Pappus-Desargues digraph confrontation

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Abstract

Like the Coxeter graph became reattached into the Klein graph in
[3], the Levi graphs of the 9_3 and 10_3 self-dual configurations, known
as the Pappus and Desargues (k-transitive) graphs $\mathcal{P}$ and $\mathcal{D}$ (where
$k = 3$), also admit reattachments of the distance-$(k-1)$ graphs of
half of their oriented shortest cycles via orientation assignments on
their common $(k-1)$-arcs, concurrent for $\mathcal{P}$ and opposite for $\mathcal{D}$, now
into 2 disjoint copies of their corresponding Menger graphs. Here,
$\mathcal{P}$ is the unique cubic distance-transitive (or CDT) graph with the
concurrent-reattachment behavior while $\mathcal{D}$ is one of 7 CDT graphs
with the opposite-reattachment behavior, that include the Coxeter
graph. Thus, $\mathcal{P}$ and $\mathcal{D}$ confront each other in these respects, ob-
tained via $\mathcal{C}$-ultrahomogeneous graph techniques [4, 5] that allow to
characterize the obtained reattachment Menger graphs in the same
terms.

1   Preliminaries

Given a collection $\mathcal{C}$ of (di)graphs closed under isomorphisms, a (di)graph
G is said to be *$\mathcal{C}$-ultrahomogeneous* (or *$\mathcal{C}$-UH*) [4, 5] if every isomorphism
between 2 induced members of $\mathcal{C}$ in G extends to an automorphism of G.
If $\mathcal{C} = \{H\}$ is the isomorphism class of a (di)graph H, we say that such a
G is $\{H\}$-*UH* (or *H-UH*). In [5], $\mathcal{C}$-UH graphs are studied when $\mathcal{C}$ is the
collection of either **(a)** the complete graphs, or **(b)** the disjoint unions of
complete graphs, or **(c)** the complements of those unions.

We consider any undirected graph G as a digraph by taking each edge e
of G as a pair of oppositely oriented (or *O-O*) arcs $\vec{e}$ and $(\vec{e})^{-1}$. Then
cohering) (or *fastening*, or *zipping*) $\vec{e}$ and $(\vec{e})^{-1}$ (meaning that we take the

union of $\vec{e}$ and $(\vec{e})^{-1}$) allows to obtain precisely e, a simple technique to be used below. In other words, G is a graph taken as a digraph, that is, for any 2 adjacent vertices $u, v \in V(G)$, the arcs $\vec{e} = (u, v)$ and $(\vec{e})^{-1} = (v, u)$ are both present in the set $A(G)$ of arcs of G, with the union $\vec{e} \cup (\vec{e})^{-1}$ interpreted as the edge $e \in E(G)$ of G. If we write $\vec{f} = (\vec{e})^{-1}$, then clearly $(\vec{f})^{-1} = \vec{e}$ and $f = e$.

1.1 Coherent $\mathcal{C}$-(ultra)homogeneous graphs

Let M be an induced subgraph of a graph H and let G be both an M-UH and an H-UH graph. We say that G is an $\{H\}_M$-UH graph if, for each copy H_0 of H induced in G and containing a copy M_0 of M, there exists exactly one copy $H_1 \neq H_0$ of H induced in G such that $V(H_0) \cap V(H_1) = V(M_0)$ and $E(H_0) \cap E(H_1) = E(M_0)$. These vertex and edge conditions can be condensed as $H_0 \cap H_1 = M_0$. We say that such a G is *coherent*. This is generalized by saying that an $\{H\}_M$-UH graph G is an ℓ-*coherent* $\{H\}_M$-UH graph if, given a copy H_0 of H induced in G and containing a copy M_0 of M, there exist exactly ℓ copies $H_i \neq H_0$ of H induced in G such that $H_i \cap H_0 \supseteq M_0$, for each $i = 1, 2, \ldots, \ell$, with $H_1 \cap H_0 = M_0$.

If G is coherent $\{H\}_M$-UH and K is both subgraph of H and supergraph of M, we say that G is $\{K \not\subset H\}_M$-UH if every isomorphism between 2 induced copies of K in G not contained in any copy of H in G extends to an automorphism of G. If, under these conditions, each copy of M induced in G coincides with the intersection of exactly one copy of H and exactly one copy of $K \not\subset H$, then we say that G is *coherent* $\{H, K\}_M$-UH. This concept is used in Theorem 3 below for the Desargues graph $G = \mathcal{D}$.

Let G be an M-UH graph but not H-UH and assume the isomorphism class $\mathcal{H}$ of H in G decomposes as $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1$ so that every isomorphism between 2 members of $\mathcal{H}_i$ induced in G extends to an automorphism of G, $(i = 0, 1)$. If H_i is a representative of $\mathcal{H}_i$, for $i = 0, 1$, then we say that G is an $\{H_0, H_1\}^M$-*homogeneous graph* if, for each copy H_i induced in G and containing a copy M_0 of M, there exists exactly one copy H_j induced in G, $(i, j \in \{0, 1\}, i \neq j)$, with $H_i \cap H_j = M_0$. This concept is likewise extended to a decomposition $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ in Theorem 5, where $G = \mathcal{P}$ is the Pappus graph.

1.2 Coherent $\mathcal{C}$-(ultra)homogeneous digraphs

Let $\vec{M}$ be an induced subdigraph of a digraph $\vec{H}$ and let G be both an $\vec{M}$-UH and an $\vec{H}$-UH digraph. We say that G is an $\{\vec{H}\}_{\vec{M}}$-UH digraph (resp. an $\{\vec{H}\}^{\vec{M}}$-UH digraph), if for each copy $\vec{H}_0$ of $\vec{H}$ induced in $\vec{G}$ and containing a copy $\vec{M}_0$ of $\vec{M}$, there exists exactly one copy $\vec{H}_1 \neq \vec{H}_0$ of $\vec{H}$ induced in G such that $V(\vec{H}_0) \cap V(\vec{H}_1) = V(\vec{M}_0)$ and $A(\vec{H}_0) \cap \bar{A}(\vec{H}_1) = A(\vec{M}_0)$ (resp. $A(\vec{H}_0) \cap A(\vec{H}_1) = A(\vec{M}_0)$), where $\bar{A}(\vec{H}_1)$ is formed by those arcs $(\vec{e})^{-1}$ whose orientations are reversed with respect to the orientations of the arcs $\vec{e}$ of $A(\vec{H}_1)$. In either case, we may say that such a G is *coherent*.

Let G be an $\vec{M}$-UH graph but not $\vec{H}$-UH and assume that the isomorphism class $\mathcal{H}$ of $\vec{H}$ in G decomposes as $\mathcal{H} = \vec{\mathcal{H}}_0 \cup \vec{\mathcal{H}}_1$ so that every isomorphisms between 2 members of $\mathcal{H}_i$ induced in G extends to an automorphism of G, $(i = 0, 1)$. If $\vec{H}_i$ is a representative of $\vec{\mathcal{H}}_i$, for $i = 0, 1$, then we say that G is an $\{\vec{H}_0, \vec{H}_1\}^{\vec{M}}$-*homogeneous* graph if, for each copy $\vec{H}_i$ induced in G and containing a copy $\vec{M}_0$ of M, there exists exactly one induced copy $\vec{H}_j$ in G, $(i, j \in \{0, 1\}, i \neq j)$, with $\vec{H}_i \cap \vec{H}_j = \vec{M}_0$.

1.3 Strongly coherent $\mathcal{C}$-ultrahomogeneous graphs

Given a finite graph H and a subgraph M of H with $|V(H)| > 3$, we say that a graph G is *strongly coherent* (or *SC*) $\{H\}_M$-UH if there is a descending sequence of connected subgraphs $M = M_1, M_2 \ldots, M_t \equiv K_2$ such that: (a) M_{i+1} is obtained from M_i by the deletion of a vertex, for $i = 1, \ldots, t-1$ and (b) G is a $(2^i - 1)$-coherent $\{H\}_{M_i}$-UH graph, for $i = 1, \ldots, t$.

Some parameters of $\mathcal{P}$ and $\mathcal{D}$ (see for example [1]) can be displayed as follows:

G	n	d	g	k	η	a	b	h
$\mathcal{P}$	18	4	6	3	18	216	1	1
$\mathcal{D}$	20	5	6	3	20	240	1	1

where n, d, g, k, η and a are respectively: order, diameter, girth, largest ℓ such that G is ℓ-arc transitive, number of g-cycles and number of automorphisms, with b (resp. h) = 1 if G is bipartite (resp. hamiltonian) and = 0 otherwise. Theorem 1 below asserts that both the Pappus graph $\mathcal{P}$ and the Desargues graph $\mathcal{D}$ are SC $\{C_6\}_{P_3}$-UH graphs, (which is also the case of the other 10 CDT graphs, see [3, 4]).

1.4 Plan of the subsequent sections

Given a (di)graph Γ, the *distance-$(k-1)$ (di)graph* Γ^{k-1} of Γ has $V(\Gamma^{k-1}) = V(\Gamma)$ and an arc (u, v) for each shortest $(k-1)$-arc in Γ from u to $v \neq u$. If Γ is a cycle, then Γ^2 is said to be a *square*. Theorem 2 below establishes that $\mathcal{D}$ is a $\{\vec{C}_g\}_{\vec{P}_k}$-UH digraph and that $\mathcal{P}$ is a $\{\vec{C}_g\}^{\vec{P}_k}$-UH digraph; it deals with just a pair of the 12 cubic distance-transitive (or CDT) graphs treated in Theorem 3 of [4] and is given, together with its proof, in part for the needs of the constructions in Sections 3-6. However, we stress here that $\mathcal{P}$ is the only CDT graph that is a $\{\vec{C}_g\}^{\vec{P}_k}$-UH digraph, while $\mathcal{D}$ is the second most interesting of 7 CDT graphs G that are $\{\vec{C}_g\}_{\vec{P}_k}$-UH digraphs [4] after the Coxeter graph, where g is the girth of k-transitive G. (Petersen, Heawood, Foster and Biggs-Smith graphs excluded here. K_4, $K_{3,3}$, the 3-cube and the dodecahedral graphs and Tutte 8-cage have either $g = 2(k-1)$ or $k = 2$, so the equivalent of the composed operation (2) in Section 3 below or in Section 3 of [3] for the Coxeter graph is less interesting).

In Sections 3-4, the squares of oriented cycles of $\mathcal{D}$ yield a coherent

$\{K_4, K_3\}_{K_2}$-UH graph by means of the O-O 2-arcs shared (as 2-paths) by the 6-cycle square pairs. In Theorem 3, this is shown to be the disjoint union of 2 copies of $L(K_5)$, the Menger graph of the self-dual (10_3)-configuration [2], (whose Levi graph is $\mathcal{D}$, [2]. Compare with [3], yielding the Klein graph from the Coxeter graph. Recall that the Menger graph $\mathcal{M}$ of a self-dual configuration $\mathcal{S}$ has as vertices its points, with any 2 determining an edge of $\mathcal{M}$ if and only if their representative points in $\mathcal{S}$ are colinear [2]. We note that 2 different configurations may have the same Menger graph, unless each line of $\mathcal{S}$ determines a maximal clique in $\mathcal{M}$, which is the case of the coherent $\mathcal{C}$-UH graphs in this paper.) We finish Section 4 noting that Theorem 3 yields an infinite nested sequence of geometric realizations of $L(K_5)$ (or of its complement, the Petersen graph) via taking barycenters of participating tetrahedra as vertices of subsequent tetrahedra. Generalizing, Theorem 4 in Section 5 asserts that for $n \geq 4$ the line graph $L(K_n)$ is a coherent $\{K_{n-1}, K_3\}_{K_2}$-UH graph containing n copies of K_{n-1} and $\binom{n}{3}$ copies of K_3. An adaptation of the previous considerations to $\mathcal{P}$ makes it yield, in Theorem 5 of Section 6, $\mathcal{P}$ in 2 complementary ways as the Menger graph of the self-dual (9_3)-configuration (whose Levi graph is $\mathcal{P}$, [2]) and as the object of application of the concepts of $\mathcal{C}$-homogeneous graphs and digraphs given above.

2 (C_6, P_3)-UH properties of $\mathcal{P}$ and $\mathcal{D}$

Theorem 1 *Let G be either $\mathcal{P}$ or $\mathcal{D}$. Then G is an SC $\{C_6\}_{P_3}$-UH graph.*

Proof. We must see that each of $G = \mathcal{P}$ and $G = \mathcal{D}$ is a $(2^{i+1} - 1)$-coherent $\{C_6\}_{P_{3-i}}$-UH graph, for $i = 0, 1$. Taking into account details in the proof of Theorem 2 below, each $(2 - i)$-path $P = P_{3-i}$ of G is seen to be shared by exactly 2^{i+1} 6-cycles of G, for $i = 0, 1$. It follows that G is an SC $\{C_6\}_{P_3}$-UH graph. □

In both $\mathcal{P}$ and $\mathcal{D}$, there are just 2 6-cycles shared by each 2-path. If G is a $\{\vec{C}_6\}_{\vec{P}_3}$-UH digraph, then there is an assignment of an orientation to each 6-cycle of G so that the 2 6-cycles shared by each 2-path receive opposite orientations. We say that such an assignment is a $\{\vec{C}_6\}_{\vec{P}_3}$-*O-O assignment* (or $\{\vec{C}_6\}_{\vec{P}_3}$-*OOA*). The collection of η oriented 6-cycles corresponding to the η 6-cycles of G, for a particular $\{\vec{C}_6\}_{\vec{P}_3}$-OOA, is called an $\{\eta\vec{C}_6\}_{\vec{P}_3}$-*OOC*. Each such cycle is written with their successive vertices between parentheses but without separating commas, where as usual the vertex that succeeds the last vertex of the cycle is the first vertex. Arcs are written (u, v) and 2-arcs (u, v, w). Figure 1 contains representations of $\mathcal{P}$ and $\mathcal{D}$ using the vertex notation in the proof of Theorem 2 (plus extra-features for $\mathcal{D}$ related to Figure 2 and the treatment of $\mathcal{D}$ in Section 4 below).

Theorem 2 *$\mathcal{D}$ is a $\{\vec{C}_6\}_{\vec{P}_3}$-UH digraph but $\mathcal{P}$ is a $\{\vec{C}_6\}^{\vec{P}_3}$-UH digraph.*
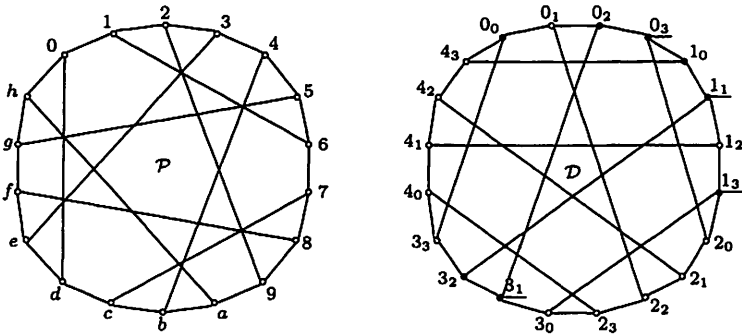

Figure 1: Representations of $\mathcal{P}$ and $\mathcal{D}$

Proof. For each positive integer n, let I_n stand for the n-cycle $(0, 1, \ldots, n-1)$. $\mathcal{P}$ can be obtained from I_{18} by adding the edges $(1 + 6x, 6 + 6x), (2 + 6x, 9 + 6x), (4 + 6x, 11 + 6x)$, for $x \in \{0, 1, 2\}$, where operations are taken

mod 18. Then G admits the following collection of 6-cycles: $A_0 = (123456)$, $B_0 = (3210de)$, $C_0 = (34bcde)$, $D_0 = (165gh0)$, $E_0 = (329ab4)$, (where octodecimal notation is used, up to $h = 17$), as well as A_x, B_x, C_x, D_x, E_x obtained by uniformly adding $6x$ mod 18 to the vertices of A_0, B_0, C_0, D_0, E_0, where $x \in \mathbf{Z}_3 \setminus \{0\}$, in addition to $F_0 = (3298fe)$, $F_1 = (hg54ba)$, $F_2 = (167cd0)$. These 6-cycles cannot be oriented into a $(18\,\vec{C}_6)_{\vec{P}_3}$-OOC, for the following sequence of alternating 6-cycles and 2-paths (with orientation reversed between each 6-cycle and its corresponding succeeding 6-cycle) reverses orientation from its initial 6-cycle to its terminal one:

$$D_1^{-1}654\,A_0\,123\,B_0\,210\,C_1\,h01\,D_0^{-1}\,g56\,C_2^{-1}765\,D_1$$

$$=(654bc7)\,654\,(123456)\,123\,(3210de)\,210\,(0129ah)\,h01\,(10hg56)\,g56\,(5gf876)\,765(cb4567).$$

Another way to see this is via an auxiliary table for $\mathcal{P}$, where $x = 0, 1, 2$ (mod 3), presenting the form in which the 6-cycles above share the 2-arcs, which are not always O-O for $\mathcal{P}$, indicated by a minus sign in front of the heading of each line of the table to distinguish it from the situation in $\mathcal{D}$, shown below. Each η_j in the table has subindex j indicating the equality of initial vertices $\eta_j = \xi_{i+2}$ of those 2-arcs, for $i = 0, \ldots, 5$:

$$
\begin{array}{ll}
-A_x:(B_x,E_x\quad,E_{x+2},D_{x+1},D_x\quad,B_{x+1}) & -F_0:(E_0,B_1,E_1,B_2,E_2,B_0) \\
-B_x:(A_x,C_{x+1},F_2\quad,A_{x+2},C_x\quad,F_0\quad) & -F_1:(D_0,E_2,D_1,E_0,D_2,E_1) \\
-C_x:(E_x,D_{x+1},D_{x+2},B_{x+2},B_x\quad,E_{x+2}) & -F_2:(B_1,D_1,B_2,D_2,B_0,D_0) \\
-D_x:(A_x,C_{x+2},F_1\quad,A_{x+2},C_{x+1},F_2\quad) & \\
-E_x:(F_0,C_{x+1},A_{x+1},F_1\quad,C_x\quad,A_x\quad) &
\end{array}
\qquad (1)
$$

This proves that $\mathcal{P}$ is a $\{\vec{C}_6\}^{\vec{P}_3}$-UH digraph.

$\mathcal{D}$ can be obtained from I_{20}, with vertices $4x, 4x+1, 4x+2, 4x+3$ redenoted alternatively x_0, x_1, x_2, x_3 respectively, for $x \in \mathbf{Z}_5$ by adding the edges $(x_3, (x+2)_0)$ and $(x_1, (x+2)_2)$, with operations taken mod 5. Then G admits a $\{20\,\vec{C}_6\}_{\vec{P}_3}$-OOC formed by the oriented 6-cycles A_x, B_x, C_x, D_x, for $x \in \{0, \ldots, 4\}$, where

$$
\begin{array}{ll}
A_x=(x_0x_1x_2x_3(x+1)_0(x+4)_3) & B_x=(x_1x_0(x+4)_3(x+4)_2(x+2)_1(x+2)_2) \\
C_x=(x_2x_1x_0(x+3)_3(x+3)_2(x+3)_1) & D_x=(x_0(x+4)_3(x+1)_0(x+1)_1(x+3)_2(x+3)_3)
\end{array}
$$

The successive copies of $\vec{P}_3$ here, when reversed, in each case, must belong to the following remaining oriented 6-cycles:

$$
\begin{array}{ll}
A_x:(C_x,C_{x+2},B_{x+1},D_{x+1},D_x,B_x) & B_x:(A_x,A_{x+4},D_{x+1},C_{x+4},C_{x+2},D_{x+4}) \\
C_x:(A_x,D_{x+4},D_x,A_{x+3},B_{x+1},B_{x+3}) & D_x:(A_x,C_{x+1},B_{x+1},B_{x+4},C_x,A_{x+4})
\end{array}
$$

showing that they constitute effectively an $\{\eta\vec{C}_6\}_{\vec{P}_3}$-OOC. $\qquad\square$

3   Cohering the distance-2 graphs of 6-cycles

We use the construction and notation of $\mathcal{D}$ and its associated $\{\eta\vec{C}_6\}_{\vec{P}_3}$-OOC, as in the proof of Theorem 2. Consider the collection $(\vec{C}_6)^2(\mathcal{D})$ of squares of oriented 6-cycles in the $\{\eta\vec{C}_6\}_{\vec{P}_3}$-OOC of $\mathcal{D}$ in that proof. Each arc $\vec{e}$ of a member $\vec{C}^2$ of $(\vec{C}_6)^2(\mathcal{D})$ can be indicated by the middle vertex of the 2-arc $\vec{E}$ in $\vec{C}$ for which $\vec{e}$ stands, while the tail and head of $\vec{e}$ are indicated by the tail and head of $\vec{E}$, respectively. We cohere such $\vec{C}^2$s along their O-O arc pairs in order to obtain a corresponding graph $Y(\mathcal{D})$ with the $\{K_4, K_3\}_{K_2}$-UH property claimed in Subsection 1.4. For such a setting, the following composed operation is performed, where ϕ assigns to each 6-cycle in $\{\eta\vec{C}_6\}_{\vec{P}_3}$-OOC its corresponding square:

$$\mathcal{D} \;\rightarrow\; \{\eta\vec{C}_6\}_{\vec{P}_3}\text{-OOC}(\mathcal{D}) \;\overset{\phi}{\rightarrow}\; (\vec{C}_6)^2(\mathcal{D}) \;\rightarrow\; Y(\mathcal{D}). \tag{2}$$

We will explain in Section 4 how this operation $\mathcal{D} \rightarrow Y(\mathcal{D})$ is performed.

As mentioned in the table (1), in any oriented 6-cycle ξ of $\mathcal{P}$, each participating copy of $\vec{P}_3$, when reversed in each case, must belong to a corresponding oriented 6-cycle η. In particular, each 6-cycle following such a copy of $\vec{P}_3$ has its orientation reversed with respect to the one of the preceding 6-cycle. This results in the second alternate 6-cycles being considered with their orientation reversed with respect to the first alternate 6-cycles. Because of this, we say that there are 2 *alternate* O-O $\{\frac{1}{2}\eta\vec{C}_6\}_{\vec{P}_3}$-OOCs, in the absence of just one $\{\eta\vec{C}_6\}_{\vec{P}_3}$-OOC for $\mathcal{P}$. This allows 2 corresponding alternate half-operations similar in nature to (2), above. See Section 6 below.

The 2 versions of $\vec{C}_6^2(\mathcal{P})$ here and the only one of $Y(\mathcal{D}) = \vec{B}_6^2(\mathcal{D})$ are formed by oriented triangles that determine 2 corresponding graphs $Y_1(\mathcal{P})$ and $Y_2(\mathcal{P})$ and a single graph $Y(\mathcal{D})$ with 2 components $Y_1(\mathcal{D})$ and $Y_2(\mathcal{D})$.

4   Desargues reattachment Menger graph

For $i = 1, 2$, it is a matter of checking that $Y_i(\mathcal{D})$ is an isomorphic coherent $\{K_4, K_3\}_{K_2}$-UH graph formed by 5 copies of K_4 and 10 of $K_3 \not\subset K_4$, with each such copy of K_3 having its edges indicated by a constant symbol, as shown in Figure 2. Each of the 5 copies T of K_4 in $Y_i(\mathcal{D})$ has any one of its six edges as a pair of O-O arcs, say $\vec{e}$ and $(\vec{e})^{-1}$, arising from corresponding
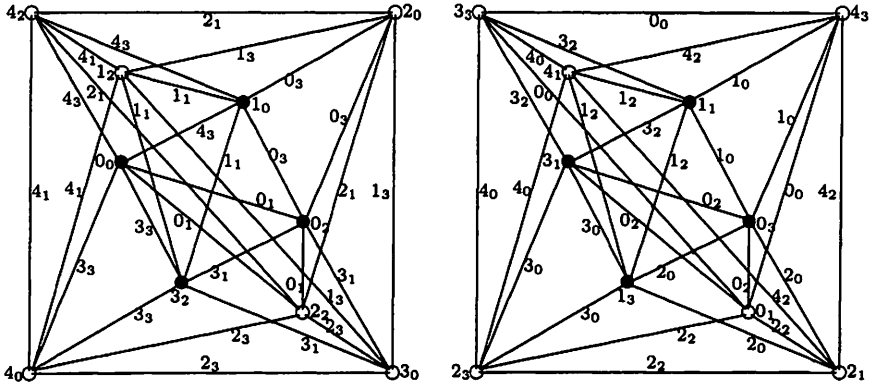
Figure 2: Representations of $Y_1(\mathcal{D})$ and $Y_2(\mathcal{D})$

O-O 2-arcs separating 2 oriented 6-cycles of the $\{20\,\vec{C}_6\}_{\bar{P}_3}$-OOC($\mathcal{D}$), as obtained in the proof of Theorem 2. Moreover, these 2 oriented 6-cycles have images in $(\vec{C}_6)^2(G)$ via the map ϕ displayed in (2) that are the 2 oriented triangles that share e in $\mathcal{T}$, or in other words, having $\vec{e}$ and $(\vec{e})^{-1}$ as O-O arcs.

For example, each of the 2 'central' copies of K_4 with black vertices in either side of Figure 2 has the 4 composing vertex triples, namely $(0_0, 0_2, 1_0)$, $(0_2, 1_0, 3_2)$, $(1_0, 3_2, 0_0)$ and $(3_2, 0_0, 0_2)$, for $Y_1(\mathcal{D})$, (resp. $(0_3, 1_1, 1_3)$,

$(1_1, 1_3, 3_1)$, $(1_3, 3_1, 0_3)$ and $(3_1, 0_3, 1_1)$, for $Y_2(\mathcal{D})$) as alternate vertices of 4 corresponding 6-cycles in $\mathcal{D}$, as can be checked on the right side of Figure 1, where the corresponding vertices in $\mathcal{D}$ are also black and those corresponding to vertices of $Y_2(\mathcal{D})$ underlined for distinction. Now, the edge $(0_0, 0_2)$ in $Y_1(\mathcal{D})$, corresponding to the 2-path $(0_0, 0_1, 0_2)$ in $\mathcal{D}$, has its 2 composing arcs separating the oriented triangles $\phi(A_0) = (0_0, 0_2, 1_0)$ and $\phi(C_0) = (0_0, 0_2, 3_2)$, corresponding to the oriented 6-cycles $A_0 = (0_0, 0_1, 0_2, 0_3, 1_0, 4_3)$ and $C_0 = (0_2, 0_1, 0_0, 3_3, 3_2, 3_1)$.

We notice that the 10 vertices and 10 copies of $K_3 \not\subset K_4$ in either $Y_i(\mathcal{D})$, $(i = 1, 2)$, may be considered as the points and lines of the Desargues self-dual (10_3) configuration, and that the Menger graph of this coincides with $Y_i(\mathcal{D})$ [2]. Each vertex of $Y_i(\mathcal{D})$ is the meeting vertex of 2 copies of K_4 and 3 copies of K_3 not forming part of a copy of K_4.

Theorem 3 $Y_1(\mathcal{D})$ *and* $Y_2(\mathcal{D})$ *are coherent* $\{K_4, K_3\}_{K_2}$-*UH graphs composed by 5 copies of* K_4 *and* 10 *copies of* $K_3 \not\subset K_4$ *each. Moreover, the* 10

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vertices and 10 *copies of* $K_3 \not\subset K_4$ *in either graph constitute the Desargues self-dual* (10_3) *configuration whose Levi graph is* $\mathcal{D}$ *and whose Menger graph is equal to both* $Y_1(\mathcal{D})$ *and* $Y_2(\mathcal{D})$. *Furthermore, both graphs are isomorphic to* $L(K_5)$, *whose complement is the Petersen graph.* □

Deleting a copy H of K_4 from such $Y_i(\mathcal{D})$ $(i = 1, 2)$ yields a copy J of $K_{2,2,2}$, 4 of whose composing copies of K_3, with no common edges, are faces of corresponding copies of $K_4 \neq H$. The other 4 copies of K_3 are among the 10 mentioned copies of K_3 in G. A realization of $Y_i(\mathcal{D})$ in 3-space can be obtained from a regular octahedron O_3 with 1-skeleton J via the midpoints, say x_1, x_2, x_3, x_4, of the 4 segments joining the barycenters of 4 edge-disjoint alternate triangles, say T_1, T_2, T_3, T_4, in O_3 to the barycenter of O_3: just construct the tetrahedron Δ_j determined by each T_j and corresponding x_j, as well as the tetrahedron Δ_0 determined by the 4 x_i s.

A realization κ of K_5 in 3-space is obtained whose vertices are the barycenters of $\Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4$ and whose edges are the segments that join those barycenters. By taking the midpoints of the segments realizing the edges of κ and joining each two of them, say midpoints P and Q of respective segments p and q, by a new segment whenever p and q have an end in common in κ, a realization $L(\kappa)$ of $L(K_5)$ is obtained. This $L(\kappa)$ is a smaller realization of $L(K_5)$ than that of $Y_i(\mathcal{D})$ in the previous paragraph and leads to an octahedron $O'_3 \subset O_3$ by the deletion of its central copy of K_4. This procedure may be repeated indefinitely, generating a nested sequence of realizations of $Y_i(\mathcal{D})$ in 3-space. Since $Y_1(\mathcal{D})$ and $Y_2(\mathcal{D})$ are isomorphic to $L(K_5)$, whose complement is the Petersen graph, this sequence yields a corresponding infinite sequence of realizations of the Petersen graph in 3-space.

5   Generalization of Theorem 3

Theorem 3 can be partly generalized by replacing $L(K_5)$ by $L(K_n)$ $(n \geq 4)$. This produces a coherent $\{K_{n-1}, K_3\}_{K_2}$-UH graph.

Theorem 4 *The line graph* $L(K_n)$, *with* $n \geq 4$, *is a coherent* $\{K_{n-1}, K_3\}_{K_2}$-*UH graph with* n *copies of* K_{n-1} *and* $\binom{n}{3}$ *copies of* $K_3 \not\subset K_{n-1}$.

Proof. Each vertex v of K_n is taken as a color of edges of $L(K_n)$ under the following rule: color all the edges between vertices of $L(K_n)$ representing edges incident to v with color v. Then, each triple of edge colors of $L(K_n)$

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corresponds to the edges of a well determined copy of $K_3 \not\subseteq K_{n-1}$ in $L(K_n)$. Thus, there are exactly $\binom{n}{3}$ copies of $K_3 \not\subseteq K_{n-1}$ intervening in $L(K_n)$ looked upon as a coherent $\{K_{n-1}, K_3\}_{K_2}$-UH graph. $\qquad\square$
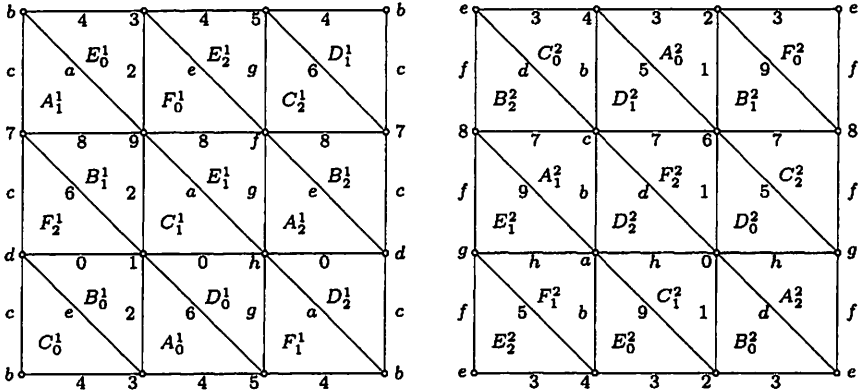
6 Pappus reattachment Menger graph

Figure 3: Toroidal cutouts of $Y_1(\mathcal{P})$ and $Y_2(\mathcal{P})$

Both $Y_1(\mathcal{P})$ and $Y_2(\mathcal{P})$ are embeddable into a closed orientable surface T_1 of genus 1, or 1-torus. Toroidal cutouts of $Y_1(\mathcal{P})$ and $Y_2(\mathcal{P})$ are as in Figure 3, which we consider composed by oriented triangles taken with their orientations derived from those of the 6-cycles of $\mathcal{P}$ in the proof of Theorem 2, according to the 2 alternate operations for $\mathcal{P}$ mentioned at the end of Section 3 similar to (2). These oriented copies of K_3 are contractible in T_1. They form 2 collections $\vec{\mathcal{H}}_1, \vec{\mathcal{H}}_2$ of oriented copies y_i^j of K_3 closed under parallel translation, where $y = A, B, C, D, E, F$; $i = 0, 1, 2$ and $j = 1, 2$, namely: the 9 oriented triangles of $\vec{\mathcal{H}}_1$ (resp. $\vec{\mathcal{H}}_2$) each with horizontal arc below (resp. above) its opposite vertex. There is also a collection $\vec{\mathcal{H}}_0$ of 9 non-contractible oriented triangles in G traceable linearly in 3 different parallel directions, 3 such triangles per direction, with: (a) the orientation of each participating arc $\vec{e}$ equal to the orientation of the arc of an oriented triangle in $\vec{\mathcal{H}}_1$ having the same end-vertices as $\vec{e}$; (b) the arcs of each such oriented triangle indicated by the (common) middle vertex of the corresponding 2-arcs in $\mathcal{P}$, as in Section 3. There are embeddings of $Y_1(\mathcal{P})$ and $Y_2(\mathcal{P})$ in T_1 for which $\vec{\mathcal{H}}_0$ (resp. $\vec{\mathcal{H}}_0^{-1}$) and $\vec{\mathcal{H}}_1$ (resp. $\vec{\mathcal{H}}_2$) provide the composing faces. In addition, each of $\vec{\mathcal{H}}_1$, $\vec{\mathcal{H}}_2$ and $\vec{\mathcal{H}}_0$ (or $\vec{\mathcal{H}}_0^{-1}$) is formed by 3 classes of parallel oriented triangles, such that any 2 triangles in a

class are disjoint. The self-dual (9_3)-configuration in the following theorem is the Pappus 9_3 [2]. Let $\mathcal{H}_i$ be an undirected version of $\vec{\mathcal{H}}_i$, for $i = 0, 1, 2$. Let H_i and $\vec{H}_i$ be respective representatives of $\mathcal{H}_i$ and $\vec{\mathcal{H}}_i$, for $i = 0, 1, 2$, and $\vec{H}_0^{-1}$ be a representative of $\vec{\mathcal{H}}_0^{-1}$.

Theorem 5 $Y_1(\mathcal{P})$ and $Y_2(\mathcal{P})$ are isomorphic tightly coherent $\{H_0, H_1 \cup H_2\}^{P_2}$-homogeneous graphs, as well as $\{\vec{H}_1, \vec{H}_2\}^{\vec{P}_2}$-, $\{\vec{H}_1, \vec{H}_0\}^{\vec{P}_2}$- and $\{\vec{H}_2, \vec{H}_0^{-1}\}^{\vec{P}_2}$-homogeneous digraphs. Moreover, each of $Y_1(\mathcal{P})$ and $Y_2(\mathcal{P})$ can be taken as the Menger graph of the Pappus self-dual (9_3)-configuration, in 12 different fashions, by selecting the point set $\mathcal{P}$ and the line set $\mathcal{L} \neq \mathcal{P}$ so that $\{\mathcal{P}, \mathcal{L}\} \subset \{V(\mathcal{P}), \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2\}$ and the point-line incidence relation either as the inclusion of a vertex in a triangle or as the containment by a triangle of a vertex or as the sharing of an edge by 2 triangles.

Proof. The claimed 12 different forms correspond to the arcs of the complete graph on vertex set $\{V(\mathcal{P}), \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2\}$. □

References

[1] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance–Regular Graphs, Springer-Verlag, New York 1989.

[2] H. S. M. Coxeter, *Self-dual configurations and regular graphs*, Bull. Amer. Math. Soc., **56** (1950), 413–455.

[3] I. J. Dejter, *From the Coxeter graph to the Klein graph*, Journal of Graph Theory, Wyley Online Library.

[4] I. J. Dejter, *Orienting and separating distance-transitive graphs*, arXiv 0904.1086.

[5] D. C. Isaksen, C. Jankowski and S. Proctor, *On K_*-ultrahomogeneous graphs*, Ars Combinatoria, Volume LXXXII, (2007), 83–96.