Bistellar equivalences of two families of simplicial complexes

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Abstract

In this paper, we study a pair of simplicial complexes, which we denote by $\mathcal{B}(k,d)$ and $\mathcal{ST}(k+1,d-k-1)$, for all nonnegative integers k and d with $0 \le k \le d-2$. We conjecture that their underlying topological spaces $|\mathcal{B}(k,d)|$ and $|\mathcal{ST}(k+1,d-k-1)|$ are homeomorphic for all such k and d. We answer this question when $k = d-2$ by relating the complexes through a series of well studied combinatorial operations that transform a combinatorial manifold while preserving its PL-homeomorphism type.

1 Introduction

A common problem in combinatorial topology asks for the minimal number of vertices that are required to triangulate a given topological manifold. In general, this is a very difficult problem as it requires both algebraic techniques to give lower bounds on the minimal number of vertices that are required and explicit constructions to establish the tightness of these bounds. Lutz's thesis [9], together with the references therein, provide an excellent survey of the progress that has been made on this general problem.

In a recent paper, Novik and the second author [5] defined a simplicial complex $\mathcal{B}(k,d)$ for all integers k and d with $0 \le k \le d-2$. These

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complexes are combinatorial manifolds with boundary such that $\partial\mathcal{B}(k,d)$ triangulates $\mathbb{S}^k \times \mathbb{S}^{d-k-2}$ [5, Theorem 1.2(e)]. Moreover, the boundary complexes $\partial\mathcal{B}(k,d)$ provide centrally symmetric, vertex-minimal triangulations of $\mathbb{S}^k \times \mathbb{S}^{d-k-2}$ for all d. As such, and based on algebraic invariants of these complexes [5, Theorem 1.2(d)], we are led to conjecture that $\mathcal{B}(k,d)$ triangulates $\mathbb{S}^k \times \mathbb{B}^{d-k-1}$. From this, it would immediately follow that the complexes $\mathcal{B}(k,d)$ are centrally symmetric, vertex-minimal triangulations of $\mathbb{S}^k \times \mathbb{B}^{d-k-1}$ for all d.

One approach to solving this problem is to study a family of operations known as bistellar flips, stellar exchanges, and (inverse) shellings, which locally transform the combinatorial structure of a manifold triangulation while preserving its PL-homeomorphism type. Our first goal in this paper is to introduce another family of simplicial complexes, which we denote by $\mathcal{ST}(m,n)$. By studying a modification of the classical staircase triangulation (see [2, 3, 4, 12]), we are easily able to show that $\mathcal{ST}(m,n)$ triangulates $\mathbb{S}^{m-1} \times \mathbb{B}^n$. This leads us to ask the following question.

Question 1.1 *Can $\mathcal{B}(k,d)$ be obtained from $\mathcal{ST}(k+1, d-k-1)$ through a series of bistellar moves, stellar exchanges, elemetary shellings, and their inverses?*

If the answer to this question is "yes," then it follows that $\mathcal{B}(k,d)$ triangulates $\mathbb{S}^k \times \mathbb{B}^{d-k-1}$ as we had hoped. In this paper, we answer Question 1.1 in the affirmative for two infinite classes of complexes when $k = 0$ (see Section 2) and $k = d-2$ (see Section 3). In Section 4, we answer Question 1.1 for two other cases when $k = 1$ and $d = 4$ or $d = 5$. These two cases illustrate part of the difficulty in answering Question 1.1 in the general case.

The problem of testing whether or not two fixed simplicial complexes are bistellar equivalent is highly computational. The software package BISTELLAR_EQUIVALENT [8] is quite efficient in solving this problem, and it was very useful for us in checking small examples during the early stages of this project. For our purposes, however, there were two main obstacles to overcome in studying Question 1.1. First, the package BISTELLAR_EQUIVALENT uses a randomized annealing algorithm, meaning that there is no inherent structure to the series of bistellar flips that connects two triangulations. In contrast, the series of bistellar operations outlined in Section 3 is highly structured and hence is unlikely to be found by a randomized computer search. Second, our goal is to find a series of bistellar operations connecting $\mathcal{B}(k,d)$ to $\mathcal{ST}(k+1, d-k-1)$ for all k and d.

We begin by defining all of the necessary definitions related to simplicial complexes and combinatorial manifolds in Section 1.1. We then proceed to define the main complexes of interest in Sections 1.2 and 1.3 and to prove our main results.

Acknowledgments

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1.1 Simplicial complexes and combinatorial manifolds

Definition 1.2 *An (abstract)* ***simplicial complex*** *Δ on vertex set $V(\Delta)$ is a collection of subsets $F \subseteq V$ (called* ***faces****) satisfying the following two properties:*

1. *$\{v\} \in \Delta$ for all $v \in V$, and*

2. *if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.*

The **dimension** of a face F in Δ is $\dim F := |F| - 1$, and the dimension of Δ is $\dim \Delta := \max\{\dim F : F \in \Delta\}$. A simplicial complex Δ is **pure** if all of its **facets** (maximal faces under inclusion) have the same dimension. If Δ is a pure $(d-1)$-dimensional simplicial complex, a $(d-2)$-dimensional face of Δ is called a **ridge**. Unless otherwise specified, we will assume that all of our simplicial complexes Δ are pure and $(d-1)$-dimensional.

For any abstract simplicial complex Δ, there is a corresponding topological space $|\Delta|$, called the **geometric realization** of Δ, which contains a geometric i-simplex for each i-dimensional face F of Δ. For a face F in Δ, we let $\overline{F} := \{G : G \subseteq F\}$ denote the simplex whose vertices belong to F. The **boundary** of $\overline{F}$ is defined as $\partial\overline{F} := \{G : G \subsetneq F\}$.

Definition 1.3 *Let F be a face in the simplicial complex Δ. The* ***link*** *of F in Δ is*

$$\mathrm{lk}_\Delta(F) := \{G \in \Delta : F \cap G = \emptyset \text{ and } F \cup G \in \Delta\}.$$

We note that if Δ is a pure simplicial complex of dimension $d-1$, then $\mathrm{lk}_\Delta(F)$ is a pure $(d - |F| - 1)$-dimensional simplicial complex for any face $F \in \Delta$.

Definition 1.4 *Let Γ and Δ be simplicial complexes such that $V(\Gamma) \cap V(\Delta) = \emptyset$. The* ***join*** *of Γ and Δ is*

$$\Gamma * \Delta = \{F \cup G : F \in \Gamma, G \in \Delta\}.$$

Next, we define a certain family of simplicial complexes known as combinatorial manifolds. For further information on combinatorial manifolds, see [1] or [6].

We say that a simplicial complex Γ is a **combinatorial n-ball** if $|\Gamma|$ is piecewise linear (PL) homeomorphic to the standard n-simplex σ^n. Specifically, this means that there is a homeomorphism $\varphi : |\Gamma| \to \sigma^n$ with the property that the restriction of φ to any face in the realization of Γ is a piecewise linear map; and that the inverse map φ^{-1} is also a PL map. Similarly, we say that Γ is a **combinatorial n-sphere** if Γ is PL homeomorphic to $\partial\sigma^{n+1}$.

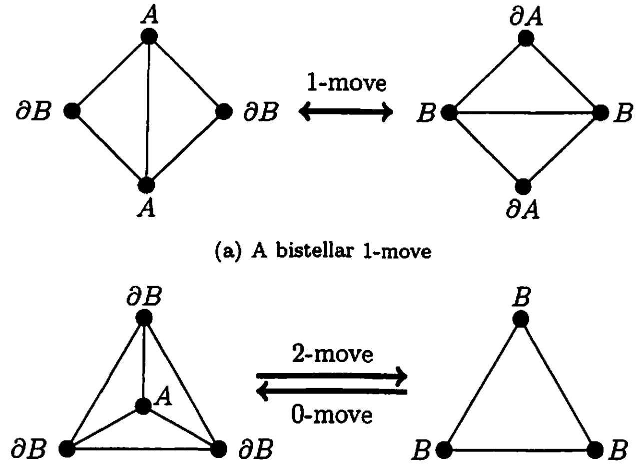
A **combinatorial $(d-1)$-manifold** is a $(d-1)$-dimensional simplicial complex Δ with the property that $\mathrm{lk}_\Delta(v)$ is either a combinatorial $(d-2)$-ball or a combinatorial $(d-2)$-sphere for all vertices $v \in \Delta$. We say that a face F in a combinatorial $(d-1)$-manifold Δ is a **boundary face** if $\mathrm{lk}_\Delta(F)$ is a combinatorial $(d-|F|-1)$-ball, and F is an **interior face** if $\mathrm{lk}_\Delta(F)$ is a combinatorial $(d-|F|-1)$-sphere.

The problem of determining whether or not two geometric simplicial complexes are PL homeomorphic may seem to be (and in fact is) quite difficult. Fortunately, there is a finite collection of local combinatorial operations such that two combinatorial manifolds are PL homeomorphic if and only if one can be obtained from the other through a finite sequence of these operations (see Theorem 1.10). Now we define these operations.

Definition 1.5 *Suppose that A is an r-simplex in a $(d-1)$-dimensional combinatorial manifold Δ and that $\mathrm{lk}_\Delta(A) = \partial\overline{B}$ for some $(d-r-1)$-simplex $B \notin \Delta$. The* ***bistellar move*** *$\chi(A,B)$ consists of changing Δ by removing $\overline{A} * \partial\overline{B}$ and inserting $\partial\overline{A} * \overline{B}$. We say that $\chi(A,B)$ is a* ***bistellar i-move*** *if the size of B is $i+1$. By interchanging the roles of A and B, we see that the inverse of a bistellar i-move is a bistellar $(d-i-1)$-move.*

Example 1.6 *In the 2-dimensional case (i.e. when $d=3$), there are three possible bistellar flips. In Figure 1(a), $|A| = |B| = 2$; this is called a 1-move. In Figure 1(b), $|A| = 1$ and $|B| = 3$; this is called a 2-move. The inverse moves also exist. In Figure 1(a) the inverse is a 1-move, and in Figure 1(b) the inverse is a 0-move.*

Definition 1.7 *Let A be a nonempty face in a combinatorial $(d-1)$-manifold Δ such that $\mathrm{lk}_\Delta(A) = \partial\overline{B} * L$ for some nonempty simplex B with $B \notin \Delta$ and some subcomplex $L \subseteq \Delta$. Then Δ is related to Δ' by the* ***stellar exchange*** *$\kappa(A,B)$, if Δ' is obtained by removing $\overline{A} * \partial\overline{B} * L$ from Δ and inserting $\partial\overline{A} * \overline{B} * L$.*



(a) A bistellar 1-move

(b) A bistellar 2-move and its inverse

Figure 1: The 2-dimensional bistellar flips

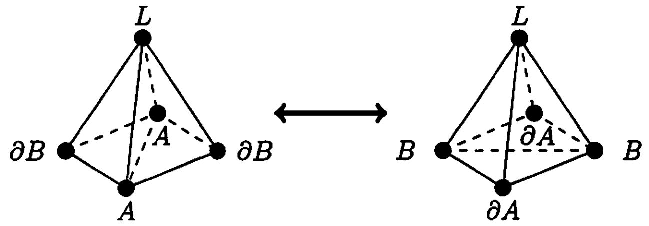
Example 1.8 *In Figure 2(a), we illustrate a stellar exchange with* $|A| = |B| = 2$ *and* $|L| = 1$. *In Figure 2(b), we illustrate a stellar exchange with* $|A| = 1$, $|B| = 3$, *and* $|L| = 1$.

Definition 1.9 *Suppose that* A *and* B *are faces of a combinatorial* $(d-1)$-*manifold* Δ *with boundary* $\partial\Delta$, *that* $A \cup B$ *is a facet of* Δ, *and that* $\overline{A} \cap \partial\Delta = \partial\overline{A}$ *and* $\overline{B} * \partial\overline{A} \subsetneq \partial\Delta$. *The manifold* Δ' *obtained from* Δ *by an* ***elementary shelling*** *from* $\overline{B}$ *is obtained from* Δ *by removing all faces of* Δ *containing* B.

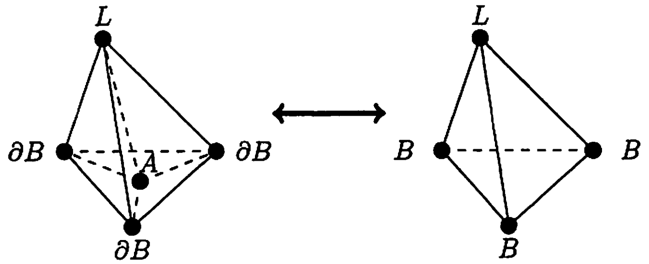
The fundamental property of these moves is that if Δ' is obtained from Δ by either a bistellar fip, a stellar exchange, or an elementary shelling, then $|\Delta|$ is PL homeomorphic to $|\Delta'|$. In fact, the converse to this is true as well, as is illustrated by the following theorem, which was originally proved by Newman [10] and Pachner [11].

Theorem 1.10 ([6, Theorem 5.10]) Two connected combinatorial $(d-1)$-manifolds with non-empty boundary are piecewise linear homeomorphic if and only if they are related by a sequence of elementary shellings, inverse shellings and a simplicial isomorphism.

In proving this theorem, Lickorish shows that any bistellar flip or stellar exchange can be written as a finite sequence of shelling and inverse



(a) A stellar exchange



(b) Another stellar exchange

Figure 2: Stellar exchanges

shelling operations. Thus, in order to prove that the geometric realizations of $\mathcal{ST}(k+1, d-k-1)$ and $\mathcal{B}(k,d)$ are homeomorphic, we need only show that they are related by a sequence of bistellar operations, stellar exchanges, and shelling/inverse shelling operations.

Definition 1.11 *Let Γ and Δ be simplicial complexes with vertex sets $V(\Gamma)$ and $V(\Delta)$ respectively. We say that Γ and Δ are* ***isomorphic*** *if there is a* ***bijection*** *$\varphi : V(\Gamma) \to V(\Delta)$ with inverse $\psi : V(\Delta) \to V(\Gamma)$ such that:*

- *For all faces $F = \{v_{i_1}, \dots, v_{i_k}\} \in \Gamma$, $\varphi(F) = \{\varphi(v_{i_1}), \dots, \varphi(v_{i_k})\}$ is a face of Δ.*
- *For all faces $G = \{u_{j_1}, \dots, u_{j_\ell}\} \in \Delta$, $\psi(G)$ is a face of Γ.*

At this point, we must introduce an additional property of combinatorial manifolds that will be used later in the proof of our main theorem.

Definition 1.12 *Let Δ be a pure $(d-1)$-dimensional simplicial complex. The* ***dual graph*** *of Δ, denoted $\mathcal{G}(\Delta)$, is the graph defined as follows. The vertices of $\mathcal{G}(\Delta)$ correspond to the facets of Δ, and two vertices in $\mathcal{G}(\Delta)$ are connected by an edge if and only if their corresponding facets intersect* *along a ridge. We say that Δ is* ***strongly connected*** *if the dual graph $\mathcal{G}(\Delta)$ is connected.*

Definition 1.13 *Let Δ be a pure $(d-1)$-dimensional simplicial complex. We say that Δ is a* ***pseudomanifold*** *if each ridge in Δ is contained in either one or two facets.*

Any combinatorial manifold is a pseudomanifold, but in general one should not expect a pseudomanifold to be a combinatorial manifold. In particular, if Δ is a combinatorial manifold, then $\mathrm{lk}_\Delta(F)$ is a strongly connected pseudomanifold for any nonempty face $F \in \Delta$.

We use this fact to prove the following lemma.

Lemma 1.14 *Let Δ be a $(d-1)$-dimensional combinatorial manifold. Suppose A and B are disjoint sets of vertices in Δ such that*

1. $|A| + |B| = d + 1$,
2. $A \in \Delta$, *and*
3. $\mathrm{lk}_\Delta(A) \supseteq \partial\overline{B}$.

Then $\mathrm{lk}_\Delta(A) = \partial\overline{B}$. Specifically, if we further assume that $B \notin \Delta$, then it is possible to perform the bistellar operation $\chi(A, B)$ on Δ.

Proof: Suppose that $\mathrm{lk}_\Delta(A)$ is $(r-1)$-dimensional so that $|B| = r+1$. We first observe that any vertex in $\mathcal{G}(\mathrm{lk}_\Delta(A))$ has degree at most r since any facet of $\mathrm{lk}_\Delta(A)$ contains r-many ridges and each such ridge is incident to at most one other facet. Moreover, since $\mathrm{lk}_\Delta(A)$ contains $\partial\overline{B}$, it follows that $\mathcal{G}(\partial\overline{B}) \subseteq \mathcal{G}(\mathrm{lk}_\Delta(A))$; and we can easily check that $\mathcal{G}(\partial\overline{B})$ is the complete graph on $r+1$ vertices.

Suppose now that there is a facet $\sigma \in \mathrm{lk}_\Delta(A)$ that does not belong to $\partial\overline{B}$, and let F be a facet of $\partial\overline{B}$. Since $\mathcal{G}(\mathrm{lk}_\Delta(A))$ is a connected graph, there is a path $F = F_0, F_1, \ldots, F_t = \sigma$ of vertices in $\mathcal{G}(\mathrm{lk}_\Delta(A))$ such that F_{i-1} is adjacent to F_i for all i. Consider the smallest index j such that F_j is a facet of $\partial\overline{B}$ but F_{j+1} is not. The vertex F_j has degree at least $r+1$ (r neighbors in $\partial\overline{B}$ in addition to F_{j+1}), which contradicts the degree bound established earlier. □

1.2 The complex $\mathcal{ST}(m, n)$

Fix nonnegative integers m and n. In this section, we define a modification of the staircase triangulation of a product of two simplices to give a triangulation of $\partial\sigma^m \times \sigma^n$. We begin by defining the staircase triangulation of the Cartesian product of two simplices.

Let $p_0, \ldots, p_m$ be the vertices of the m-dimensional simplex σ^m, and let $q_0, \ldots, q_n$ be the vertices of the n-dimensional simplex σ^n. The vertices of $\sigma^m \times \sigma^n$ all have the form (p_i, q_j) with $0 \leq i \leq m$ and $0 \leq j \leq n$.

Let $\mathcal{L}$ denote the $n \times m$ grid in the xy-plane whose lower-left corner is $(0,0)$ and whose upper-right corner is (m,n). We can identify each integer lattice point (i,j) in $\mathcal{L}$ with the vertex (p_i, q_j) in $\sigma^m \times \sigma^n$.

Definition 1.15 (see, e.g., [2]) The **staircase triangulation** of $\sigma^m \times \sigma^n$ is the simplicial complex whose facets correspond to all lattice paths from $(0,0)$ to (m,n) in $\mathcal{L}$ with steps in directions $\langle 1,0 \rangle$ or $\langle 0,1 \rangle$.

Let Γ and Δ be simplicial complexes on totally ordered vertex sets. Having defined the staircase triangulation of a product of simplices, we can define a simplicial complex called the **Cartesian product** [3] or **staircase refinement** [12, 4] of $|\Gamma| \times |\Delta|$ as follows. Let F be a d_1-dimensional face in Γ and let G be a d_2-dimensional face in Δ. We triangulate the cell $|F| \times |G| \subseteq |\Gamma| \times |\Delta|$ by using the staircase triangulation arising from the $d_2 \times d_1$ lattice whose columns are indexed by the vertices of F, and whose rows are indexed by the vertices of G, ordered according to the total order on the vertex sets of $V(\Gamma)$ and $V(\Delta)$.

We define a simplicial complex $\mathcal{ST}(m,n)$ on vertex set $\{(p_i, q_j) : 0 \leq i \leq m, 0 \leq j \leq n\}$ to be the staircase refinement of $\partial\sigma^m \times \sigma^n$. Specifically, the facets of $\mathcal{ST}(m,n)$ are described as follows. For each integer $0 \leq r \leq m$, let $\mathcal{L}'_r$ be the $n \times (m-1)$ lattice whose columns are labeled $0, 1, \ldots, r-1, r+1, \ldots, m$. For each lattice path L in the lattice $\mathcal{L}'_r$ starting in the lower-left corner, ending in the upper right corner, and taking only north and east steps, we form a facet in $\mathcal{ST}(m,n)$ whose vertices are the coordinates of integer points (p_i, q_j) on the lattice path L.

Example 1.16 *We label the vertices of the* 1×2 *lattice* $\mathcal{L}$ *as shown in Figure 3 . The resulting simplicial complex* $\mathcal{ST}(2,1)$ *is shown in Figure 4.*

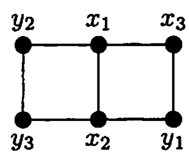


Figure 3: The lattice $\mathcal{L}$ defining $\mathcal{ST}(2,1)$

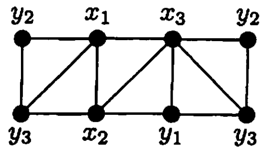


Figure 4: The simplicial complex $\mathcal{ST}(2,1)$

1.3 The complex $\mathcal{B}(k,d)$

Next we define a family of simplicial complexes denoted $\mathcal{B}(k,d)$ for all nonnegative integers k and d with $0 \le k < d$. See [5] for further information on the complexes $\mathcal{B}(k,d)$.

The boundary of the d-dimensional *cross-polytope*, which we denote by C_d^*, has vertex set $V(C_d^*) = \{x_1,\dots,x_d,y_1,\dots,y_d\}$ and its facets are all sets of the form $\{Z_1,...,Z_d\}$ such that $Z_i \in \{x_i,y_i\}$ for all i. As such, we may identify each facet F of C_d^* with a word $W(F) = W_1\cdots W_d$ in the letters x and y with $W_i = x$ if $Z_i = x_i$ and $W_i = y$ if $Z_i = y_i$. We define the **switch** set of such a word to be

$$\mathcal{S}(W(F)) := \{i : W_i \neq W_{i+1}, 1 \le i \le d-1\},$$

and we say that the facet F has m switches if $|\mathcal{S}(W(F))| = m$.

With this notation established, we define $\mathcal{B}(k,d)$ to be the simplicial complex on vertex set $\{x_1,\dots,x_d,y_1,\dots,y_d\}$ whose facets are all facets of C_d^* with *at most* k switches.

Example 1.17 *The complex $\mathcal{B}(1,3)$ is shown in Figure 5.*

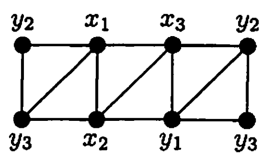


Figure 5: The complex $\mathcal{B}(1,3)$.

Notice that $\mathcal{B}(1,3)$ (Figure 5) can be obtained from $\mathcal{ST}(2,1)$ (Figure 4) by performing the bistellar operation $\chi(A,B)$ with $A = \{x_3,y_3\}$ and $B = \{y_1,y_2\}$. Geometrically, this is a bistellar 1-move as shown in Figure 1(a).

2 An isomorphism between $\mathcal{B}(0,d)$ and $\mathcal{ST}(1,d-1)$

We begin by observing that $\mathcal{ST}(1,d-1)$ is isomorphic to $\mathcal{B}(0,d)$ when we choose an appropriate labeling of the lattice points in the $(d-1)\times 1$ lattice. Specifically, when the vertices of $\mathcal{ST}(1,d-1)$ are labeled as in Figure 6, the facets of $\mathcal{ST}(1,d-1)$ are $\{x_1, x_2, ..., x_d\}$ and $\{y_1, y_2, ..., y_d\}$. These are also the facets of $\mathcal{B}(0,d)$. Therefore the labeling of the lattice in Figure 6 gives an isomorphism between $\mathcal{B}(0,d)$ and $\mathcal{ST}(1,d-1)$.

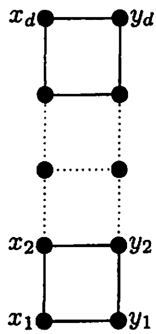


Figure 6: The lattice defining $\mathcal{ST}(1,d-1)$

3 The bistellar equivalence of $\mathcal{ST}(1,d-1)$ and $\mathcal{B}(d-2,d)$

In this section we will define an algorithm that generates a bistellar equivalence between $\mathcal{ST}(d-1,1)$ and $\mathcal{B}(d-2,d)$ for all $d \geq 3$. First, we must introduce the reverse lexicographic (revlex) order on the collection of subsets of $[N] := \{1,\dots,N\}$.

Definition 3.1 *The reverse lexicographic order on the collection of subsets of $[N]$ is defined by declaring that $F \prec G$ if and only if the maximum element of the symmetric difference of F and G belongs to G.*

Example 3.2 *The revlex order on subsets of $\{1,2,3\}$ is:*

$$\{1\} \prec \{2\} \prec \{1,2\} \prec \{3\} \prec \{1,3\} \prec \{2,3\} \prec \{1,2,3\}.$$

Label the vertices of $\mathcal{ST}(d-1,1)$ according to the lattice $\mathcal{L}$ shown in Figure 7, so that for all $1 \leq i \leq d$,

- $z_i := \begin{cases} y_i & \text{if } i \text{ is odd} \\ x_i & \text{if } i \text{ is even;} \end{cases}$

- $w_i := \begin{cases} x_i & \text{if } i \text{ is odd} \\ y_i & \text{if } i \text{ is even.} \end{cases}$

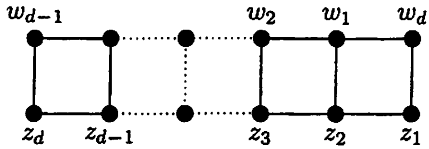


Figure 7: The lattice $\mathcal{L}$ defining $\mathcal{ST}(d-1,1)$.

Definition 3.3 *Let $\mathcal{T} := \{T \subseteq [d-1] : |T| \geq 2\}$. For each $T = \{t_1 < \cdots < t_\ell\} \in \mathcal{T}$, we define sets A_T and B_T by*

- $B_T := \{z_{t_1}, z_{t_2}, \ldots, z_{t_{\ell-1}}, w_{t_\ell}\}$, *and*
- $A_T := \{w_i : i \notin T, i < t_\ell\} \cup \{z_i : t_\ell < i \leq d-1\} \cup \{z_d, w_d\}$.

As our main result of this paper, we claim that performing the bistellar flips $\chi(A_T, B_T)$ sequentially according to the revlex order on $\mathcal{T}$ gives a bistellar equivalence between $\mathcal{ST}(1, d-1)$ and $\mathcal{B}(d-2, d)$.

Before we go on to state and prove our main result, let us pause to discuss the motivation behind this choice of labeling and these choices of A_T and B_T. First, observe that $\mathcal{B}(d-2, d)$ is generated by all facets of C_d^* with at most $d-2$ switches. In other words, it has all facets except $\{x_1, y_2, x_3, y_4, \ldots\}$ and $\{y_1, x_2, y_3, x_4, \ldots\}$, which are the facets with exactly $d-1$ switches. We choose to label the top and bottom rows of $\mathcal{L}$ with these unwanted facets, since the modified staircase triangulation method will not produce them. Additionally, we shift the top row so that x_i and y_i are not contained in a common face of $\mathcal{ST}(1, d-1)$ for $1 \leq i \leq d-1$. This is because the facets in $\mathcal{ST}(1, d-1)$ are indexed by north/east lattice paths, and w_i lies northwest of z_i for $1 \leq i \leq d-1$.

In order to motivate the seemingly complicated sets A_T and B_T, we appeal to the labeling of the lattice $\mathcal{L}$ shown in Figure 7. In addition to the issue that $\{x_d, y_d\}$ is a face in $\mathcal{ST}(d-1, 1)$, we also observe that, for example, $\{y_1, y_2\}$ is not a face of $\mathcal{ST}(d-1, 1)$, but it is a face of $\mathcal{B}(d-2, d)$. More generally, any face σ in $\mathcal{B}(d-2, d)$ that does not belong to $\mathcal{ST}(d-1, 1)$ contains a pair of vertices z_i, w_j with $i < j$ (i.e. such that w_j lies northwest of z_i). We have made a canonical choice of missing faces B_T with the property that B_T contains one vertex from the top row of $\mathcal{L}$ that lies to the northwest of its other vertices, all of which lie in the bottom row of $\mathcal{L}$.

Having fixed this method for describing B_T, we now describe the corresponding face A_T. In order to justify this choice, it is actually easier to work backwards. If we consider $A_T \cup B_T$ as a set of vertices on the lattice $\mathcal{L}$, the element w_{t_ℓ} is the left-most element in the top row of $\mathcal{L}$. All other elements belonging to the top row of $\mathcal{L}$ belong to A_T; all elements positioned strictly southeast of w_{t_ℓ} belong to B_T; and all elements positioned either south or southwest of w_{t_ℓ} belong to A_T. Given a collection W of vertices from $\mathcal{ST}(d-1,1)$ such that $x_d \in W$, $y_d \in W$, and either x_i or y_i belongs to W for all $1 \leq i \leq d-1$, then either (1) W can be uniquely decomposed into corresponding sets A_T and B_T or (2) the vertices of W lie on a north/east lattice path in $\mathcal{L}$ from the lower-left to upper-right corner.

The vertices x_d and y_d are originally connected by an edge in $\mathcal{ST}(1,d-1)$, but they are not connected by an edge in $\mathcal{B}(d-2,d)$. In the last step $\chi(A_{[d-1]}, B_{[d-1]})$ corresponding to the revlex-maximal set $T = [d-1]$, we have $A_{[d-1]} = \{x_d, y_d\}$. Performing this bistellar operation disconnects x_d and y_d, and hence we can think of this sequence of bistellar operations as slowly disintegrating the link of the edge $\{x_d, y_d\}$.

The following theorem makes this argument rigorous.

Theorem 3.4 *Fix a positive integer $d \geq 3$. For all $T \subseteq [d-1]$ with $|T| \geq 2$, let A_T and B_T be the sets defined in Definition 3.3. Under the revlex order on the collection of such subsets T, the sequence of bistellar flips $\chi(A_T, B_T)$ transforms $\mathcal{ST}(1,d-1)$ into $\mathcal{B}(d-2,d)$.*

Before proving Theorem 3.4, we give an example illustrating this sequence of bistellar operations in the case that $d = 4$.

Example 3.5 *The following is an example of the bistellar equivalence between $\mathcal{ST}(3,1)$ and $\mathcal{B}(2,4)$.*

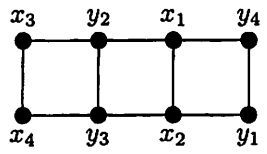


Figure 8: The lattice $\mathcal{L}$ defining $\mathcal{ST}(3,1)$

The facets of $\mathcal{ST}(3,1)$ according to Figure 8 are listed in the following array. The r-th column shows the facets obtained by removing the r-th column of $\mathcal{L}$.

$\{x_1, y_2, y_3, y_4\}$ $\{x_1, x_3, x_4, y_4\}$ $\{y_2, x_3, x_4, y_4\}$ $\{x_1, y_2, x_3, x_4\}$
$\{x_1, x_2, y_3, y_4\}$ $\{x_1, x_2, x_4, y_4\}$ $\{y_2, y_3, x_4, y_4\}$ $\{x_1, y_2, y_3, x_4\}$
$\{y_1, x_2, y_3, y_4\}$ $\{y_1, x_2, x_4, y_4\}$ $\{y_1, y_3, x_4, y_4\}$ $\{x_1, x_2, y_3, x_4\}$.

Now we define B_T and A_T according to Definition 3.3. After performing the following bistellar flips, we are left with the facets of $\mathcal{B}(2,4)$.

Bistellar Moves				
T	B_T	A_T	*Facets Removed*	*Facets Gained*
$\{1,2\}$	$\{y_1, y_2\}$	$\{y_3, x_4, y_4\}$	$\{y_1, y_3, x_4, y_4\}$ $\{y_2, y_3, x_4, y_4\}$	$\{y_1, y_2, x_4, y_4\}$ $\{y_1, y_2, y_3, x_4\}$ $\{y_1, y_2, y_3, y_4\}$
$\{1,3\}$	$\{y_1, x_3\}$	$\{y_2, x_4, y_4\}$	$\{y_1, y_2, x_4, y_4\}$ $\{y_2, x_3, x_4, y_4\}$	$\{y_1, x_3, x_4, y_4\}$ $\{y_1, y_2, x_3, x_4\}$ $\{y_1, y_2, x_3, y_4\}$
$\{2,3\}$	$\{x_2, x_3\}$	$\{x_1, x_4, y_4\}$	$\{x_1, x_2, x_4, y_4\}$ $\{x_1, x_3, x_4, y_4\}$	$\{x_2, x_3, x_4, y_4\}$ $\{x_1, x_2, x_3, x_4\}$ $\{x_1, x_2, x_3, y_4\}$
$\{1,2,3\}$	$\{y_1, x_2, x_3\}$	$\{x_4, y_4\}$	$\{y_1, x_2, x_4, y_4\}$ $\{y_1, x_3, x_4, y_4\}$ $\{x_2, x_3, x_4, y_4\}$	$\{y_1, x_2, x_3, x_4\}$ $\{y_1, x_2, x_3, y_4\}$

In order to simplify the proof of Theorem 3.4, we prove some technical lemmas here.

Lemma 3.6 *Let F be a facet of $\mathcal{ST}(d-1,1)$ that contains both x_d and y_d. Then there is a unique $T \in \mathcal{T}$ such that F is a facet of $\overline{A}_T * \partial\overline{B}_T$.*

Proof: We view F as a north/east lattice path obtained from $\mathcal{L}$ by removing the column whose vertices are w_j and z_{j+1} for some $1 \le j < d-1$. We claim that exactly one of z_j and w_{j+1} belongs to F. This is because F contains both x_d and y_d, so there is only one index $1 \le p \le d-1$ such that F contains neither x_p nor y_p. Since z_j lies southeast of w_{j+1} in $\mathcal{L}$, it is not possible that both z_j and w_{j+1} belong to F. We examine these two possibilities separately.

Case 1: $z_j \in F$

Let i be the smallest index in $[d-1]$ such that $z_i \in F$ and let $T := \{i, i+1, \ldots, j+1\}$. Then

$$\begin{aligned} B_T &= \{z_i, z_{i+1}, \ldots, z_j, w_{j+1}\}, \text{ and} \\ A_T &= \{z_d, z_{d-1}, \ldots, z_{j+2}\} \cup \{w_{i-1}, w_{i-2}, \ldots, w_1, w_d\}, \end{aligned}$$

as shown in Figure 9 with the vertices of B_T colored black, the vertices of A_T colored white, and the corresponding lattice path shown as a dashed line. We see that $F = A_T \cup (B_T \setminus \{w_{j+1}\}) \in \overline{A}_T * \partial\overline{B}_T$.

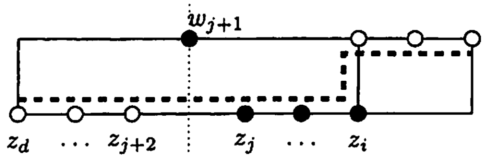


Figure 9:

Case 2: $w_{j+1} \in F$

Let q be the largest index in $[d-1]$ such that $w_q \in F$ and let $T = \{j, q\}$. Then

$$\begin{aligned} B_T &= \{z_j, w_q\}, \text{ and} \\ A_T &= \{z_d, z_{d-1}, \ldots, z_{q+1}\} \cup \{w_{q-1}, \ldots, w_{j+1}, w_{j-1}, \ldots, w_1, w_d\}, \end{aligned}$$

as shown in Figure 10 with the vertices of B_T colored black, the vertices of A_T colored white, and the corresponding lattice path shown as a dashed line.

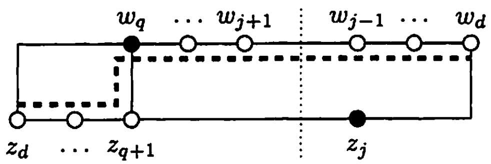


Figure 10:

Again we see that $F = A_T \cup (B_T \setminus \{z_j\}) \in \overline{A}_T * \partial\overline{B}_T$. □

Lemma 3.7 *Let F be a facet of $\overline{A}_T * \partial\overline{B}_T$ for some $T \in \mathcal{T}$. Then either*

1. *F is a facet of $S\mathcal{T}(d-1,1)$; or*
2. *there is a unique $S \in \mathcal{T}$ such that $S \prec T$ and $F \in \partial\overline{A}_S * \overline{B}_S$.*

Proof: We write $T = \{t_1 < \cdots < t_\ell\}$ so that $B_T = \{z_{t_1}, \ldots, z_{t_{\ell-1}}, w_{t_\ell}\}$. We must examine two possibilities based on the element of B_T that is removed from $A_T \cup B_T$ to form the facet F.

Case 1: $F = A_T \cup (B_T \setminus \{w_{t_\ell}\})$

Suppose first that there is no index $1 \leq j < t_\ell$ such that $w_j \in A_T$. In this case, $F = \{z_d, \ldots, z_{t_\ell+1}, z_{t_\ell-1}, \ldots, z_1, w_d\}$ is a facet of $ST(d-1,1)$.

Otherwise, consider the largest index $1 \le j < t_\ell$ such that $w_j \in A_T$, and let $S = \{t_i \in T : t_i < j\} \cup \{j\}$. We see that $S \prec T$ since t_ℓ is the largest element of the symmetric difference of S and T. Then

$$\begin{aligned} B_S &= \{z_{t_i} : t_i \in T, t_i < j\} \cup \{w_j\}, \text{ and} \\ A_S &= \{z_i : i > j\} \cup \{w_i : i < j, i \notin T\} \cup \{x_d, y_d\}, \end{aligned}$$

and we may check that $F = (A_S \setminus \{z_{t_\ell}\}) \cup B_S \in \partial\overline{A}_S * \overline{B}_S$.

Case 2: $F = A_T \cup (B_T \setminus \{z_{t_j}\})$ for some $1 \le j < \ell$

If $|T| = 2$, then $F = \{z_d, \ldots, z_{t_2+1}, w_{t_2}, \ldots, w_{t_1+1}, w_{t_1-1}, \ldots, w_1, w_d\}$ is a facet of $\mathcal{ST}(d-1, 1)$ since F contains neither w_{t_1} nor z_{t_1+1}. This is illustrated in Figure 11 with the vertices in B_T colored black, the vertices in A_T colored white, the eliminated column shown with a dotted line, and the corresponding lattice path shown as a dashed line.

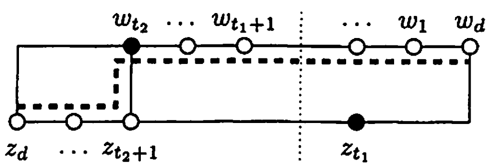


Figure 11:

Otherwise, if $|T| > 2$, consider $S := T \setminus \{t_j\}$. Then $|S| \ge 2$ so that $S \in \mathcal{T}$ and $S \prec T$ since S is a subset of T. In this case, we see that

$$\begin{aligned} B_S &= B_T \setminus \{z_{t_j}\}, \text{ and} \\ A_S &= A_T \cup \{w_{t_j}\}, \end{aligned}$$

so that $F = A_T \cup (B_T \setminus \{z_{t_j}\}) = (A_S \setminus \{w_{t_j}\}) \cup B_S \in \partial\overline{A}_S * \overline{B}_S$. □

Lemma 3.8 *Let F be a facet of $\partial\overline{A}_S * \overline{B}_S$ for some $S \in \mathcal{T}$ such that x_d and y_d belong to F. Then there is a unique $T \in \mathcal{T}$ such that $S \prec T$ and F is a facet of $\overline{A}_T * \partial\overline{B}_T$.*

Proof: Since F contains both x_d and y_d, there is exactly one index $1 \le j \le d-1$ such that neither x_j nor y_j belongs to F. Note that exactly one of x_j or y_j belongs to A_S. Say $S = \{s_1 < \cdots < s_m\}$.

Case 1: $j < s_m$

In this case, $w_j \in A_S$. Consider $T = S \cup \{j\}$. Clearly $S \prec T$ since S is a subset of T, and we see that

$$\begin{aligned} B_T &= B_S \cup \{z_j\}, \text{ and} \\ A_T &= A_S \setminus \{w_j\}. \end{aligned}$$

Thus $F = (A_S \setminus \{w_j\}) \cup B_S = A_T \cup B_T \setminus \{z_j\} \in \overline{A}_T * \partial \overline{B}_T$.

Case 2: $j > s_m$

In this case, $z_j \in A_S$. Consider $T = \{s_1, \ldots, s_{m-1}\} \cup \{s_m + 1, \ldots, j\}$. Then $S \prec T$ since $j > s_m$, and we see that

$$\begin{aligned} B_T &= (B_S \setminus \{w_{s_m}\}) \cup \{z_{s_m+1}, \ldots, z_{j-1}, w_j\}, \text{ and} \\ A_T &= (A_S \setminus \{z_{s_m+1}, \ldots, z_{j-1}\}) \cup \{w_{s_m}\}. \end{aligned}$$

Thus $F = A_T \cup B_T \setminus \{w_j\} \in \overline{A}_T * \partial \overline{B}_T$.

□

Lemma 3.9 *Let F be a facet of $\mathcal{B}(d-2, d)$. Then either*

1. *F is a facet of $\mathcal{ST}(d-1, 1)$, or*

2. *there is a unique $T \in \mathcal{T}$ such that F is a facet of $\partial \overline{A}_T * \overline{B}_T$.*

Proof: Since $\{z_1, \ldots, z_d\}$ and $\{w_1, \ldots, w_d\}$ are the two facets of C_d^* that do not belong to $\mathcal{B}(d-2, d)$, we see that F must contain at least one vertex from the top row of $\mathcal{L}$ and at least one vertex from the bottom row of $\mathcal{L}$.

Consider the largest index t such that $w_t \in F$. If $t = d$, then $F = \{z_{d-1}, z_{d-2}, \ldots, z_1, w_d\}$ is a facet of $\mathcal{ST}(d-1, 1)$. Otherwise, if $t \leq d-1$, consider the set $I := \{i < t : z_i \in F\} \subseteq [d-1]$.

Case 1: $I = \emptyset$

In this case, F contains $\{z_{d-1}, \ldots, z_{t+1}, w_t, \ldots, w_1\}$ and either z_d or w_d. In either case, F corresponds to a lattice path obtained by removing either the first or last column from $\mathcal{L}$. This is illustrated in Figure 12, where the dotted circles indicate that either x_d or y_d can be added, and the corresponding lattice path is shown as a dashed line.

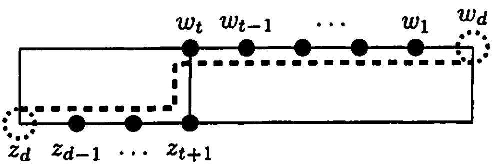


Figure 12:

Case 2: $I \neq \emptyset$

In this case, we let $T := I \cup \{t\}$. Clearly F is either $(A_T \setminus \{x_d\}) \cup B_T$ or $(A_T \setminus \{y_d\}) \cup B_T$ and hence $F \in \partial \overline{A}_T * \overline{B}_T$. □

Proof of Theorem 3.4:

Let us begin by establishing the notation that will be used for the remainder of the proof. As above, let $\mathcal{L}$ be the lattice shown in Figure 7. Let $\mathcal{T}$ denote the collection of all subsets of $[d-1]$ of size at least two, and let $M := 2^{d-1} - d = |\mathcal{T}|$. Order the sets in $\mathcal{T}$ under the revlex order as $T_1 \prec T_2 \prec T_3 \prec \cdots \prec T_M$. Let $\Delta_0 = \mathcal{ST}(d-1,1)$, and for all $1 \leq j \leq M$, let Δ_j be the simplicial complex obtained from Δ_{j-1} by performing the bistellar operation $\chi(A_{T_j}, B_{T_j})$.

In order to prove this theorem, we must prove that for all j,

1. $A_{T_j} \in \Delta_{j-1}$,

2. $B_{T_j} \notin \Delta_{j-1}$,

3. $\text{lk}_{\Delta_{j-1}}(A_{T_j}) = \partial\overline{B}_{T_j}$, and

4. $\Delta_M = \mathcal{B}(d-2,d)$.

Conditions (1)–(3) say that the bistellar operation $\chi(A_{T_j}, B_{T_j})$ can be performed on the complex Δ_{j-1} for all $1 \leq j \leq M$; condition (4) says that only those facets belonging to $\mathcal{B}(d-2,d)$ remain after performing the bistellar operations $\chi(A_{T_1}, B_{T_1}), \cdots, \chi(A_{T_M}, B_{T_M})$.

Let F be a facet of $\overline{A}_{T_j} * \partial\overline{B}_{T_j}$. Since F contains all the vertices in A_{T_j}, both x_d and y_d belong to F. Thus by Lemma 3.7, F was either originally a facet of $\mathcal{ST}(d-1,1)$ or was created as a facet of $\partial\overline{A}_S * \overline{B}_S$ for some $S \prec T_j$. By Lemma 3.6 (in the former case) and Lemma 3.7 (in the latter case), T_j is the unique subset of $[d-1]$ such that F is a facet of $\overline{A}_{T_j} * \partial\overline{B}_{T_j}$. In particular, this means that F is a facet of Δ_{j-1}, which proves that A_{T_j} is a face of Δ_{j-1} as well. Moreover, by the structure of our choices of the sets B_T, we see that B_{T_j} is not a face of Δ_{j-1}.

Next, we show that $\text{lk}_{\Delta_{j-1}}(A_{T_j}) = \partial\overline{B}_{T_j}$. By the argument in the previous paragraph, we see that each facet of $\overline{A}_{T_j} * \partial\overline{B}_{T_j}$ belongs to Δ_{j-1}. Thus $\text{lk}_{\Delta_{j-1}}(A_{T_j}) \supseteq \partial\overline{B}_{T_j}$ and $\text{lk}_{\Delta_{j-1}}(A_{T_j}) = \partial\overline{B}_{T_j}$ by Lemma 1.14.

Finally, suppose σ is a facet of $\mathcal{B}(d-2,d)$. By Lemma 3.9, either σ is a facet of $\mathcal{ST}(d-1,1)$ or σ was created as a facet of $\partial\overline{A}_S * \overline{B}_S$ for some $S \in \mathcal{T}$. Since σ contains either x_d or y_d, but not both, σ will not be removed as a facet of $\overline{A}_T * \partial\overline{B}_T$ for any $T \in \mathcal{T}$. Thus σ is a facet of Δ_M. Moreover, any facet of Δ_M contains exactly one of x_i and y_i for all $1 \leq i \leq d$. Since $\{z_1, \ldots, z_d\}$ and $\{w_1, \ldots, w_d\}$ are not created as facets of $\partial\overline{A}_T * \overline{B}_T$ for any set $T \in \mathcal{T}$, we conclude that $\Delta_M = \mathcal{B}(d-2,d)$.

□

4 Other cases

Theorem 4.1 *The complex $\mathcal{B}(1,4)$ can be obtained from $\mathcal{ST}(2,2)$ through a series of bistellar flips, stellar exchanges, and elementary shellings.*

Proof: We label the vertices of $\mathcal{ST}(2,2)$ as shown in Figure 13

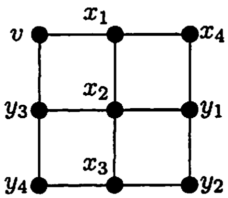


Figure 13:

Under this labeling, the facets of $\mathcal{ST}(2,2)$ are

$$\begin{array}{lll}\{x_1,x_2,x_3,x_4\}, & \{v,y_3,x_4,y_4\}, & \{v,x_1,y_3,y_4\}, \\ \{y_1,x_2,x_3,x_4\}, & \{y_1,y_3,x_4,y_4\}, & \{x_1,x_2,y_3,y_4\}, \\ \{y_1,y_2,x_3,x_4\}, & \{y_1,y_2,x_4,y_4\}, & \{x_1,x_2,x_3,y_4\}.\end{array}$$

In comparison with the previously studied cases, we now have two issues to overcome in proving this theorem. We still must disintegrate the link of the edge $\{x_4,y_4\}$ while adding in the missing faces $\{y_2,y_3\}$ and $\{x_1,y_2\}$ (amongst others). We also must remove the vertex v by using either elementary shellings or a bistellar 4-move. We begin by performing the following bistellar operations and stellar exchanges.

Step	A	B	L	Facets Removed	Facets Gained
1.	$\{y_1,x_4,y_4\}$	$\{y_2,y_3\}$	–	$\{y_1,y_2,x_4,y_4\}$ $\{y_1,y_3,x_4,y_4\}$	$\{y_1,y_2,y_3,x_4\}$ $\{y_1,y_2,y_3,y_4\}$ $\{y_2,y_3,x_4,y_4\}$
2.	$\{x_4,y_4\}$	$\{v,y_2\}$	$\{y_3\}$	$\{v,y_3,x_4,y_4\}$ $\{y_2,y_3,x_4,y_4\}$	$\{v,y_2,y_3,x_4\}$ $\{v,y_2,y_3,y_4\}$
3.	$\{v,y_4\}$	$\{x_1,y_2\}$	$\{y_3\}$	$\{v,x_1,y_3,y_4\}$ $\{v,y_2,y_3,y_4\}$	$\{v,x_1,y_2,y_3\}$ $\{x_1,y_2,y_3,y_4\}$

Let us denote by Δ the simplicial complex obtained from $\mathcal{ST}(2,2)$ by performing these three operations. We observe that the vertex v is only contained in the facets $\{v,x_1,y_2,y_3\}$ and $\{v,y_2,y_3,x_4\}$ in Δ. By using elementary shellings, we wish to remove these two facets, which will remove the vertex v and leave us with precisely those facets in $\mathcal{B}(1,4)$.

We begin by considering the facet $\{v, x_1, y_2, y_3\}$. The link of the face $F := \{v, x_1\}$ is the simplex on $G := \{y_2, y_3\}$. We claim that G is an interior face of Δ, that its boundary faces $\{y_2\}$ and $\{y_3\}$ belong to the boundary of Δ, and that $\overline{F} * \partial\overline{G}$ is contained in the boundary of Δ. Since the boundary of Δ is a subcomplex of Δ, we need only check that F is an interior face of Δ and that $\overline{F} * \partial\overline{G}$ is contained in the boundary of Δ.

First we show that F is an interior face of Δ. This can be easily checked as $\text{lk}_\Delta(F)$, when viewed as a graph, is a cycle on the vertices v, x_4, y_1, y_4, x_1, v. To see that $\overline{F} * \partial\overline{G}$ is contained in the boundary of Δ, we check that $\{v, x_1, y_2, y_3\}$ is the unique face of Δ that contains the two-dimensional faces of $\overline{F} * \partial\overline{G}$: $\{v, x_1, y_2\}$ and $\{v, x_1, y_3\}$. Thus we may remove the facet $\{v, x_1, y_2, y_3\}$ using an elementary shelling to obtain a new complex Δ'.

Now we consider the facet $\{v, y_2, y_3, x_4\}$, which is the only remaining facet of Δ' that contains v. As before, we let $F' = \{v\}$ and $G' = \{y_2, y_3, x_4\}$. We can check that G' is an interior face of Δ' since it is contained in two facets, $\{v, y_2, y_3, x_4\}$ and $\{y_1, y_2, y_3, x_4\}$, and that $\overline{F}' * \partial\overline{G}'$ is contained in the boundary of Δ'.

□

Theorem 4.2 *The complex $\mathcal{B}(1,5)$ can be obtained from $\mathcal{ST}(2,3)$ through a series of bistellar flips, stellar exchanges, and elementary shellings.*

Proof:

We label the vertices of $\mathcal{ST}(2,3)$ as shown in Figure 14.

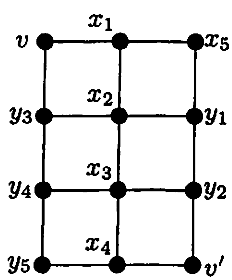


Figure 14:

Under this labeling, the facets of $\mathcal{ST}(2,3)$ are

$\{v, x_1, y_3, y_4, y_5\}$, $\{v, y_3, y_4, x_5, y_5\}$, $\{x_1, x_2, x_3, x_4, x_5\}$,
$\{x_1, x_2, y_3, y_4, y_5\}$, $\{y_1, y_3, y_4, x_5, y_5\}$, $\{y_1, x_2, x_3, x_4, x_5\}$,
$\{x_1, x_2, x_3, y_4, y_5\}$, $\{y_1, y_2, y_4, x_5, y_5\}$, $\{y_1, y_2, x_3, x_4, x_5\}$,
$\{x_1, x_2, x_3, x_4, y_5\}$, $\{v', y_1, y_2, x_5, y_5\}$, $\{v', y_1, y_2, x_4, x_5\}$.

We begin by viewing the 3×2 lattice in Figure 14 as two overlapping copies of the 2×2 lattice from Figure 13. We perform the following bistellar/stellar operations, which are motivated by the three initial operations used in the proof of Theorem 4.1.

Step	A	B	L	Facets Removed	Facets Gained
1.	$\{y_1, y_4, x_5, y_5\}$	$\{y_2, y_3\}$	–	$\{y_1, y_2, y_4, x_5, y_5\}$ $\{y_1, y_3, y_4, x_5, y_5\}$	$\{y_1, y_2, y_3, y_4, y_5\}$ $\{y_1, y_2, y_3, y_4, x_5\}$ $\{y_1, y_2, y_3, x_5, y_5\}$ $\{y_2, y_3, y_4, x_5, y_5\}$
2.	$\{y_4, x_5, y_5\}$	$\{v, y_2\}$	$\{y_3\}$	$\{v, y_3, y_4, x_5, y_5\}$ $\{y_2, y_3, y_4, x_5, y_5\}$	$\{v, y_2, y_3, y_4, y_5\}$ $\{v, y_2, y_3, y_4, x_5\}$ $\{v, y_2, y_3, x_5, y_5\}$
3.	$\{v, y_4, y_5\}$	$\{x_1, y_2\}$	$\{y_3\}$	$\{v, x_1, y_3, y_4, y_5\}$ $\{v, y_2, y_3, y_4, y_5\}$	$\{x_1, y_2, y_3, y_4, y_5\}$ $\{v, x_1, y_2, y_3, y_4\}$ $\{v, x_1, y_2, y_3, y_5\}$
4.	$\{y_1, x_5, y_5\}$	$\{v', y_3\}$	$\{y_2\}$	$\{v', y_1, y_2, x_5, y_5\}$ $\{y_1, y_2, y_3, x_5, y_5\}$	$\{v', y_1, y_2, y_3, x_5\}$ $\{v', y_1, y_2, y_3, y_5\}$ $\{v', y_2, y_3, x_5, y_5\}$
5.	$\{v', y_1, x_5\}$	$\{y_3, x_4\}$	$\{y_2\}$	$\{v', y_1, y_2, y_3, x_5\}$ $\{v', y_1, y_2, x_4, x_5\}$	$\{y_1, y_2, y_3, x_4, x_5\}$ $\{v', y_1, y_2, y_3, x_4\}$ $\{v', y_2, y_3, x_4, x_5\}$

Let Δ denote the simplicial complex obtained by performing these five operations. In addition to all of the facets of $\mathcal{B}(1,5)$, Δ contains the following facets, which may be removed from Δ in the order that they are listed through a series of elementary shellings. We list the decomposition of each facet F into the interior face A and boundary face B such that $\mathrm{lk}(B) = \overline{A}$ and $\overline{B} * \partial\overline{A} \subseteq \partial\Delta$.

Step	Facet	A	B
6.	$\{v', y_2, y_3, x_4, x_5\}$	$\{y_2, y_3\}$	$\{v', x_4, x_5\}$
7.	$\{v', y_1, y_2, y_3, x_4\}$	$\{y_1, y_2, y_3\}$	$\{v', x_4\}$
8.	$\{v', y_1, y_2, y_3, y_5\}$	$\{y_2, y_3, y_5\}$	$\{v', y_1\}$
9.	$\{v', y_2, y_3, x_5, y_5\}$	$\{y_2, y_3, x_5, y_5\}$	$\{v'\}$
10.	$\{v, y_2, y_3, x_5, y_5\}$	$\{v, y_2, y_3\}$	$\{x_5, y_5\}$
11.	$\{v, y_2, y_3, y_4, x_5\}$	$\{y_2, y_3, y_4\}$	$\{v, x_5\}$
12.	$\{v, x_1, y_2, y_3, y_4\}$	$\{x_1, y_2, y_3\}$	$\{v, y_4\}$
13.	$\{v, x_1, y_2, y_3, y_5\}$	$\{x_1, y_2, y_3, y_5\}$	$\{v\}$

After removing these facets by elementary shellings, we are left with preciscely those facets of $\mathcal{B}(1,5)$.

□

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