Maximizing the size of planar graphs under girth constraints

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Abstract. In 1975, Erdős proposed the problem of determining the maximal number of edges in a graph on n vertices that contains no triangles or squares. In this paper we consider a generalized version of the problem, i.e. what is the maximum size, $ex(n;t)$, of a graph of order n and girth at least $t+1$ (containing no cycles of length less than $t+1$). The set of those extremal C_t-free graphs is denoted by $EX(n;t)$. We consider the problem on special types of graphs, such as pseudotrees, cacti, graphs lying in a square grid, Halin, generalized Halin and planar graphs. We give the extremal cases, some constructions and we use these results to obtain general lower bounds for the problem in the general case.

Introduction

Graphs arise in many areas of mathematics and computer science having applications in many other fields as well. Extremal graph theory problems usually ask for the max/min size of a graph having certain characteristics. Such questions are often quite natural in the construction of networks or circuits.

We consider the EX-problem which simply asks given a graph of order n what is the maximal number of edges, denoted by $ex(n;t)$, that can exist in the graph such that it contains no cycle C_k, where $3 \leq k \leq t$. The

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set of those extremal C_t-free graphs is denoted by $EX(n;t)$. Erdős [12] initially posed the problem with $t = 4$. Since then a lot of research has been done trying to obtain exact solutions for the problem on general graphs, or obtaining good lower and upper bounds [1,2,3,4,5,7,10,11,15,16,17,19,20].

Research has also concentrated around special types of graphs, such as bipartite graphs [14,6]. For $n \leq t$, it is easy to show that all trees including stars $K_{1,n-1}$ and paths P_n are extremal graphs and $ex(n;t) = n-1$. For $t+1 \leq n \leq \lfloor 3t/2 \rfloor$ the cycles C_n are extremal graphs and $ex(n;t) = n$. These values are included in the tables as folklore. The problem of finding the extremal number $ex(n;3)$ is solved by Mantel's theorem [18] which states that the maximum size of a triangle free graph of order n is $\lfloor n^2/4 \rfloor$. The extremal graphs $EX(n;3)$ are the complete bipartite graphs $K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$. For $t \geq 4$ no exact general formula is known for $ex(n;t)$. There are however general lower and upper bounds for $t = 4$. Currently the best lower bound is $\frac{n^{3/2}}{2\sqrt{2}}$, given by Garnick and Neuwejaar [16] and the best upper bound is $\frac{1}{2}n\sqrt{n-1}$, given by Garnick, Kwong and Lazebnik [15].

A relevant problem is whether or not a given graph contains cycles of all lengths [8]. Such graphs are called pancyclic and much research has been concentrated around them as well. There are also obvious connections with the cage problem [13], also known as the degree/girth problem, which simply asks for the minimum order of a regular graph of degree d and girth g.

In this paper we consider the EX-problem for planar graphs and special types of planar graphs such as pseudotrees, cacti, Halin, generalized Halin graphs and graphs lying in an infinite square grid, giving the extremal numbers and some constructions.

The rest of the paper is structured as follows. In Section 1 we present the basic definitions used throughout the paper and we define the problems. In Sections 2-5 we consider the C_t-free problem for special types of graphs. Finally, we give some future proposals and a brief conclusion in Section 6.

1 Definitions and Problems

Throughout this paper we consider an undirected graph $G(V, E)$, where V is the set of vertices, also called nodes, and E is the set of edges. The *complement* graph $\overline{G}(V, \overline{E})$ of G has the same vertices as G but edges that appear in G do not appear in $\overline{G}$ and edges that do not appear in G appear in $\overline{G}$. The *order* of a graph is the number of its vertices. The *size* of a graph is the number of its edges. A *path* $P_n(V, E)$ is a graph with $V = \{x_1, x_2, \ldots, x_n\}$ and $E = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. Its end vertices are x_1, x_n and its *length* ℓ is equal to $n - 1$. A *cycle* $C_n(V, E)$, where $n \geq 3$, is a graph with $V = \{x_1, x_2, \ldots, x_n\}$ and $E = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. Its *length* ℓ

is equal to n. A *cycle* is called *odd/even* if its length is *odd/even*. The *girth* g of a graph G is the length of its shortest cycle. A graph containing no cycles is called an *acyclic* graph. The *degree* of a vertex $v \in G$ is denoted by $d(v)$ and is equal to the number of vertices to which v is connected. A *regular* graph is a graph in which all its vertices have the same degree. A graph is *planar* if it can be drawn in a plane without its edges crossing. A *face* is a region surrounded by a cycle in a planar embedding of a graph without any path crossing the cycle. A *tree* T_n is a maximal acyclic graph on n vertices. A *forest* is a disconnected acyclic graph. A *rooted* tree has a node which is called the *root*. In such a tree, each of the nodes that is one graph edge further away from a given node (*parent*) and its distance to the root is one more than its *parent* is called a *child*. Nodes having the same parent node are called *siblings*. The *height* of a tree T_n, denoted by *Height*(T_n), is defined as the maximum length of a path from the root of T_n to a leaf of T_n.

A *pseudotree* is a connected graph with exactly one cycle. A *cactus* graph, sometimes also called a cactus tree, is a connected graph in which any two cycles have no edge in common. Equivalently, it is a connected graph in which any two (simple) cycles have at most one vertex in common. A *Halin* graph is a graph constructed from a plane drawing of a tree having four or more vertices, no vertices of degree two, by connecting all leaves of the tree by a cycle (see Figure 1). It is important to mention that leaves are connected in the order they are found in a post order traversal of the ancestor tree of the Halin graph. The first leaf is then connected to the last one (Figure 2 illustrates that different orderings of the nodes of a tree may produce different Halin graphs). A *generalized Halin* graph is a Halin graph where we allow vertices to have degree two. Obviously in the case that only one leaf is present at the ancestor tree of the Halin graph there is no cycle joining its leafs.

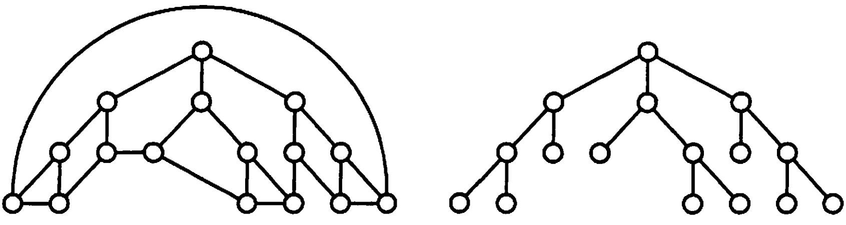

Fig. 1: A Halin graph with its ancestor tree at its right

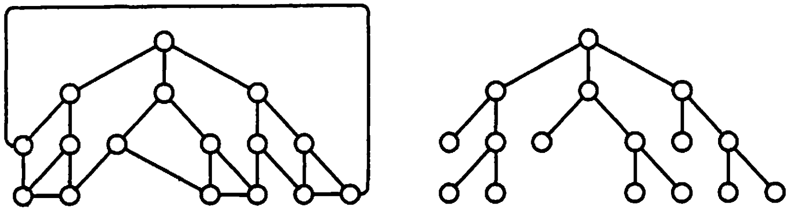

Fig. 2: A different ordering of the tree of Figure 1 produces a different Halin graph

A graph G is an *extremal C_t-free* graph if G has maximum size and girth g at least $t+1$. The set of extremal C_t-free graphs is denoted by $EX(n;t) = EX(n; C_3, C_4, \ldots, C_t)$ and the size of the graph is the extremal number $ex(n;t) = ex(n; C_3, C_4, ..., C_t)$. The largest known lower bound for $ex(n;t)$ is denoted by $ex_l(n;t)$ and the smallest upper bound known by $ex_u(n;t)$.
We consider the following problem:

Problem 1. Given natural numbers n, t find $ex(n,t)$ for the following classes of graphs: pseudotrees, cacti, generalized Halin graphs, rectangular grid graphs and planar graphs.

2 Pseudotrees and Cacti

For a pseudotree the situation is easy to handle as the removal of one edge from the cycle of the pseudotree leaves a tree.

Theorem 1. *An extremal pseudotree of size n and girth at least $t+1$ has $ex(n,t) = n$, where $3 \le t \le n-1$.*

Proof. The only limitation is that the cycle of the pseudotree must have length at least $t+1$. It is then clear that $|E| = n$ as removal of an edge from the cycle of the pseudotree leaves a tree which has $n-1$ edges, no matter the arrangement of the vertices lying on the hanging subtrees of the cycle. □

Cacti graphs are more complex than pseudotrees as they are allowed to have more than one cycle.

Theorem 2. *An extremal cactus graph of size n and girth at least $t+1$ has:*

$$ex(n,t) = \begin{cases} \lfloor \frac{n-t-1}{t} \rfloor + n, & 3 \le t \le n-1 \\ n-1, & t \ge n \end{cases}$$

Proof. The number of edges of the cactus graph is:

$$|E| = (\text{number of cycles in the graph}) + n - 1,$$

as removal of an edge from each cycle of the cactus graph leaves a tree which has $n-1$ edges (or consider Euler's formula with n vertices and number of faces equal to the number of cycles in the graph plus the unbounded face).
Maximizing E is then equivalent to maximizing the number of cycles in the cactus graph.
Cycles of the cactus graph must have length at least $t+1$.
Therefore we can have at most $\lfloor \frac{n-t-1}{t} \rfloor + 1$ cycles ($t+1$ vertices make the first cycle, then the remaining cycles are made by using t vertices).
So $ex(n,t) = \lfloor \frac{n-(t+1)}{t} \rfloor + n$. □

3 Generalized Halin graphs

Halin graphs are another tree-like structure. The limitation for the minimum degree of their vertices to be more than 2 forces the existence of a C_3. Actually it has been proved that Halin graphs are almost pancyclic, i.e. they contain cycles of all lengths $3 \leq l \leq n$ except possibly for one even value m of l [9]. Following is an elementary proof of this result.

Theorem 3. *There are no extremal C_t-free Halin graphs for any $t \geq 3$.*

Proof. Every vertex in a Halin graph has degree at least 3.
That means every non leaf node in its ancestor tree has at least 2 children, i.e. every node in its ancestor tree except the root has at least one sibling.
Consider a leaf at depth equal to the height of the tree.
By the above arguments that must have at least one sibling.
Those two siblings are connected in the Halin graph giving a cycle of length 3, C_3, together with their parent.
Therefore any Halin graph contains a C_3. □

As the restriction for the minimum degree of a Halin graph to be more than 2 leaves no space for our problem we consider the generalized Halin graph, where we allow vertices of the graph to have degree 2. We relate the number of edges of the graph to the number of leaves of the ancestor tree of the Halin graph thus being able to derive conclusions.

Lemma 1. *A generalized Halin graph of order n and girth at least $t+1$ has at most $\lfloor \frac{2n-2}{t} \rfloor + n - 1$ edges if $3 \leq t \leq \lfloor \frac{2n-2}{t} \rfloor - 1$.*

Proof. The number of edges of a generalized Halin graph is $|E|$ =length of cycle joining the leaves of the Halin graph + number of edges in the ancestor tree = number of leaves in the ancestor tree + $n-1$,
as the generalized Halin graph is made from an ancestor tree which has $n-1$ edges and one edge per leaf which join the tree leaves in a cycle.
Maximizing E is then equivalent to maximizing the number of leaves in the ancestor tree.
The number of leaves of the generalized Halin graph is the number of cycles found at the bottom of the Halin graph, which are created by connecting the leaves of the tree plus the outer cycle joining the leftmost leaf with the rightmost leaf. Maximizing E is then equivalent to maximizing the number of those cycles.
One can observe that every generalized Halin graph has all its nodes attached to the cycles mentioned above except maybe its root and a path hanging from it.
That path gives no contribution to the number of cycles, therefore we can consider only cases without that path, i.e. generalized Halin graphs that all their nodes are attached on those cycles.
Let us denote the cycle based on the ith and $i+1$th leaf as CY_i (leaves are ordered in the way they are found in a post order traversal of the ancestor tree of the Halin graph).
By counting the number of nodes of each cycle we observe that each node is counted twice except maybe their roots.
The root is counted as many times as is the number of cycles hanging from it.
Inner roots are counted as many times as is the number of cycles hanging from them plus 2 (for their neighbour cycles).
Let the number of the cycles mentioned above be c. Then $\sum_{i=1}^{c} |CY_i| =$ 2(nodes that are counted twice) + number of roots= $2(n-1)+c = 2n+c-2$
Cycles of the generalized Halin graph must have length at least $t+1$.
Therefore $2n+c-2 \geq c(t+1)$
and $c \leq \lfloor \frac{2n-2}{t} \rfloor$ □

Lemma 2. *A generalized Halin graph of size n and girth at least $t+1$ can have $\lfloor \frac{2n-2}{t} \rfloor + n - 1$ edges if $3 \leq t \leq \lfloor \frac{2n-2}{t} \rfloor - 1$.*

Proof. It is easy to see that the bound introduced in the theorem above is achieved by the following construction:
Consider the root connected to paths of length $\lceil \frac{t}{2} \rceil$, $\lfloor \frac{t}{2} \rfloor$, $\lceil \frac{t}{2} \rceil$, $\lfloor \frac{t}{2} \rfloor$,...
Any remaining vertices are added to the last path.(see also Figure 3)
Then $c = 2\lfloor \frac{n-1}{t} \rfloor + I((n-1) \mod t \geq \lceil \frac{t}{2} \rceil) = \lfloor \frac{2n-2}{t} \rfloor$
So $ex(n,t) = \lfloor \frac{2n-2}{t} \rfloor + n - 1$.
(We remind the reader that $I((n-1) \mod t \geq \lceil \frac{t}{2} \rceil)$ is the indicator

function returning 1 if the condition in the brackets is true or 0 otherwise) □

Theorem 4. *A generalized Halin graph of size n and girth at least t + 1 has*

$$ex(n,t) = \begin{cases} \lfloor \frac{2n-2}{t} \rfloor + n - 1, & 3 \leq t \leq \lfloor \frac{2n-2}{t} \rfloor - 1 \\ n-1, & t \geq \lfloor \frac{2n-2}{t} \rfloor \end{cases}$$

Proof. As of Lemmas 1 and 2 we get the above result for $3 \leq t \leq \lfloor \frac{2n-2}{t} \rfloor - 1$. However if $t \geq \lfloor \frac{2n-2}{t} \rfloor$ the only valid construction is P_n, a path starting from the root ending at a single leaf, as it has no connected leaves which would result to a cycle of length at least $\lfloor \frac{2n-2}{t} \rfloor$. □

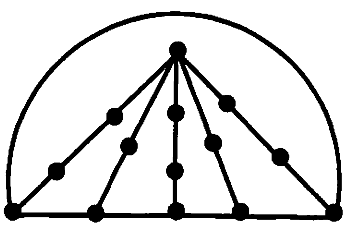

Fig. 3: An extremal generalized Halin graph of order 14 and girth at least 5

4 Rectangular grid graphs

It appears that to maximize the number of edges of a graph of order n in the grid we need to minimize the perimeter of a closed shape containing all n vertices. Thus the situation becomes easier to handle and we are able to draw conclusions for extremal numbers and give some constructions. We note that we consider only cases for odd t, as due to the fact that no odd cycles can exist in the grid $ex(n, 2k) = ex(n, 2k+1)$ for $k \geq 2$.

Theorem 5. *A graph of order n and girth at least t + 1 that lies in an infinite square grid can have at most:*

- $n-1$ *edges ,if* $1 \leq n \leq t$
- $\lfloor \frac{2n-t-3}{t-1} \rfloor + n - 1$ *edges, if* $t+1 \leq n \leq \lceil \frac{t+1}{4} \rceil \lfloor \frac{t+1}{4} \rfloor$
- $\lfloor \frac{2n+2-2\lceil\sqrt{n}\rceil - 2\lfloor\sqrt{n}\rfloor}{t-1} \rfloor + n - 1$ *edges, if* $\lceil \frac{t+1}{4} \rceil \lfloor \frac{t+1}{4} \rfloor + 1 \leq n \leq \lceil\sqrt{n}\rceil\lfloor\sqrt{n}\rfloor$

– $\lfloor\frac{2n+2-4\lceil\sqrt{n}\rceil}{t-1}\rfloor+n-1$ *edges, if* $\lceil\frac{t+1}{4}\rceil\lfloor\frac{t+1}{4}\rfloor+1\leq n$ *and* $\lceil\sqrt{n}\rceil\lfloor\sqrt{n}\rfloor+1\leq n\leq\lceil\sqrt{n}\rceil\lceil\sqrt{n}\rceil$

Proof. For $1 \leq n \leq t$ our graph must be acyclic and any maximal acyclic is by definition a tree with $n-1$ edges. If $t+1 \leq n$ there must be cycles present in our graph.
Let us denote the number of simple cycles of the graph by c.
Using Euler's formula for planar graphs we get $|E| = c+n-1$.
By summing the edges of each cycle we get:
$\sum_{i=1}^{c}|Cy_i| = 2x+y$, where x is the number of edges of the graph that are counted twice, y is the number of edges of the graph that are counted once and $\{Cy_1, Cy_2, \ldots, Cy_c\}$ is the set of simple cycles present in our graph.
As $|E| = x+y$, $\sum_{i=1}^{c}|Cy_i| = 2|E|-y$.
By Euler's formula we get: $\sum_{i=1}^{c}|Cy_i| = 2c+2n-2-y$
As cycles must have length at least $t+1$:
$c(t+1) \leq \sum_{i=1}^{c}|Cy_i| = 2c+2n-2-y$
i.e. $c \leq \lfloor\frac{2n-2-y}{t-1}\rfloor$
Obviously to maximize $|E|$ our graph must be connected.
Then y is the perimeter of a two dimensional shape in the grid.
It is easy to see that any such shape can be expanded to a rectangle which touches its leftmost, rightmost, top and bottom edges. This rectangle has same or smaller perimeter than the original shape and maybe some more area.
It is crucial that the shape contains at least n vertices.
Minimizing y means that we choose a rectangle with sides $\lceil\sqrt{n}\rceil-1$, $\lfloor\sqrt{n}\rfloor-1$ and if we can not fit all n vertices in it we choose a square with sides of length $\lceil\sqrt{n}\rceil-1$. □

The following theorem suggests that the above upper bounds can be achieved if we somehow avoid no non-simple cycles of length less than $t+1$ in our constructions.

Theorem 6. *The upper bounds of Theorem 5 can be achieved if no non-simple cycles of length less than* $t+1$ *are present in our constructions.*

Proof. To achieve the upper bounds of Theorem 5 we need to tessellate the rectangle mentioned in Theorem 5 with as many cycles of length $t+1$ as possible. That is considering as many cycles of area $\frac{t-1}{2}$. Then we get $c \leq \lfloor\frac{\text{Area of rectangle}}{(t-1)/2}\rfloor$ and hence the above upper bounds. (as long as no non-simple cycles of length less than $t+1$ are present in our constructions) □

Below we show that the upper bounds that we have introduced can be achieved in certain cases.

Theorem 7. *The above upper bound can be achieved for $n \geq (\frac{t+1}{2})^2$. Therefore for $n \geq (\frac{t+1}{2})^2$:*

$$ex(n;t) = \begin{cases} \lfloor \frac{2n+2-2\lceil\sqrt{n}\rceil-2\lfloor\sqrt{n}\rfloor}{t-1} \rfloor + n - 1, & n \leq \lceil\sqrt{n}\rceil\lfloor\sqrt{n}\rfloor \\ \lfloor \frac{2n+2-4\lceil\sqrt{n}\rceil}{t-1} \rfloor + n - 1, & \lceil\sqrt{n}\rceil\lfloor\sqrt{n}\rfloor + 1 \leq n \leq \lceil\sqrt{n}\rceil\lceil\sqrt{n}\rceil \end{cases}$$

Proof. To achieve the upper bound introduced in the above theorem we need to tessellate the rectangle mentioned above with as many cycles of length $t+1$ as possible as Theorem 6 suggests. This can be obtained by tiling the central $(\frac{t+1}{2}-1)$x$(\frac{t+1}{2}-1)$ square with $\frac{t+1}{2}-1$ cycles in the form of a column of thickness 1. We then spiral the rest of the cycles around it until all vertices are exhausted as shown in Figure 4. It is easy to see that no cycle of length less than $\frac{t+1}{2}$ is formed due to cycles touching each other. □

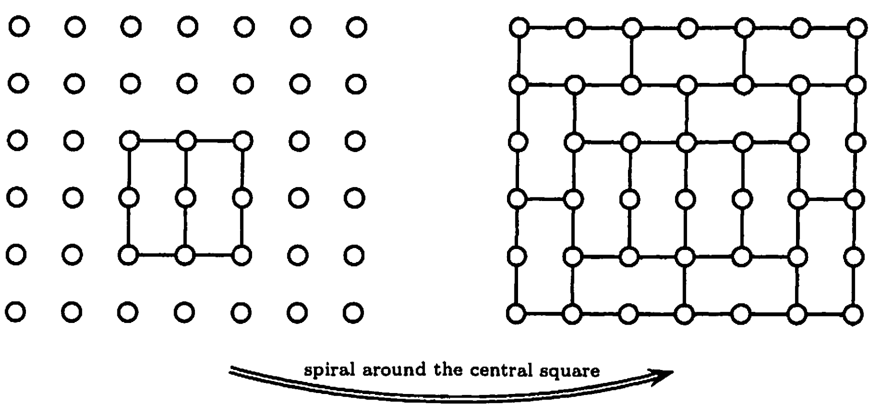

Fig. 4: Construction of an extremal graph for $n = 42$ and girth at least 6

5 Planar graphs

Using similar arguments we are now able to give the extremal numbers and some extremal graphs for the problem on any planar graph.

Lemma 3. *A planar graph of order n and girth at least $t+1$ can have at most:*

– $n-1$ *edges, if* $1 \leq n \leq t$
– $\lfloor\frac{2n-t-3}{t-1}\rfloor + n - 1$ *edges, if* $t+1 \leq n$

Proof. For $1 \leq n \leq t$ our graph must be acyclic and any maximal acyclic is by definition a tree with $n-1$ edges. If $t+1 \leq n$ there must be cycles present in our graph.
Let us denote the number of simple cycles of the graph by c.
Using Euler's formula for planar graphs we get $|E| = c + n - 1$.
By summing the edges of each cycle we get:
$\sum_{i=1}^{c} |Cy_i| = 2x + y$, where x is the number of edges of the graph that are counted twice and y the number of edges of the graph that are counted once.
As $|E| = x + y$, $\sum_{i=1}^{c} |Cy_i| = 2|E| - y$.
By Euler's formula we get: $\sum_{i=1}^{c} |Cy_i| = 2c + 2n - 2 - y$
As cycles must have length at least $t+1$:
$c(t+1) \leq \sum_{i=1}^{c} |Cy_i| = 2c + 2n - 2 - y$
i.e. $c \leq \lfloor\frac{2n-2-y}{t-1}\rfloor$
Obviously to maximize $|E|$ our graph must be connected.
Then y is the perimeter of a two dimensional shape.
Following the restriction for the girth we get $y \geq t+1$ and hence the above upper bound. □

Theorem 8. *The above upper bound can be achieved for any* $n \geq 1$. *Therefore:*

$$ex(n;t) = \begin{cases} n-1, & 1 \leq n \leq t \\ \lfloor\frac{2n-t-3}{t-1}\rfloor + n - 1, & n \geq t+1 \end{cases}$$

Proof. For $1 \leq n \leq t$ our graph must be acyclic and any maximal acyclic is by definition a tree with $n-1$ edges, so any tree on n vertices gives a lower bound construction.
For $n \geq t+1$ we consider the following cases:

– If t is odd the following construction gives the upper bound. Connect the first $t+1$ vertices to make a C_{t+1}. Number those vertices from 1 to $t+1$. Keep connecting paths of length $fract - 12$ on vertices 1 and $2 + \frac{t-1}{2}$. Insert any remaining vertices in the latest added path, obtaining $\lfloor\frac{n-\frac{t-1}{2}-2}{\frac{t-1}{2}}\rfloor = \lfloor\frac{2n-t-3}{t-1}\rfloor$ cycles(see also Figure 5).
– If t is even the following construction gives the upper bound (actually it works also for odd t but the above construction is much simpler). Connect the first $t+1$ vertices to make a C_{t+1}. Number those vertices from 1 to $t+1$. Connect a path of length $\frac{t}{2}$ on vertices 1 and $1 + \frac{t}{2}$ meeting the outer face. Connect a path of length $\frac{t}{2} - 1$ starting from the

second vertex of the latest added path (count from the start of the path not from the vertex $1+\frac{t}{2}$ or $2+\frac{t}{2}$ which is its end) ending at vertex $2+\frac{t}{2}$ if latest added path was ending at vertex $1+\frac{t}{2}$ or otherwise at vertex $1+\frac{t}{2}$, meeting the outer face. Connect a path of length $\frac{t}{2}$ on the same vertices as the previous path, meeting the outer face. Repeat until no vertices are left to make such paths. Insert any remaining vertices in the latest added path, obtaining $2\lfloor\frac{n-\frac{t-1}{2}-2}{t-1}\rfloor + I(n-\frac{t-1}{2}-2 \bmod t-1 \geq \frac{t}{2}-1)$ cycles.
The number of cycles can be at most $\lfloor\frac{2n-t-3}{t-1}\rfloor = \lfloor\frac{2n-t-4}{t-1}\rfloor$
$= 2\lfloor\frac{n-\frac{t}{2}-2}{t-1}\rfloor + I(\frac{n-\frac{t-1}{2}-2}{t-1} \geq \frac{1}{2})$
$= 2\lfloor\frac{n-\frac{t}{2}-2}{t-1}\rfloor + I(n-\frac{t-1}{2}-2 \bmod t-1 \geq \frac{t}{2}-1)$
(see also Figure 6).

It is easy to see that no cycle of length less than $\frac{t+1}{2}$ is formed due to cycles touching each other. □

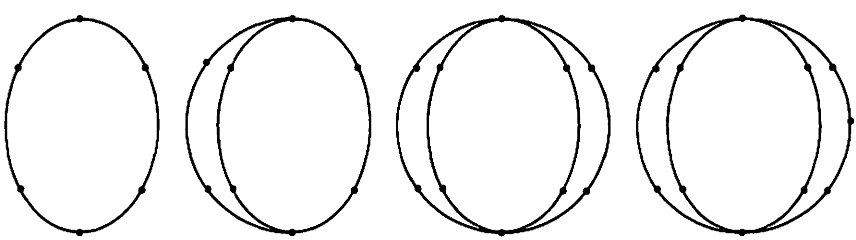

Fig. 5: Step by step construction of an extremal planar graph of order 11 and girth at least 6

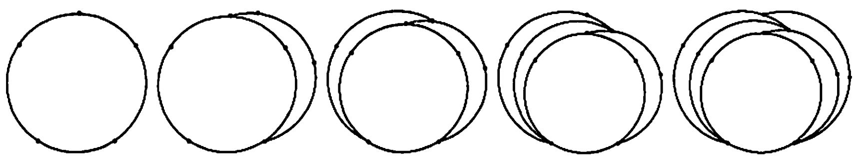

Fig. 6: Step by step construction of an extremal planar graph of order 11 and girth at least 5

By considering planar graphs we are now able to give a lower bound for the problem on its general case.

Theorem 9. *A graph of order n and girth at least $t+1$ has $ex(n,t)$ at least:*

- $n-1$, *if* $1 \leq n \leq t$
- $\lfloor \frac{2n-t-3}{t-1} \rfloor + n - 1$, *if* $t+1 \leq n$

Proof. Consider the planar graph constructions of Theorem 8. □

6 Conclusion and Future Work

In this paper we have given the $ex(n;t)$ numbers for pseudotrees, cacti graphs, graphs lying in a square grid, Halin, generalized Halin graphs and planar graphs, together with some constructions for the extremal cases. Future research might concentrate on finding the extremal values for other special types of graphs and on obtaining better bounds for the problem on general graphs.

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