-function. Then $|V_2 \cap W_i| \leq 1$ for some i, say $i = n(G)$. To dominate the vertices in $W_{n(G)}$, E_1 must contain at least $n(H)$ arcs which goes from some vertices in V_2 to vertices in W_i (note that some vertex in W_i can belong to V_1). Hence, $|E_1| \geq n(H)$ and thus, $r_R(G\vec{\partial}H) = |E_1| \leq n(H)$. Therefore $r_R(G\vec{\partial}H) = n(H)$.

Case 2 Assume that $A(G) \neq \emptyset$.
Assume, without loss of generality, that $(v_1, v_{n(G)}) \in A(G)$. Let $V(H_{n(G)}) = \{w_1, w_2, \ldots, w_{n(H)}\}$ and let $A' = \{(v_1, u)\colon u \in V(H_{n(G)}) - \{w_1\}\}$. Then obviously,

$$(\cup_{i=1}^{n(G)} V(H_i) \cup \{v_{n(G)}\}, \{w_1\}, \{v_1, v_2, \ldots, v_{n-1}\})$$

is an RDF of $(G\vec{\partial}H) + A'$ and thus, $\gamma_R((G\vec{\partial}H) + A') < \gamma_R(G\vec{\partial}H)$. Hence, $r_R(G\vec{\partial}H) \leq n(H) - 1$. An similar argument as in Case 1, shows that $r_R(G\vec{\partial}H) = n(H) - 1$. This completes the proof. □

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