Polyhedral Approach to Integer Partitions

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Abstract

This paper develops the polyhedral approach to integer partitions. We consider the set of partitions of an integer n as a polytope $P_n \subset \mathbb{R}^n$. Vertices of P_n form the class of partitions that provide the first basis for the whole set of partitions of n. Moreover, we show that there exists a subclass of vertices, from which all others can be generated with the use of two combinatorial operations. The calculation demonstrates considerable decrease in the cardinality of these classes of basic partitions as n grows. We focus on the vertex enumeration problem for P_n. We prove that vertices of all partition polytopes form a partition ideal of the Andrews partition lattice. This allows us to construct vertices of P_n by a lifting method, which requires examining only certain partitions of n. A criterion of whether a given partition is a convex combination of two others connects vertices with knapsack partitions, sum-free sets, Sidon sets, and Sidon multisets introduced in the paper. All but a few non-vertices for small n's were recognized with its help. We also prove several easy-to-check necessary conditions for a partition to be a vertex.

1 Introduction

We develop the polyhedral approach to integer partitions proposed in [6]. A partition of a positive integer n is any finite non-decreasing sequence of positive integers $n_1, n_2, \ldots, n_r$ such that

$$\sum_{j=1}^{r} n_j = n. \tag{1}$$

Informally, a partition of n is any its representation of the form (1). The integers $n_1, n_2, \ldots, n_r$ are called the parts of a partition [1].

The polyhedral approach is based on the n-dimensional geometrical interpretation of integer partitions [14] that is common in Diophantine analysis but seldom used in the partition theory. A partition of n is referred to as a non-negative integer point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ whose components x_i, $i = 1, \ldots, n$, indicate the numbers of times the parts i enter the partition. So, x is a solution to the equation $x_1 + 2x_2 + \ldots + nx_n = n$. We keep on writing $x \vdash n$ to indicate that $x \in \mathbb{R}^n$ is a partition of n. For example, the partition $8 = 4 + 2 + 1 + 1$ with three distinct parts 1, 2, 4 is considered as the point $x = (2, 1, 0, 1, 0, 0, 0, 0) \in \mathbb{R}^8$.

The polytope $P_n \subset \mathbb{R}^n$ of partitions of n is defined as the convex hull of the set T_n of all partitions of n :

$$P_n = \operatorname{conv} T_n = \operatorname{conv} \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x \vdash n\}.$$

The conversion from set to polytope brings geometry into arithmetic of partitions and raises new problems concerned with the geometrical structure of integer partition polytopes. The well-known 2-dimensional interpretation of partitions as Young tables, which proved to be extremely useful for studying connections between individual partitions, hardly provides tools to treat the set of partitions of an integer as a whole.

There are two ways to describe any polytope: 1) to indicate its facets, i.e. faces of the maximal dimension, and 2) to enumerate its vertices. Facets of P_n were described in [6] as all but one coordinate hyperplanes and certain solutions of a system of subadditive inequalities and equalities. This was done with the use of a representation of P_n as a polytope on a partial algebra and a technique borrowed from the group theoretic approach to the integer linear programming problem and generalized for the case.

This article focuses on vertices of partition polytopes. As for any polytope, a point $x \in P_n$ is its vertex if it cannot be expressed as a convex combination $x = \sum_{j=1}^{k} \lambda_j y^j$, $\sum_{j=1}^{k} \lambda_j = 1$, $\lambda_j > 0$, of some other points $y^j \in P_n$, $j = 1, \ldots, k$. Vertices of P_n are of importance because they form a kind of basis for T_n as each $x \vdash n$ is a convex combination of some vertices.

The first partition that is not a vertex appears for $n = 4$. There are five partitions of 4: $x_1 = (4, 0, 0, 0)$, $x_2 = (2, 1, 0, 0)$, $x_3 = (1, 0, 1, 0)$, $x_4 = (0, 2, 0, 0)$, and $x_5 = (4, 0, 0, 0)$. P_4 is a tetrahedron with vertices x_1, x_3, x_4, x_5 since $x_2 = \frac{1}{2}(x_1 + x_4)$ is not a vertex. For larger n's, the vertex recognition problem for P_n "Is a given partition $x \vdash n$ a vertex of P_n?" cannot be solved the same easily. However, the calculation shows that the gap between the number of vertices and the number of partitions rapidly increases as n grows.

Another motif to study vertices of P_n is linked to optimization problems on partitions: these are vertices who provide their optimal solutions in the linear case. Anyways, we believe that any result on the topic is of interest for its own sake.

The paper is organized as follows. Section 2 contains notation and a few quotes of previous results. In section 3, we prove that vertices of partition polytopes form a partition ideal of the partition lattice introduced by Andrews [1]. This property is crucial for constructing vertices of P_n by a lifting method: they should be selected from only those partitions of n that are induced by certain vertices of some polytopes P_j, $j < n$. The main result of section 4 is a criterion of whether a given partition can be expressed as a convex combination of two others. It generalizes all known necessary conditions for vertices and provides some new ones, in particular the exact bound on the number of distinct parts of vertices. A great part of partitions that are not vertices can be recognized and rejected with the help of this criterion, however, for some $n \geq 15$, there exist non-vertices that need three partitions for their convex representations.

In Section 5, we show that integer partition polytopes possess an intriguing feature: there exists a subset of support vertices, from which all others can be generated by two operations of merging parts. Numerical data testify that the subset of support vertices is small by comparison with the set of all vertices. Some results of Sections 4 and 5 were announced in rapid publications [7], [8].

In Section 6, we discuss relations of vertices of P_n to knapsack partitions, sum-free sets, Sidon sets, and Sidon multisets. In conclusion, we sketch the most promising directions for the future study. In particular, we suggest that the number of vertices of P_n inversely depends on the number of divisors of n.

2 Preliminaries

Throughout the paper, $\mathbb{Z}_+$ denotes the set of nonnegative integers, $[1, m]$ denotes the set of integers $\{1, 2, \dots, m\}$, $0 < m \in \mathbb{Z}$. For $\alpha \in \mathbb{R}$, $\lfloor \alpha \rfloor$ and $\lceil \alpha \rceil$ denote the greatest (respectively, the least) integer not greater (respectively, not less) than α. We denote by $S(x)$ the set $\{i \in [1, n] \mid x_i > 0\}$ of distinct parts of x, write $\operatorname{vert} P$ for the set of vertices of a polytope P and 0^k for the sequence of k zeroes. Symbol $\biguplus$ denotes the union of disjoint sets.

Recall some results from [6]. We study the polytope P_n in $\mathbb{R}^n$, though, in fact, it is $(n-1)$-dimensional since it belongs to the hyperplane

$$x_1 + 2x_2 + \ldots + nx_n = n.$$

It is not hard to see that P_n is a pyramid with the point $(0^{n-1}, 1)$ as the apex and the base lying in the hyperplane $x_n = 0$.

Define transformations $\varphi_i : \mathbb{R}^{n-i} \to \mathbb{R}^n$, $i = 1, 2, \ldots, n-1$, as

$$\varphi_i(y_1, y_2, \ldots y_{n-i}) = (y_1, y_2, \ldots, y_{i-1}, y_i + 1, y_{i+1}, \ldots, y_{n-i}, 0^i).$$

Each φ_i is the composition of the translation by 1 along the i-axis and the embedding of $\mathbb{R}^{n-i}$ into $\mathbb{R}^n$. It is easy to see that if $y \vdash n-i$ then $\varphi_i(y) \vdash n$. Conversely, for $x \vdash n$ with some $x_i > 0$, $i < n$, the preimage $\varphi_i^{-1}(x)$ is well-defined and $\varphi_i^{-1}(x) \vdash n-i$.

Some necessary and some sufficient conditions for $x \in \operatorname{vert} P_n$ were obtained in [6]. One of those is as follows.

Theorem 1 ([6]) *Let $1 = i_1 < i_2 < \ldots < i_k \leq n$ be an increasing sequence of integers. Define $n_k = n$, $x_{i_k} = \lfloor \frac{n_k}{i_k} \rfloor$; $n_{k-1} = n_k - x_{i_k} i_k$, $x_{i_{k-1}} = \lfloor \frac{n_{k-1}}{i_{k-1}} \rfloor$; $\ldots$; $n_1 = n_2 - x_{i_2} i_2$, $x_1 = x_{i_1} = \lfloor \frac{n_1}{i_1} \rfloor = n_1$; and $x_i = 0$ for $i \neq i_1, i_2, \ldots, i_k$. Then $x = (x_1, x_2, \ldots, x_n)$ is a vertex of P_n.* □

One can see that the theorem holds for the case $i_1 > 1$ and $\frac{n_1}{i_1}$ integer.

Partitions of n with parts in a given subset $M \subset [1, n]$ are often studied. They form the polytope $P_n(M) = \operatorname{conv}\{x \vdash n \mid S(x) \subseteq M\}$.

Theorem 2 ([6]) *A vertex x of P_n is a vertex of $P_n(M)$ if and only if $x_i = 0$ for all $i \notin M$.* □

3 Vertex ideals and generating vertices

In this section, we show how one can construct all vertices of P_n provided certain vertices of some polytopes P_j, $j < n$, are known. This method is based on a lattice property of vertices.

It is shown in [1] that the set of all partitions of all numbers forms a lattice $\mathcal{P}$ relative to the partial order that can be defined as follows. Let $x \vdash n$ and $y \vdash m$, where $n, m \in \mathbb{N}$, $n \geq m$. Consider $y \preceq x$ if $y_i \leq x_i$ for all $i \in [1, m]$. Then the lower bound $u \wedge v$ of two partitions $u, v \in \mathcal{P}$ is the partition with parts in $S(u) \cap S(v)$ that contains each part i $\min(u_i, v_i)$ times (we consider $u_i, v_i = 0$ for all i larger than the partitioned numbers). The upper bound $u \vee v$ is the partition with parts in $S(u) \cup S(v)$ that contains each part i $\max(u_i, v_i)$ times. One can check that these operations satisfy the lattice identities.

Remind that for any lattice $\mathcal{L}$ with the partial order $\preceq_L$, a subset $\mathcal{M} \subset \mathcal{L}$ is called an ideal of $\mathcal{L}$ if $\mathcal{M}$ contains the lower bound of any its two elements and satisfies the condition: $m \in \mathcal{M}$, $l \in \mathcal{L}$, $l \leq_{\mathcal{L}} m$ imply $l \in \mathcal{M}$ [2]. Sometimes the terms "the order ideal" and "semi-ideal" are used [12] but we follow Andrews [1] and, in the case $\mathcal{L} = \mathcal{P}$ we deal with, call such $\mathcal{M}$ a partition ideal.

For any integer $k \geq 2$, denote by $\mathcal{V}_k$ (respectively, $\mathcal{V}_{\leq k}$) the set of partitions $x \vdash n$ of all $n \in \mathbb{N}$ that cannot be expressed as convex combinations of exactly k (respectively, at most k) partitions of n.

Proposition 1 *$\mathcal{V}_k$ and $\mathcal{V}_{\leq k}$ are partition ideals of $\mathcal{P}$ for $k \geq 2$.*

Proof. Prove the assertion for the case of $\mathcal{V}_k$. Let $x \in \mathcal{V}_k$, $y \in \mathcal{P}$, and $y \preceq x$. It is sufficient to show that x without any its part i belongs to $\mathcal{V}_k$. Then one can apply this claim consequently for all parts $i \in S(x)$ that are extra relative to y and conclude that $y \in \mathcal{V}_k$.

Deleting a part i from x results in the partition $z = \varphi_i^{-1}(x)$ of $n-i$. Assume $z \notin \mathcal{V}_k$. Then z is a convex combination $z = \sum_{t=1}^k \lambda_t z^t, \sum_{t=1}^k \lambda_t = 1$, $\lambda_t > 0$, of some k partitions $z^t \vdash n-i$, $1 \leq t \leq k$. Define integer points $x^t \in \mathbb{R}^n$, $1 \leq t \leq k$, with the components: $x_i^t = z_i^t + 1$; $x_j^t = z_j^t$, $1 \leq j \leq n-i$, $j \neq i$; $x_j^t = 0$, $n-i < j \leq n$. It is clear that all $x^t \vdash n$ and we have the convex representation $x = \sum_{t=1}^k \lambda_t x^t$ since

$$\sum_t \lambda_t x_i^t = \sum_t \lambda_t (z_i^t + 1) = 1 + \sum_t \lambda_t z_i^t = 1 + z_i = x_i,$$

$$\sum_t \lambda_t x_j^t = \sum_t \lambda_t z_j^t = z_j = x_j \quad \text{for} \quad 1 \leq j \leq n-i, \quad j \neq i,$$

$$\sum_t \lambda_t x_j^t = 0 \quad \text{for} \quad n-i < j \leq n.$$

This contradicts $x \in \mathcal{V}_k$, yields $z \in \mathcal{V}_k$, and ends the proof of the first claim. The case of $\mathcal{V}_{\leq k}$ can be considered similarly, the difference is that the number of partitions of $n-i$ is some k_1, $2 \leq k_1 \leq k$. □

Denote by $\mathcal{V}$ the set of vertices of all integer partition polytopes,

$$\mathcal{V} = \bigcup_{n \in \mathbb{N}} \operatorname{vert} P_n.$$

Theorem 3 *The following statements are true:*

(i) $\mathcal{V}$ *is a partition ideal of* $\mathcal{P}$;

(ii) $\mathcal{V} = \bigcap_{k \geq 2} \mathcal{V}_k$;

(iii) $\mathcal{V} = \lim_{k \to \infty} \mathcal{V}_{\leq k}$.

Proof. (ii) follows from the definition of vertices of P_n : these are those partitions $x \vdash n$ that cannot be expressed as convex combinations of any $k \geq 2$ partitions of n. Inclusion $\mathcal{V}_{\leq k+1} \subseteq \mathcal{V}_{\leq k}$, $k \geq 2$, implies (iii). Statement (i) can be proved directly but it follows from (ii) and the fact that the intersection of any two ideals is an ideal. □

We will see in Section 4 that $\mathcal{V}_2 \setminus \mathcal{V}_3 \neq \emptyset$, so that $\mathcal{V} \subseteq \mathcal{V}_{\leq 3} \subset \mathcal{V}_2$. However, it is not known yet whether $\mathcal{V}_{\leq k} \setminus \mathcal{V}_{\leq k+1} \neq \emptyset$ for any $k > 2$.

Corollary 1 $\mathcal{V}$ *is a lower sublattice of* $\mathcal{P}$ *but not its sublattice.*

Proof. The statement (i) implies $u \wedge v \in \mathcal{V}$ for all $u, v \in \mathcal{V}$. The instance $u = (1,1,0) \vdash 3$ and $v = (2,0) \vdash 2$ shows that $\mathcal{V}$ is not a lattice since $u \vee v = (2,1,0,0) \vdash 4$ is the half-sum of two partitions of 4, $(2,1,0,0) = \frac{1}{2}((0,2,0,0) + (4,0,0,0))$, whence $u \vee v \notin \mathcal{V}$. □

The next corollary extends Theorem 2 [6].

Corollary 2 *For any* $x \in \operatorname{vert} P_n$ *with a part* $i < n$, *the inclusion* $\varphi_i^{-1}(x) \in \operatorname{vert} P_{n-i}$ *holds.* □

Now we show that the most complicated case of the vertex recognition problem for P_n is that when all parts of the partition are small.

Proposition 2 *The following assertions are true:*

(i) *a partition* $x \vdash n$ *with some* $x_i > 0$, $\lfloor \frac{n}{2} \rfloor < i < n$, *is a vertex of* P_n *if and only if* $y = \varphi_i^{-1}(x)$ *is a vertex of* P_{n-i};

(ii) *for any even* n, *the partition* $n = \frac{n}{2} + \frac{n}{2}$ *is the unique vertex of* P_n *with* $x_{\frac{n}{2}} > 0$.

Proof. In view of Corollary 2, to prove (i), it remains to prove only one implication: if $x \vdash n$, $x_i > 0$, $\lfloor \frac{n}{2} \rfloor < i < n$, and $y \in \operatorname{vert} P_{n-i}$, then $x \in \operatorname{vert} P_n$. First, note that in this case $x_i = 1$. Assume $x \notin \operatorname{vert} P_n$. Then x is a convex combination of some $x^1, \ldots, x^k \vdash n$ with $x_i^t = 1$ and $x_j^t = 0$ for all

$j > n-i, j \neq i, t = 1, \dots, k$, since $x_j^t > 0$ would imply $x_j^t j + x_i^t i > n$. Therefore, $y^1 = \varphi_i^{-1}(x^1), \dots, y^k = \varphi_i^{-1}(x^k)$ are well-defined and are partitions of $n-i$. One can check that y is a convex combination of $y^1, \dots, y^k$, whence $y \notin \operatorname{vert} P_{n-i}$, as in the proof of Proposition 1. The contradiction completes the proof of (i).

Prove (ii). Notice that $(0^{\frac{n}{2}-1}, x_{\frac{n}{2}} = 2, 0^{\frac{n}{2}}) \in \operatorname{vert} P_n$. Further, if $x = (x_1, x_2, \dots, x_{\frac{n}{2}} = 1, \dots, x_n) \in \operatorname{vert} P_n$ then $x_{\frac{n}{2}+1} = \dots = x_n = 0$ and x is the half-sum of partitions $(2x_1, 2x_2, \dots, 2x_{\frac{n}{2}-1}, x_{\frac{n}{2}} = 0, 0^{\frac{n}{2}})$ and $(0^{\frac{n}{2}-1}, x_{\frac{n}{2}} = 2, 0^{\frac{n}{2}})$. Thus any $x \in \operatorname{vert} P_n$ with $x_{\frac{n}{2}} > 0$ satisfies $x_{\frac{n}{2}} = 2$. □

For $m \in \mathbb{N}$, denote $T_m[\geq i] = \{x \vdash m \mid x_j = 0, j < i\}$ and consider the polytope $P_m[\geq i] = \operatorname{conv} T_m[\geq i]$ of partitions of m with all parts $\geq i$.

Theorem 4 *The set T_n of partitions of n and the set of vertices of P_n satisfy the following recurrence relations:*

(2)
$$T_n = \left(\biguplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} \varphi_i(T_{n-i}[\geq i]) \right) \bigcup (0^{n-1}, 1),$$

(3)
$$\operatorname{vert} P_n \subseteq \left(\biguplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} \varphi_i(\operatorname{vert} P_{n-i}[\geq i]) \right) \bigcup (0^{n-1}, 1).$$

Proof. Note that $(0^{n-1}, 1) \in \operatorname{vert} P_n$. Let $x \neq (0^{n-1}, 1)$ be a partition of n, and let i be its least part. Then $i \leq \lfloor \frac{n}{2} \rfloor$ and $x = \varphi_i(y)$ for some $y \vdash n-i$. This implies inclusion $\subseteq$ in (2). The opposite inclusion is obvious, as well as disjointness, so (2) is proved.

The proof of (3) is similar. For any $(0^{n-1}, 1) \neq x \in \operatorname{vert} P_n$ with the least part $i \leq \lfloor \frac{n}{2} \rfloor$, Theorem 3 (i) implies $x = \varphi_i(y)$ for some $y \in \operatorname{vert} P_{n-i}$ with $y_j = 0$, $j = 1, \dots, i-1$. By Theorem 2, y is a vertex of $P_{n-i}[\geq i]$. □

(3) can be used as the base for the lifting method to construct vertices of P_n. It states that the set of partitions, for which the vertex recognition problem should be solved, can be reduced to the set of φ_i-images of all vertices of the polytopes P_{n-i}, $i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$, with parts $\geq i$. In subsequent sections we consider how to treat this problem.

4 Convex combination of two partitions

In this section, we characterize partitions that are convex combinations of two partitions of the same number and deduce new easy-to-check necessary conditions for a partition to be a vertex.

Theorem 5 ([7]) *A partition $x \vdash n$ is a convex combination of two partitions of n (whence $x \notin \operatorname{vert} P_n$) if and only if there exist two disjoint subsets S_1, S_2 of parts of x and two tuples of integers $u = \langle u_j \in \mathbb{N}; j \in S_1 \rangle$, $v = \langle v_k \in \mathbb{N}; k \in S_2 \rangle$ satisfying relations*

$$(4) \qquad \sum_{j \in S_1} u_j j = \sum_{k \in S_2} v_k k, \quad u_j \le x_j, \quad v_k \le x_k.$$

Proof. Given the subsets $S_1, S_2 \subset S(x)$ and the tuples u and v, one can build partitions $y, z \vdash n$, of which x is the half-sum, by setting

$$\begin{array}{lll} y_j = x_j + u_j, & z_j = x_j - u_j, & j \in S_1; \\ y_i = x_i, & z_i = x_i, & i \notin S_1 \cup S_2; \\ y_k = x_k - v_k, & z_k = x_k + v_k, & k \in S_2. \end{array}$$

Conversely, if $x \vdash n$ is a convex combination $x = z + \lambda(y - z)$, $0 < \lambda < 1$, of two partitions $y, z \vdash n$ then λ is rational and we can consider that $\lambda = \frac{p}{q}$, with p and q coprime. Then q divides all components of $y - z$ and x is the half-sum of partitions $z + \frac{p-1}{q}(y - z)$ and $z + \frac{p+1}{q}(y - z)$ of n. So we can consider that $x = \frac{1}{2}(y + z)$. Define subsets $S_1 = \{j \in S(x) \mid x_j < y_j\}$ and $S_2 = \{k \in S(x) \mid x_k > y_k\}$. It is easy to see that $S_1, S_2 \subseteq S(x)$, $S_1 \cap S_2 = \emptyset$, $S_1 = \{j \in S(x) \mid x_j > z_j\}$, $S_2 = \{k \in S(x) \mid x_k < z_k\}$. The tuples u and v can be constructed by setting $u_j = y_j - x_j$, $j \in S_1$, and $v_k = x_k - y_k$, $k \in S_2$. The equality $x = \frac{1}{2}(y + z)$ and nonnegativity of x, y, z imply $u_j < x_j$ and $v_k < x_k$, and $x, y \vdash n$ implies equality (4). □

Corollary 3 *For a given $x \in \operatorname{vert} P_n$, no integer $k < n$ of the form $k = \sum_{i \in S(x)} q_i i$, $q_i \in \mathbb{Z}_+$, $q_i \le x_i$, except for the trivial case $k = 1 \cdot i$, is a part of x.* □

Corollary 4 *Replacing requirement $S_1 \cap S_2 = \emptyset$ by $S_1 \neq S_2$ transforms Theorem 5 to an equivalent form.*

Proof. Disjoint sets are nonequal, so the new version of the statement follows from the original one. To prove the opposite implication, note that if some S_1, S_2, $S_1 \neq S_2$, satisfy (4) and $i \in S_1 \cap S_2$, $u_i \le v_i$ then i can be excluded from S_1 and either (a) left in S_2 while v_i replaced by $v_i - u_i$ if $u_i < v_i$, or (b) excluded from S_2 as well if $u_i = v_i$. □

Theorem 5 has a simple interpretation. Given some $x \vdash n$, consider that for every $i \in S(x)$, one has x_i weights of i grams each. Then (4) means that there exists some weight that can be weighed in two different ways with the use of the given weights.

The criterion (4) successfully determines all partitions of $n \leq 20$ that are not vertices of P_n, except one for $n = 15$. The deviant partition $15 = 2 \cdot 3 + 4 + 5$ with parts 3, 4, 5 is the convex combination of three partitions, $\frac{1}{3}(3 \cdot 5) + \frac{1}{3}(3 + 3 \cdot 4) + \frac{1}{3}(5 \cdot 3)$, but not of any two. For $n = 21$, there are three partitions of this kind: $1 + 4 + 7 + 9$, $3 + 5 + 6 + 7$, and $3 \cdot 3 + 5 + 7$. The next proposition shows that these partitions are not exclusions.

Proposition 3 *For any integer $k \geq 0$, partitions $1 + (4 + k) + (7 + 2k) + (9 + 3k)$ and $3 + (5 + k) + (6 + k) + (7 + k)$ of the numbers $n = 21 + 6k$ and $n = 21 + 3k$, respectively, are convex combinations of three partitions of n but not of any two ones.*

Proof. Each partition of the two series above can be expressed as a convex combination of three partitions: $3 \cdot (7 + 2k)$, $3 \cdot (4 + k) + 1 \cdot (9 + 3k)$, $3 \cdot 1 + 2 \cdot (9 + 3k)$ for the first, and $3 \cdot (7 + k)$, $1 \cdot 3 + 3 \cdot (6 + k)$, $2 \cdot 3 + 3 \cdot (5 + k)$, for the second. Not harder is it to see that these partitions do not satisfy condition (4), whence not any two partitions are sufficient for their convex representations. □

Theorem 6 *$\mathcal{V}_2 \neq \mathcal{V}_3$ and $\mathcal{V} \subset \mathcal{V}_2$.*

Proof follows from Proposition 3 and Theorem 3. □

Theorem 5 induces new necessary conditions for vertices of P_n.

Theorem 7 ([7]) *Every $x \in \mathrm{vert} P_n$ satisfies conditions:*

(i) $\prod_{i \in S(x)} (x_i + 1) \leq n + 1,$

(ii) *the number of distinct parts of x is not greater than $\lfloor \log(n + 1) \rfloor$ and this bound is sharp.*

Proof. Let $x \in \mathrm{vert} P_n$. Then x does not satisfy (4) and all sums

$$\sum_{i \in S(x)} u_i i, \quad 0 \leq u_i \leq x_i, \tag{5}$$

are pairwise different. The number of such sums is $\prod_{i\in S(x)}(x_i+1)$, all of them are less than or equal to n, where the least sum is zero. By the Pigeonhole principle, this implies (i).

Denote the number of distinct parts of $x \in \text{vert}\, P_n$ by d. The inequality $\prod_{i\in S(x)}(x_i+1) \geq 2^d$ obviously holds, so the estimate in (ii) follows from (i). By Theorem 1, the partition $1 + 2 + 2^2 + \ldots + 2^m = n = 2^{m+1} - 1$ is a vertex of P_n, thus the bound is sharp. □

5 Merging parts and support vertices

We show in this section that all vertices of each P_n can be generated from some subset of support vertices using two combinatorial operations of merging parts of partitions. This means that support vertices of P_n constitute an even smaller basis of the set of all partitions of n. At the end, we compare the numbers of vertices and support vertices of some partition polytopes with the total numbers of partitions of the corresponding integers. Define these operations.

Operation $\mu_{u,v}$. Let $x \vdash n$ and let $u, v \in S(x)$, $u \neq v$, be two distinct parts of x; assume that $x_u \leq x_v$. Build the point $y = \mu_{u,v}(x) \in \mathbb{Z}_+^n$ with the components $y_u = 0$, $y_v = x_v - x_u$, $y_{u+v} = x_{u+v} + x_u$, and $y_j = x_j$ for $1 \leq j \leq n$, $j \neq u, v, u+v$.

Operation μ_u. Let $x \vdash n$ and a part $u \in S(x)$ enter x more than once, i.e. $x_u > 1$. Build the point $y = \mu_u(x) \in \mathbb{Z}_+^n$ with the components $y_u = 0$, $y_{x_u u} = x_{x_u u} + 1$, and $y_j = x_j$ for $1 \leq j \leq n$, $j \neq u, x_u u$.

Theorem 8 *Let a vertex x of the polytope P_n have distinct parts $u, v \in S(x)$ and $x_u \leq x_v$. Then $y = \mu_{u,v}(x)$ is a vertex of P_n.*

Proof. At first, we show that $y \vdash n$. Indeed,

$$\sum_{i=1}^{n} y_i i = \sum_{\substack{j=1,\\ j\neq u,\, v,\, u+v}}^{n} y_j j + (x_v - x_u)v + (x_{u+v} + x_u)(u+v) =$$

$$\sum_{\substack{j=1,\\ j\neq u,\, v,\, u+v}}^{n} x_j j + x_v v + x_u(u + v - v) + x_{u+v}(u+v) =$$

$$\sum_{i=1}^{n} x_i i = n.$$

Now prove that $y \in \operatorname{vert} P_n$. By Corollary 3, $x_{u+v} = 0$. Assume $y \notin \operatorname{vert} P_n$. Then y is a convex combination $y = \sum_{t=1}^{k} \lambda_t y^t$, $\sum_{t=1}^{k} \lambda_t = 1$, $\lambda_t > 0$, of some partitions $y^t \vdash n$, $1 \leq t \leq k$. It follows from $y_u = 0$ that $y_u^t = 0$ for all t. Define integer points $x^t \in \mathbb{R}^n$, $1 \leq t \leq k$, with the components

$$x_u^t = y_{u+v}^t; \quad x_v^t = y_{u+v}^t + y_v^t; \quad x_{u+v}^t = 0; \quad x_j^t = y_j^t, j \neq u, v, u+v.$$

One can check that all x^t are partitions of n. Since $\sum_t \lambda_t x_u^t = x_u$; $\sum_t \lambda_t x_v^t = x_v$; $\sum_t \lambda_t x_{u+v}^t = x_{u+v}$; $\sum_t \lambda_t x_j^t = x_j$ for $j \neq u, v, u+v$, we have the convex representation $x = \sum_{t=1}^{k} \lambda_t x^t$, which contradicts x being a vertex of P_n. Therefore, $y \in \operatorname{vert} P_n$. □

The next theorem for μ_u can be proved similarly.

Theorem 9 *Let a part u enter a vertex x of the polytope P_n more than once. Then $y = \mu_u(x)$ is a vertex of P_n.* □

Theorems 8, 9 provide new sufficient conditions for $x \in \operatorname{vert} P_n$.

Let us illustrate application of operations of merging parts using polytope P_6 as an example. There are 7 vertices of P_6 [6]: $x^1 = (6,0,0,0,0,0)$, $x^2 = (2,0,0,1,0,0)$, $x^3 = (1,0,0,0,1,0)$, $x^4 = (0,3,0,0,0,0)$, $x^5 = (0,1,0,1,0,0)$, $x^6 = (0,0,2,0,0,0)$, and $x^7 = (0,0,0,0,0,1)$. We have $\mu_{4,1}(x^2) = x^3$ and $\mu_{1,5}(x^3) = \mu_{2,4}(x^5) = x^7$. Further, $\mu_1(x^2) = x^5$ and $\mu_2(x^4) = \mu_3(x^6) = x^7$. On the other hand, none of the vertices x^1, x^2, x^3, x^6 can be obtained from any other with the use of these operations. Therefore, all vertices of P_6 can be obtained from four vertices x^1, x^2, x^3, x^6 and this is a minimal set of this kind, relative to inclusion. The next definition is natural.

Definition 1 *A vertex of a partition polytope is called support vertex if it does not result from any other vertex of the same polytope with the use of operations $\mu_{u,v}$ or μ_u.*

The inequality $\sum_{i=1}^{n} y_i < \sum_{i=1}^{n} x_i$, provided $y = \mu_{u,v}(x)$ or $s = \mu_u(x)$, implies existence of support vertices of any P_n.

We have seen that x^1, x^2, x^3, x^6 are support vertices of P_6. Denote the numbers of partitions, vertices, and support vertices of P_n by $p(n)$, $v(n)$, and $s(n)$ respectively. The values of these functions for $6 \leq n \leq 23$ are presented in Table 1. One can observe that while the part of support vertices for $n = 6$ constitutes 36% of $p(n)$, it decreases to 19% for $n = 10$ and falls down to 5% for $n = 20$. The ratio $s(n)/v(n)$ also definitely decreases as n grows.

n	$p(n)$	$v(n)$	$s(n)$	n	$p(n)$	$v(n)$	$s(n)$	n	$p(n)$	$v(n)$	$s(n)$
6	11	7	4	12	77	25	9	18	385	75	19
7	15	11	5	13	101	41	13	19	490	117	28
8	22	12	5	14	135	41	12	20	627	99	27
9	30	17	8	15	176	57	15	21	792	146	42
10	42	19	8	16	231	56	17	22	1002	140	36
11	56	29	8	17	297	84	20	23	1255	211	42

Table 1. Numbers of partitions, vertices and support vertices.

The values of $v(n)$ and $s(n)$ calculated for all $n \leq 100$ can be found in The On-Line Encyclopedia of Integer Sequences [9, 10].

6 Additive structures related to vertices

Theorem 5 reveals relations of vertices and the vertex recognition problem for integer partition polytopes with several structures of additive combinatorics. Ehrenborg and Readdy [4] independently came to a class of knapsack partitions. These are partitions, all collections of parts of which give different sums. Theorem 5 states that knapsack partitions are just those that cannot be expressed as convex combinations of two other partitions of the same number. Therefore, knapsack partitions form the class $\mathcal{V}_2$ and we can refer to the problem "Does a given partition belong to the $\mathcal{V}_2$ class?" as the decision problem *Knapsack Partition*, "Is a given partition knapsack partition?" Ehrenborg displayed the numbers $k(n)$ of knapsack partitions for $n \leq 50$ in The On-Line Encyclopedia of Integer Sequences [3]. Comparison of $v(n)$ with $k(n)$ shows that $v(n) = k(n)$ for all $n < 24$, $n \neq 15, 21$, and $v(15) = k(15) - 1$, $v(21) = k(21) - 3$, where the differences are caused by existence of partitions that need three partitions for their convex representations.

The well-known structures of additive combinatorics related to vertices are Sidon and sum-free sets [13]. We will define Sidon sets in the way that looks most accepted now, though the original Sidon's definition [11] was slightly different and the term B_2-sequence could be more appropriate [5]. So, let a Sidon set be a set $A \subset \mathbb{N}$ such that

$$a_1 + a_2 \neq a_3 + a_4 \tag{6}$$

for all $a_i \in A$ unless $\{a_1, a_2\} = \{a_3, a_4\}$. Some authors, see [5, 13] for references, consider Sidon sets of order $h > 2$ (or B_h-sequences). In their definition, instead of pairs, both sums in (6) engage all possible h-tuples, h fixed. Considering inequalities $a_1 + a_2 \neq a_3$ instead of (6) brings us to sum-free sets $A \subset \mathbb{N}$, see [13]. Note that repetitions of numbers are allowed in all cases.

Theorem 5 states that $x \in \mathcal{V}_2$ (respectively, if $x \in \operatorname{vert} P_n$) if and only if (respectively, then) no two h- and k-tuples of elements of $S(x)$, with each element i engaged at most x_i times, have equal sums; here h and k are arbitrary integers $< \sum_{a \in A} x_a$. So, knapsack partitions differ from Sidon sets in that

(a) the lengths h and k of the tuples can be different and

(b) repetitions of any $a \in A$ in every tuple are restricted by x_a.

We introduce the notion of Sidon multiset. Recall that a multiset is a pair $\langle A, x \rangle$ of a set A and a positive integer-valued multiplicity function $x : A \to \mathbb{N}$, whose values $x_a, a \in A$, can be considered as the numbers of copies of a in the multiset.

Definition 2 *We call a multiset* $\langle A, x \rangle$, $A \subset \mathbb{N}$, *Sidon multiset if all its submultisets* $\langle B, y \rangle$, *where* $B \subseteq A$ *and* $y_b \leq x_b$ *for all* $b \in B$, *have distinct sums* $\sum_{b \in B} y_b b$ *of their elements.*

Note that all types of Sidon sets and sum-free sets can be represented as multisets with restrictions similar to those for Sidon multisets (e.g. $x_a \geq 2$, $a \in A$, and $\sum_{a \in B} y_a = 2$ for Sidon sets). Sidon multisets satisfy conditions (a) and (b) a priori. Moreover, every knapsack partition corresponds to a Sidon multiset and vice versa: a knapsack partition $x \vdash n$ automatically determines a multiset $\langle S(x), x \rangle$, while the subsets $S_1, S_2 \subset S(x)$ and the tuples u, v from Theorem 5 correspond to its submultisets $\langle S_1, u \rangle$ and $\langle S_2, v \rangle$, so that (4) does not hold for Sidon multisets by definition.

Let us summarize what we know about vertices and the additive structures considered. First, $\mathcal{V} \subset \mathcal{V}_2$. Further, $\mathcal{V}_2$ is a subclass of the classes of Sidon sets of all orders and sum-free sets. $\mathcal{V}_2$ coincides with the class of knapsack partitions and is in one-to-one correspondence with the class of Sidon multisets. The problem *Knapsack Partition* is equivalent to the decision problem "Is a given multiset Sidon multiset?"

7 Concluding remarks

One of the goals of the polyhedral approach to integer partitions is to avoid enumeration of the enormous amount of partitions by exploring the geometrical structure of the polytope they form up. Should one know all vertices of P_n, the whole set of partitions of n could be built as their integer-valued convex combinations. Moreover, one can concentrate on the subset of support vertices since all vertices can be built from these with the use of recursive application of two special operations of merging parts. The criterion of Theorem 5 and more easy-to-check necessary conditions allowed us to reject almost all non-vertices of P_n for small n's. The data demonstrate that the gaps between the numbers of support vertices, vertices, and partitions are considerable. Theorem 5 reveals connections of vertices with several other additive structures. Two new notions introduced in the paper, support vertex and Sidon multiset, deserve in our opinion further attention.

This work draws forth new questions. The first group includes the problems of characterizing vertices, support vertices, knapsack partitions (Sidon multisets), as well as those of obtaining necessary and/or sufficient conditions for a partition to belong to these classes. Exploring connections between vertices and facets would be also valuable.

The targets of the second group include asymptotic behaviour of the functions $k(n)$, $v(n)$, $s(n)$, estimates on their values, and dependence on the properties of n. Similar functions can be considered for some special classes of partitions, e.g. knapsack partitions with distinct parts and so forth. These problems are most likely to be hard — we cannot refer to any beneficial results, only a few close in subject can be found in [13] for Sidon sets. Yet, the known values of $v(n)$ and $k(n)$ demonstrate their definite dependence on the evenness of n : for all odd n's, except a few very small, the intriguing inequalities $k(n) > k(n+1)$ and $v(n) > v(n+1)$ hold. Lots of by hand and computer calculations, as well as some more formal arguments, impel us to suggest a stronger hypothesis: the values of $v(n)$ and $k(n)$ are inversely dependent on the number of divisors of n. If true this facts might have divergent consequences.

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