On the Signed (Total) k-Domination Number of a Graph*

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Abstract

Let k be a positive integer and $G = (V, E)$ be a graph of minimum degree at least $k-1$. A function $f : V \rightarrow \{-1, 1\}$ is called a *signed k-dominating function* of G if $\sum_{u \in N_G[v]} f(u) \geq k$ for all $v \in V$. The *signed k-domination number* of G is the minimum value of $\sum_{v \in V} f(v)$ taken over all signed k-dominating functions of G. The *signed total k-dominating function* and *signed total k-domination number* of G can be similarly defined by changing the closed neighborhood $N_G[v]$ to the open neighborhood $N_G(v)$ in the definition. The *upper signed k-domination number* is the maximum value of $\sum_{v \in V} f(v)$ taken over all *minimal* signed k-dominating functions of G. In this paper, we study these graph parameters from both algorithmic complexity and graph-theoretic perspectives. We prove that for every fixed $k \geq 1$, the problems of computing these three parameters are all $\mathcal{NP}$-hard. We also present sharp lower bounds on the signed k-domination number and signed total k-domination number for general graphs in terms of their minimum and maximum degrees, generalizing several known results about signed domination.

1 Introduction

All graphs considered in this paper are simple and undirected. We generally follow [4] for standard notation and terminology in graph theory. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is $|V(G)|$. For each vertex $v \in V(G)$, let $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, which are called the *open neighborhood* and *closed neighborhood* of v (in G), respectively. The *degree* of v (in G) is $d_G(v) = |N_G(v)|$. The *minimum degree* of G is $\delta(G) = \min_{v \in V(G)}\{d_G(v)\}$,

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and the *maximum degree* of G is $\Delta(G) = \max_{v\in V(G)}\{d_G(v)\}$. For an integer r, G is called *r-regular* if $\Delta(G) = \delta(G) = r$, and is called *nearly r-regular* if $\Delta(G) = r$ and $\delta(G) = r-1$. For $S \subseteq V(G)$, $G[S]$ is the subgraph of G *induced* by S; that is, $G[S]$ is a graph with vertex set S and edge set $\{uv \in E(G) \mid \{u,v\} \subseteq S\}$. For an integer $n \geq 1$, let K_n denote the complete graph of order n; i.e., K_n is an $(n-1)$-regular graph of order n. For any function $f : V(G) \to \mathbb{R}$, we write $f(S) = \sum_{v\in S} f(v)$ for all $S \subseteq V(G)$, and the *weight* of f is $w(f) = f(V(G))$.

Domination is an important subject in graph theory, and has numerous applications in other fields; see [11, 12] for comprehensive treatment and detailed surveys on (earlier) results in domination theory from both theoretical and applied perspectives. A set $S \subseteq V(G)$ is called a *dominating set* (resp. *total dominating set*) of G if $\bigcup_{v\in S} N_G[v] = V(G)$ (resp. $\bigcup_{v\in S} N_G(v) = V(G)$). The *domination number* (resp. *total domination number*) of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the minimum size of a dominating set (resp. total dominating set) of G.

Let $k \geq 1$ be a fixed integer and G be a graph of minimum degree at least $k-1$. A function $f : V(G) \to \{-1, 1\}$ is called a *signed k-dominating function* of G if $f(N_G[v]) \geq k$ for all $v \in V(G)$. The *signed k-domination number* of G, denoted by $\gamma_{kS}(G)$, is the minimum weight of a signed k-dominating function of G. When G is of minimum degree at least k, the *signed total k-dominating function* and *signed total k-domination number* of G (denoted by $\gamma^t_{kS}(G)$) can be analogously defined by changing the closed neighborhood $N_G[v]$ to the open neighborhood $N_G(v)$ in the definition. The concepts of signed k-domination number and signed total k-domination number are introduced in [16], where sharp lower bounds of these numbers are established for general graphs, bipartite graphs and r-regular graphs in terms of the order of the graphs. A related graph parameter called the *upper signed k-domination number* of G, denoted by $\Gamma_{kS}(G)$, is defined in [17] as the maximum weight of a *minimal* signed k-dominating function of G. (A signed k-dominating function f of G is called *minimal* if there exists no signed k-dominating function f' of G such that $f' \neq f$ and $f'(v) \leq f(v)$ for every $v \in V(G)$.) This parameter has also been studied in [3].

In the special case where $k = 1$, the signed k-domination number and signed total k-domination number are exactly the *signed domination number* [5] and *signed total domination number* [18], respectively. These two parameters have been extensively studied in the literature; see e.g. [1, 2, 5, 6, 7, 9, 13, 14, 18, 19] and the references therein.

In this paper, we continue the investigation of the signed k-domination number and signed total k-domination number of graphs, from both algorithmic complexity and graph theoretic points of view. In Section 2 we show that, for every fixed $k \geq 1$, the problems of computing the signed k-domination number, the signed total k-domination number, and the upper signed k-domination number of a graph are all $\mathcal{NP}$-hard. We then present, in Section 3, sharp lower bounds on the signed k-domination number and signed total k-domination number for general graphs in terms of their minimum and maximum degrees, from which several interesting results follow immediately.

2 Complexity of Signed (Total) k-Domination

In this section we first show the $\mathcal{NP}$-hardness of computing the signed k-domination number and signed total k-domination number of a graph for all $k \geq 1$. Since the proofs for the two parameters are very similar, we only detail the proof for the signed total k-domination number, and merely point out the changes that need to be made for establishing hardness for the signed k-domination number. We now formally define the two decision problems corresponding to the computation of these two graph parameters.

SIGNED k-DOMINATION PROBLEM (SkDP)
Instance: A graph $G = (V, E)$ and an integer r.
Question: Is $\gamma_{kS}(G) \leq r$?

SIGNED TOTAL k-DOMINATION PROBLEM (STkDP)
Instance: A graph $G = (V, E)$ and an integer r.
Question: Is $\gamma_{kS}^t(G) \leq r$?

Theorem 1. *For every integer $k \geq 1$, the STkDP problem is $\mathcal{NP}$-complete.*

Proof. Let $k \geq 1$ be a fixed integer. The STkDP problem is clearly in $\mathcal{NP}$. We now present a polynomial-time reduction from MINIMUM TOTAL DOMINATING SET (MTDS), which is a classical $\mathcal{NP}$-complete problem [8], to STkDP. The MTDS problem is defined as follows: Given a graph G and an integer r, decide whether G has a total dominating set of size at most r.

Let (G, r) be an instance of the MTDS problem. Construct another graph H as follows. First let H contain of a copy of G, which is denoted by G'. Also, for each vertex $v \in V(G)$, let v' denote its counterpart in G'. For each $v \in V(G)$, we add $t(v)$ disjoint copies of K_{k+2} to H, where $t(v) = d_G(v) + k - 2$; call these copies $K_{k+2}^{v,1}, K_{k+2}^{v,2}, \ldots, K_{k+2}^{v,t(v)}$. Then, for each $i \in \{1, 2, \ldots, t(v)\}$, add an edge between v' and an (arbitrary) vertex from $K_{k+2}^{v,i}$. This finishes the construction of H. It is easy to verify that $d_H(v') = 2d_G(v) + k - 2$ for all $v \in V(G)$.

Let $T = (k+2)\sum_{v \in V(G)} t(v) = (k+2)\sum_{v \in V(G)}(k + d_G(v) - 2)$ be the number of vertices in $V(H \setminus G')$. We will prove that $\gamma_t(G) \leq r$ if and only if $\gamma_{kS}^t(H) \leq 2r - |V(G)| + T$.

First consider the "if" direction. Assume that $\gamma_{kS}^t(H) \leq 2r-|V(G)|+T$, and $f : V(H) \to \{-1,1\}$ is a signed total k-dominating function of H of weight $\gamma_{kS}^t(H)$. Let $S' = \{v' \in V(G') \mid f(v') = 1\}$. It is easy to see that, for each $v \in V(G)$ and $1 \leq i \leq t(v)$, all vertices in $K_{k+2}^{v,i}$ must have function value "1" under f. It follows that $\gamma_{kS}^t(H) = w(f) = T + |S'| - (|V(G')| - |S'|) = 2|S'|-|V(G)|+T$. Since $\gamma_{kS}^t(H) \leq 2r-|V(G)|+T$, we have $|S'| \leq r$. Now define $S = \{v \in V(G) \mid v' \in S'\}$; i.e., S is the counterpart of S' in G. We show that S is a total dominating set of G. Assume to the contrary that S is not a total dominating set of G, and let $v \in V(G)$ be such that $N_G(v) \cap S = \emptyset$. By our definitions of S and S', $f(u') = -1$ for all $u \in N_G(v)$. Thus, $\sum_{x \in N_H(v')} f(x) \leq t(v) - d_G(v) = k-2$, contradicting with the fact that f is a signed total k-dominating function of H. Therefore, S' is indeed a total dominating set of G, from which $\gamma_t(G) \leq |S'| \leq r$ follows. This completes the proof for the "if" direction.

Now comes the "only if" part of the reduction. Suppose $\gamma_t(G) \leq r$ and $S \subseteq V(G)$ is a total dominating set of G of size at most r. Define a function $f : V(H) \to \{-1,1\}$ as follows: $f(x) = -1$ if $x = v'$ for some $v \in V(G) \setminus S$, and $f(x) = 1$ otherwise. The weight of f is $T + |S| - (|V(G)| - |S|) = 2|S|-|V(G)|+T \leq 2r-|V(G)|+T$. We now verify that f is a signed total k-dominating function of H. For each $x \in V(H \backslash G')$, $f(N_H(x)) \geq (k+1)-1 = k$. For each $v' \in V(G')$ (with $v \in V(G)$), since S is a total dominating set of G, $f(N_H(v')) \geq t(v)+1-(d_G(v)-1) = t(v)+2-d_G(v) = k$. Hence, f is a signed total k-dominating function of H of weight at most $2r-|V(G)|+T$. This completes the "only if" part of the reduction.

Therefore, $\gamma_t(G) \leq r$ if and only if $\gamma_{kS}^t(H) \leq 2r - |V(G)| + T$. This finishes the whole reduction, and hence concludes the proof of Theorem 1. □

Theorem 2. *For every integer $k \geq 1$, the SkDP problem is $\mathcal{NP}$-complete.*

Proof. The proof is very similar to that of Theorem 1, with two differences in the reduction. Therefore, we only describe the reduction. We reduce from the $\mathcal{NP}$-complete problem MINIMUM DOMINATING SET (which, given a graph G and an integer r, needs to decide whether G has a dominating set of size at most r) to SkDP. Let (G,r) be an instance of MINIMUM DOMINATING SET. Construct another graph H as follows. First let H contain of a copy of G, which is denoted by G'. For each vertex $v \in V(G)$, add $s(v)$ disjoint copies of K_{k+1} to H, where $s(v) = d_G(v) + k - 1$; call these copies $K_{k+1}^{v,1}, K_{k+1}^{v,2}, \ldots, K_{k+1}^{v,s(v)}$. Then, for each $i \in \{1,2,\ldots,s(v)\}$, add an edge between v' (the counterpart of v in G') and an arbitrary vertex from $K_{k+1}^{v,i}$. This finishes the construction of H. Using similar argument to that in Theorem 1, we can prove that $\gamma(G) \leq r$ if and only if $\gamma_{kS}(H) \le 2r - |V(G)| + T$, where $T = (k+1)\sum_{v \in V(G)} s(v)$. The $\mathcal{NP}$-completeness of SkDP is thus established. □

We now define the problem corresponding to the computation of the upper signed k-domination number of graphs as follows.

UPPER SIGNED k-DOMINATION PROBLEM (USkDP)
Instance: A graph $G = (V, E)$ and an integer r.
Question: Is $\Gamma_{kS}(G) \ge r$?

Theorem 3. *For every integer $k \ge 1$, the USkDP problem is $\mathcal{NP}$-complete.*

Proof. The USkDP problem is in $\mathcal{NP}$ because given a function $f : V(G) \to \{-1, 1\}$, we can verify in polynomial time whether f is a minimal signed k-dominating function of G using Lemma 4 in [3]. We will describe a polynomial time reduction from the 1-in-3 SAT problem to it. The 1-in-3 SAT problem is defined as follows: Given a Boolean formula in conjunctive normal form, each clause of which contains exactly three *positive* literals (i.e., variables with no negations), decide whether the formula is *1-in-3 satisfiable*, i.e., if there exists an assignment of the variables such that exactly one variable of each clause is assigned TRUE. This problem is known to be $\mathcal{NP}$-complete [15].

Let F be a Boolean formula with variables $\{x_1, x_2, \ldots, x_n\}$, which is an input of the 1-in-3 SAT problem. Assume $F = \bigwedge_{i=1}^{m} c_i$ where $c_i = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ for each $i \in \{1, 2, \ldots, m\}$. We construct a graph G as follows. Take m disjoint copies of K_{k+2}, each of which corresponds to a clause c_i with $i \in \{1, 2, \ldots, m\}$, and n disjoint copies of K_{k+3} (also disjoint from the copies of K_{k+2}'s) each of which corresponds to a variable x_j with $j \in \{1, 2, \ldots, n\}$. Delete one edge from each copy of K_{k+3}. We will call the copy of K_{k+2} corresponding to c_i the *i-th clause block*, and call the copy of K_{k+3} (with one edge missing) corresponding to x_j the *j-th variable block*. For each $i \in \{1, 2, \ldots, m\}$, let c'_i be an (arbitrary) vertex in the i-th clause block. For every $j \in \{1, 2, \ldots, n\}$, let x'_j and x''_j be the two vertices in the j-th variable block for which the edge $x'_j x''_j$ is removed. For each clause $c_i = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$, add three cross-block edges $c'_i x'_{i_1}, c'_i x'_{i_2}$, and $c'_i x'_{i_3}$. This finishes the construction of G. Note that $|V(G)| = (k+3)n + (k+2)m$.

We claim that $\Gamma_{kS}(G) \ge (k+1)n + (k+2)m$ if and only if F is 1-in-3 satisfiable. First consider the "if" direction, and let $\mathcal{A} : \{x_1, x_2, \ldots, x_n\} \to$ {TRUE, FALSE} be an assignment that witnesses the 1-in-3 satisfiability of F. Define $f : V(G) \to \{-1, 1\}$ as follows: For each $j \in \{1, 2, \ldots, n\}$, let

$$f(x'_j) = \begin{cases} 1 & \text{if } \mathcal{A}(x_j) = \text{TRUE}; \\ -1 & \text{if } \mathcal{A}(x_j) = \text{FALSE} \end{cases}$$

and

$$f(x_j'') = \begin{cases} -1 & \text{if } \mathcal{A}(x_j) = \text{TRUE;} \\ 1 & \text{if } \mathcal{A}(x_j) = \text{FALSE.} \end{cases}$$

Let $f(v) = 1$ for all $v \in V(G) \setminus \bigcup_{j=1}^{n} \{x_j', x_j''\}$.

Clearly, $w(f) = (k+1)n + (k+2)m$. Since exactly one of $\mathcal{A}(x_{i_1}), \mathcal{A}(x_{i_2})$ and $\mathcal{A}(x_{i_3})$ is TRUE for each $1 \leq i \leq m$, it is easy to verify that f is a signed k-dominating function of G. We next prove that f is minimal, that is, for every vertex $v \in V(G)$ with $f(v) = 1$ there exists $u \in N_G[v]$ for which $f(N_G[u]) \in \{k, k+1\}$ (see [3]). For every $j \in \{1, 2, \ldots, n\}$, there is (at least) one vertex u in the j-th variable block such that $u \notin \{x_j', x_j''\}$. This vertex u is adjacent to all other vertices in the j-th variable block, and clearly $f(N_G[u]) = k+1$. For every $i \in \{1, 2, \ldots, m\}$, c_i' is adjacent to all other vertices in the i-th clause block, and $f(N_G[c_i']) = (k+2) + (1-2) = k+1$ since exactly one of $f(x_{i_1}'), f(x_{i_2}')$ and $f(x_{i_3}')$ is 1. Therefore, f is indeed a minimal signed k-dominating function of G with weight $(k+1)n + (k+2)m$, and the correctness of the "if" direction follows.

We now turn to the "only if" part of the claim. Assume that f is a minimal signed k-dominating function of G of weight at least $(k+1)n + (k+2)m$. If for some $j \in \{1, 2, \ldots, n\}$, the vertices in the j-th variable block all have value 1 under f, then $f(N_G[v]) \geq k+2$ for every $v \neq x_j'$ in the j-th variable block. Thus, there is no $u \in N_G[x_j'']$ such that $f(N_G[u]) \in \{k, k+1\}$, which violates the minimality of f. Hence, at least one vertex from each variable block must have value -1 under f, implying that $w(f) \leq (k+1)n + (k+2)m$. We thus have $w(f) = (k+1)n + (k+2)m$, and therefore (1) $f(v) = 1$ for every vertex v in the clause blocks, and (2) for each $j \in \{1, 2, \ldots, n\}$, $f(v) = -1$ for exactly one vertex v in the j-th variable block. Now produce an assignment $\mathcal{A}$ as follows: For each $j \in \{1, 2, \ldots, n\}$, let $\mathcal{A}(x_j)$ =TRUE if $f(x_j') = 1$, and $\mathcal{A}(x_j)$ =FALSE otherwise. For every $i \in \{1, 2, \ldots, m\}$, we have $k \leq f(N_G[c_i']) = (k+2) + f(x_{i_1}) + f(x_{i_2}) + f(x_{i_3})$, and thus at least one of $f(x_{i_1}), f(x_{i_2})$ and $f(x_{i_3})$ must be 1. Assume that at least two of the three values are 1. Then $f(N_G[c_i']) \geq k+3$, and obviously $f(N_G[v]) = k+2$ for every other vertex v in the i-th clause block. This indicates, however, that a vertex $v \neq c_i'$ in the i-th clause block does not have any neighbor (including itself) whose closed-neighborhood-sum is k or $k+1$, contradicting with the minimality of f. Accordingly, exactly one of $f(x_{i_1}), f(x_{i_2})$ and $f(x_{i_3})$ is 1, and thus exactly one of $\mathcal{A}(x_{i_1}), \mathcal{A}(x_{i_2})$ and $\mathcal{A}(x_{i_3})$ is TRUE, for every $i \in \{1, 2, \ldots, n\}$. Therefore, F is 1-in-3 satisfiable, finishing the proof of the "only if" part of the reduction.

The reduction is completed and the $\mathcal{NP}$-completeness of USkDP is thus established. □

3 Sharp Lower Bounds on $\gamma_{kS}(G)$ and $\gamma_{kS}^t(G)$

In this section we present sharp lower bounds on $\gamma_{kS}(G)$ and $\gamma_{kS}^t(G)$ in terms of the minimum and maximum degrees of G. Let $k \geq 1$ be a fixed integer throughout this section. For each integer n, define $I_n = 1$ if $n \equiv k \pmod 2$, and $I_n = 0$ otherwise; that is, I_n is the indicator variable of whether n and k have the same parity.

Theorem 4. *For every graph G with $\delta(G) \geq k-1$,*

$$\gamma_{kS}(G) \geq |V(G)| \cdot \frac{\delta(G) - \Delta(G) + 2k + I_{\delta(G)} + I_{\Delta(G)}}{\delta(G) + \Delta(G) + 2 + I_{\delta(G)} - I_{\Delta(G)}}.$$

Proof. Let G be a graph of order n with $\delta(G) \geq k-1$. For notational simplicity, we write δ and Δ to denote $\delta(G)$ and $\Delta(G)$ respectively. When $\delta = \Delta$, it is easy to verify that the theorem degenerates to Theorem 5 in [16]. Thus, we assume in what follows that $\Delta \geq \delta + 1$. Let f be a signed k-dominating function of G of weight $\gamma_{kS}(G)$. We need to introduce some notations. Let $P = \{v \in V(G) \mid f(v) = 1\}$ and $Q = V(G) \setminus P = \{v \in V(G) \mid f(v) = -1\}$. Furthermore, denote $P_\delta = \{v \in P \mid d_G(v) = \delta\}$, $P_\Delta = \{v \in P \mid d_G(v) = \Delta\}$, and $P_m = P \setminus (P_\delta \cup P_\Delta)$. Define Q_δ, Q_Δ, and Q_m analogously. For each $c \in \{\delta, \Delta, m\}$, let $V_c = P_c \cup Q_c$. Notice that $V_\delta \cap V_\Delta = \emptyset$ since $\Delta > \delta$. Let $R = \{v \in V(G) \mid d_G(v) \equiv k \pmod 2\}$. Clearly $\sum_{y \in N_G[x]} f(y) \geq k+1$ for each $x \in R$. Thus, we have

$$\begin{aligned}
& kn + |R| \leq \sum_{x \in V(G)} \sum_{y \in N_G[x]} f(y) = \sum_{x \in V(G)} (d_G(x)+1) f(x) \\
= \ & (\delta+1)|P_\delta| + (\Delta+1)|P_\Delta| + \sum_{x \in P_m} (d_G(x)+1) - (\delta+1)|Q_\delta| \\
& -(\Delta+1)|Q_\Delta| - \sum_{x \in Q_m} (d_G(x)+1) \\
\leq \ & (\delta+1)|P_\delta| + (\Delta+1)|P_\Delta| + \Delta|P_m| - (\delta+1)|Q_\delta| - (\Delta+1)|Q_\Delta| \\
& -(\delta+2)|Q_m| \\
& (\text{since } \delta + 1 \leq d_G(x) \leq \Delta - 1 \text{ for each } x \in P_m \cup Q_m) \\
= \ & (\delta+1)|V_\delta| + (\Delta+1)|V_\Delta| + \Delta|V_m| - 2(\delta+1)|Q_\delta| - 2(\Delta+1)|Q_\Delta| \\
& -(\Delta+\delta+2)|Q_m| \\
= \ & (\Delta+1)n - (\Delta-\delta)|V_\delta| - |V_m| - (\Delta+\delta+2)|Q| + (\Delta-\delta)|Q_\delta| \\
& -(\Delta-\delta)|Q_\Delta| \\
& (\text{note that } n = |V_\delta| + |V_\Delta| + |V_m| \text{ and } |Q| = |Q_\delta| + |Q_\Delta| + |Q_m|).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\Delta+1-k)n \\
\geq\ & |R|+|V_m|+(\Delta-\delta)(|V_\delta|-|Q_\delta|+|Q_\Delta|)+(\Delta+\delta+2)|Q| \\
=\ & |R|+|V_m|+(\Delta-\delta)(|P_\delta|+|Q_\Delta|)+(\Delta+\delta+2)|Q|.
\end{aligned}$$

Since $R=\{v\in V(G)\mid d(v)\equiv k \pmod 2\}$, it holds that $V_\delta\subseteq R$ if $\delta\equiv k \pmod 2$, and that $V_\Delta\subseteq R$ if $\Delta\equiv k \pmod 2$. Recalling that $V_\Delta\cap V_\delta=\emptyset$, we have $|R|\geq I_\delta\cdot|V_\delta|+I_\Delta\cdot|V_\Delta|$. Thus,

$$\begin{aligned}
&(\Delta+1-k)n \\
\geq\ & I_\delta\cdot|V_\delta|+I_\Delta\cdot|V_\Delta|+|V_m|+(\Delta-\delta)(|P_\delta|+|Q_\Delta|)+(\Delta+\delta+2)|Q| \\
=\ & I_\Delta(|V_m|+|V_\delta|+|V_\Delta|)+(1-I_\Delta)|V_m|+(I_\delta-I_\Delta)|V_\delta| \\
& +(\Delta-\delta)(|P_\delta|+|Q_\Delta|)+(\Delta+\delta+2)|Q| \\
=\ & I_\Delta\cdot n+(1-I_\Delta)|V_m|+(I_\delta-I_\Delta)|V_\delta|+(\Delta-\delta)(|P_\delta|+|Q_\Delta|) \\
& +(\Delta+\delta+2)|Q|.
\end{aligned}$$

Observing that $\Delta-\delta\geq 1\geq\max\{I_\delta-I_\Delta, I_\Delta-I_\delta\}$ and $(1-I_\Delta)|V_m|\geq(1-I_\Delta)|Q_m|\geq(I_\delta-I_\Delta)|Q_m|$, we get

$$\begin{aligned}
&(\Delta+1-k-I_\Delta)n \\
\geq\ & (I_\delta-I_\Delta)|Q_m|+(I_\delta-I_\Delta)|V_\delta|+(I_\Delta-I_\delta)|P_\delta|+(I_\delta-I_\Delta)|Q_\Delta| \\
& +(\Delta+\delta+2)|Q| \\
=\ & (I_\delta-I_\Delta)(|Q_m|+|V_\delta|-|P_\delta|+|Q_\Delta|)+(\Delta+\delta+2)|Q| \\
=\ & (I_\delta-I_\Delta)(|Q_m|+|Q_\delta|+|Q_\Delta|)+(\Delta+\delta+2)|Q| \\
=\ & (\Delta+\delta+2+I_\delta-I_\Delta)|Q|.
\end{aligned}$$

Hence, we deduce that

$$|Q|\leq n\cdot\frac{\Delta-k+1-I_\Delta}{\delta+\Delta+2+I_\delta-I_\Delta},$$

from which it follows that

$$\gamma_{kS}(G)=n-2|Q|\geq n\cdot\frac{\delta-\Delta+2k+I_\delta+I_\Delta}{\delta+\Delta+2+I_\delta-I_\Delta},$$

which is exactly the desired inequality in Theorem 4. □

A vertex of degree $k-1$ or k in a graph G clearly has function value 1 under all signed k-dominating functions of G. Thus, it is natural to consider graphs with minimum degree at least $k+1$ (as is done in [3] for establishing sharp upper bounds for the upper signed k-domination number). We next show that Theorem 4 is sharp for all $\Delta\geq\delta\geq k+1$. This level of sharpness is high as it applies not only to special values of minimum and maximum degrees.

Theorem 5. *For any integers δ and Δ such that $\Delta \geq \delta \geq k+1$, there exists an infinite family $\mathcal{F}$ of graphs with minimum degree δ and maximum degree Δ, such that for every graph $G \in \mathcal{F}$,*

$$\gamma_{kS}(G) = |V(G)| \cdot \frac{\delta - \Delta + 2k + I_\delta + I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta}.$$

Proof. Fix integers Δ and δ such that $\Delta \geq \delta \geq k+1$. Let $H_1, H_2, \ldots, H_t$ be t disjoint copies of the complete bipartite graph $K_{a,b}$ with vertex partition (A, B), where $|A| = a = (\delta+k+1+I_\delta)/2$, $|B| = b = (\Delta-k+1-I_\Delta)/2$ (it is easy to verify that a and b are both integers), and t is an arbitrary *even* integer larger than Δ. It is also easy to check that $1 \leq a \leq \delta$ and $1 \leq b \leq \Delta$ (just note that $I_\delta = 0$ when $\delta = k+1$). For each $1 \leq i \leq t$, let A_i and B_i denote the vertex partition of H_i with size a and b, respectively. Let $P = \bigcup_{i=1}^t A_i$ and $Q = \bigcup_{i=1}^t B_i$. Note that each vertex in P is connected to exactly b vertices in Q, and each vertex in Q is adjacent to exactly a vertices in P.

Our desired graph G has vertex set $P \cup Q$, and contains $\bigcup_{i=1}^t H_i$ as a subgraph. Furthermore, we add some edges between vertices in P to make $G[P]$ become $(\Delta - b)$-regular (no edges need to be added if $\Delta = b$). This can be done in the following way: Imagine that there is a complete graph K whose vertex set is P. Since $|P| = ta$ is even and every complete graph of even order is 1-factorable (see e.g. Theorem 9.1 in [10]), the edges of K can be partitioned into $|P| - 1 \geq \Delta$ perfect matchings of K. Taking $\Delta - b$ of these matchings and adding them to G certainly makes $G[P]$ become $(\Delta - b)$-regular. Similarly, we add some edges between vertices in Q to make $G[Q]$ $(\delta - a)$-regular. This finishes the construction of G. Note that all vertices in P have degree Δ and those in Q have degree δ, and thus G is of minimum degree δ and maximum degree Δ. (Note also that by varying t, we get an infinite family of graphs with the desired properties.)

Define a function $f : P \cup Q \to \{-1, 1\}$ by letting $f(v) = 1$ for all $v \in P$ and $f(u) = -1$ for all $u \in Q$. Then, for each $v \in P$, $f(N_G[v]) = \Delta + 1 - 2b = k + I_\Delta \geq k$, and for each $u \in Q$, $f(N_G[u]) = 2a - (\delta + 1) = k + I_\delta \geq k$. Therefore, f is a signed k-dominating function of G. Since $|V(G)| = |P| + |Q|$ and $|P|/|Q| = a/b = \frac{\delta+k+1+I_\delta}{\Delta-k+1-I_\Delta}$, we have

$$\begin{aligned}\gamma_{kS}(G) &\leq w(f) = |P| - |Q| = \left(1 - \frac{2}{|P|/|Q| + 1}\right)|V(G)| \\ &= |V(G)| \cdot \frac{\delta - \Delta + 2k + I_\delta + I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta}.\end{aligned}$$

By Theorem 4, we know that the equality holds in the above formula, which completes the proof of Theorem 5. □

We can also derive a sharp lower bound on the signed total k-domination number of a graph as follows.

Theorem 6. *For every graph G with $\delta(G) \geq k$,*

$$\gamma_{kS}^t(G) \geq |V(G)| \cdot \frac{\delta(G) - \Delta(G) + 2k + 2 - I_{\delta(G)} - I_{\Delta(G)}}{\delta(G) + \Delta(G) + I_{\Delta(G)} - I_{\delta(G)}} .$$

Theorem 7. *For any integers δ and Δ such that $\Delta \geq \delta \geq k+2$, there exists an infinite family $\mathcal{F}$ of graphs with minimum degree δ and maximum degree Δ, such that for every graph $G \in \mathcal{F}$,*

$$\gamma_{kS}^t(G) = |V(G)| \cdot \frac{\delta - \Delta + 2k + 2 - I_\delta - I_\Delta}{\delta + \Delta + I_\Delta - I_\delta} .$$

The proofs of Theorems 6 and 7 are very similar to those of Theorems 4 and 5, and thus are put in the appendix.

Theorems 4 and 6 are generalizations of Theorem 5 in [16]. The following corollaries, which generalize some other known results regarding signed domination number and signed total domination number, are also immediate from the preceding theorems.

Corollary 1. *For any nearly r-regular graph G of order n with $r \geq k$, $\gamma_{kS}(G) \geq kn/(r + I_{r-1})$ and $\gamma_{kS}^t(G) \geq kn/(r - I_{r-1})$.*

Corollary 2. *Let c be a real number for which $-1 < c \leq 1$. Then $\gamma_{kS}(G) \geq cn$ for every graph G of order n with $\delta(G) \geq k-1$ and $\Delta(G) \leq ((1-c)\delta(G) + 2k - 2c)/(1+c)$, and $\gamma_{kS}^t(G) \geq cn$ for every graph G of order n with $\delta(G) \geq k$ and $\Delta(G) \leq ((1-c)\delta(G) + 2k)/(1+c)$.*

Corollary 3. *Let G be a graph with $\delta(G) \geq k$ and $\Delta(G) \leq \delta(G) + 2k$. Then $\gamma_{kS}(G) \geq 0$ and $\gamma_{kS}^t(G) \geq 0$.*

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A Proof of Theorem 6

Proof of Theorem 6. Let G be a graph of order n and f be a signed total k-dominating function of G. Let $\delta, \Delta, P, Q, P_\delta, P_\Delta, P_m, Q_\delta, Q_\Delta, Q_m, V_\delta, V_\Delta, V_m$ be defined in the same way as in the proof of Theorem 4. Let $R = \{v \in V(G) \mid d(v) \not\equiv k \pmod 2\}$ (which is different from the definition of R in the proof of Theorem 4). Assume $\Delta > \delta$, otherwise the theorem just becomes Theorem 5 in [16]. Since $\sum_{y\in N_G(x)} f(y) \geq k+1$ for all $x \in R$, we have:

$$\begin{aligned}
& kn + |R| \\
\leq\ & \sum_{x\in V(G)} \sum_{y\in N_G(x)} f(y) \\
=\ & \sum_{x\in V(G)} d_G(x) f(x) \\
=\ & \delta|P_\delta| + \Delta|P_\Delta| + \sum_{x\in P_m} d_G(x) - \delta|Q_\delta| - \Delta|Q_\Delta| - \sum_{x\in Q_m} d_G(x) \\
\leq\ & \delta|P_\delta| + \Delta|P_\Delta| + (\Delta-1)|P_m| - \delta|Q_\delta| - \Delta|Q_\Delta| - (\delta+1)|Q_m| \\
=\ & \delta|V_\delta| + \Delta|V_\Delta| + (\Delta-1)|V_m| - 2\delta|Q_\delta| - 2\Delta|Q_\Delta| - (\Delta+\delta)|Q_m| \\
=\ & \Delta n - (\Delta-\delta)|V_\delta| - |V_m| - (\Delta+\delta)|Q| + (\Delta-\delta)|Q_\delta| - (\Delta-\delta)|Q_\Delta| \\
& \text{(recall that } n = |V_\delta| + |V_\Delta| + |V_m| \text{ and } |Q| = |Q_\delta| + |Q_\Delta| + |Q_m|\text{)}.
\end{aligned}$$

By our definition, it holds that $|R| \geq (1-I_\delta)|V_\delta| + (1-I_\Delta)|V_\Delta|$. Therefore,

$$\begin{aligned}
& (\Delta-k)n \\
\geq\ & |R| + |V_m| + (\Delta-\delta)(|V_\delta| - |Q_\delta| + |Q_\Delta|) + (\Delta+\delta)|Q| \\
=\ & |R| + |V_m| + (\Delta-\delta)(|P_\delta| + |Q_\Delta|) + (\Delta+\delta)|Q| \\
\geq\ & (1-I_\delta)|V_\delta| + (1-I_\Delta)|V_\Delta| + |V_m| + (\Delta-\delta)(|P_\delta| + |Q_\Delta|) \\
& +(\Delta+\delta)|Q| \\
=\ & (1-I_\Delta)(|V_m| + |V_\delta| + |V_\Delta|) + I_\Delta|V_m| + (I_\Delta - I_\delta)|V_\delta| \\
& +(\Delta-\delta)(|P_\delta| + |Q_\Delta|) + (\Delta+\delta)|Q| \\
=\ & (1-I_\Delta)n + I_\Delta|V_m| + (I_\Delta - I_\delta)|V_\delta| + (\Delta-\delta)(|P_\delta| + |Q_\Delta|) \\
& +(\Delta+\delta)|Q|.
\end{aligned}$$

Noting that $I_\Delta|V_m| \geq (I_\Delta - I_\delta)|Q_m|$ and $\Delta - \delta \geq \max\{I_\Delta - I_\delta, I_\delta - I_\Delta\}$, we obtain

$$\begin{aligned}
& (\Delta - k + I_\Delta - 1)n \\
\geq\ & I_\Delta|V_m| + (I_\Delta - I_\delta)|V_\delta| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q| \\
\geq\ & (I_\Delta - I_\delta)|Q_m| + (I_\Delta - I_\delta)|V_\delta| + (I_\delta - I_\Delta)|P_\delta| + (I_\Delta - I_\delta)|Q_\Delta| \\
& +(\Delta + \delta)|Q| \\
=\ & (I_\Delta - I_\delta)(|Q_m| + |V_\delta| - |P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q| \\
=\ & (I_\Delta - I_\delta)|Q| + (\Delta + \delta)|Q| \\
=\ & (\Delta + \delta + I_\Delta - I_\delta)|Q|.
\end{aligned}$$

Hence, we have

$$|Q| \leq n \cdot \frac{\Delta - k + I_\Delta - 1}{\delta + \Delta + I_\Delta - I_\delta},$$

from which it follows that

$$\gamma_{kS}(G) = n - 2|Q| \geq n \cdot \frac{\delta - \Delta + 2k + 2 - I_\delta - I_\Delta}{\delta + \Delta + I_\Delta - I_\delta},$$

completing the proof of Theorem 6. □

B Proof of Theorem 7

Proof of Theorem 7. Fix integers Δ and δ such that $\Delta \geq \delta \geq k + 2$. We proceed with the same construction used in the proof of Theorem 5, except for setting $a = (\delta + k - I_\delta + 1)/2$ and $b = (\Delta - k + I_\Delta - 1)/2$ instead. (It is easy to check that a and b are integers satisfying that $1 \leq a \leq \delta$ and $1 \leq b \leq \Delta$.) The obtained graph G has vertex set $P \cup Q$, where $d_G(v) = \Delta$ for all $v \in P$ and $d_G(u) = \delta$ for all $u \in Q$. Furthermore, each vertex $v \in P$ is adjacent to exactly b vertices in Q and $\Delta - b$ vertices in P, while every vertex $u \in Q$ is adjacent to precisely a vertices in P and $\delta - a$ vertices in Q. Now define a function f which assigns 1 to all vertices in P and -1 to those in Q. It is easy to verify that f is a signed total k-dominating function of G with weight $|V(G)| \cdot \frac{\delta - \Delta + 2k + 2 - I_\delta - I_\Delta}{\delta + \Delta + I_\Delta - I_\delta}$, completing the proof of Theorem 7. □