On Isomorphic Graph Factoring

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Abstract. We describe an algorithm which combines a discrete optimization heuristic with the construction due to Ringel and Sachs (independently) for self-complementary graphs. The algorithm is applied to some problems from Generalized Ramsey Theory.

1. Introduction

In general our notation will follow Harary [6]. We use $r(G_1, \ldots, G_k)$ to denote the k-color Ramsey number of graphs $G_1, \ldots, G_n$, which is defined to be the smallest integer, n, such that in any k-coloring of the edge of K_n there is a monochromatic copy of G_i in color i, for some i, $1 \leq i \leq k$. If the G_i are all the same graph, we write $r(G; k)$. In discussing a given Ramsey problem, such as that of finding a lower bound for $r(G; k)$, we shall refer to k-colorings of some K_n and to their *bad subgraphs*, by which we mean monochromatic subgraphs isomorphic to G. An *isomorphic factorization* of a graph [7] is a partition of its edge set into isomorphic graphs. We call it a *factorization by k* if there are k sets in the partition.

An algorithm, well known to Graph Theorists, is the one used to construct self-complementary graphs, discovered (independently) by Ringel [11] and Sachs [12]. We will use an easy generalization of their idea to construct factorizations of K_n into any number of isomorphic graphs. The algorithm requires only that the obvious divisibility condition be satisfied. To construct an isomorphic factorization of K_n into k parts, assuming that $\binom{n}{2}$ is divisible by k, proceed as follows. Prepare a list of the edge of K_n, arranged in any order. Then choose a permutation, σ, of

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the n vertices such that the length of every cycle of σ is a multiple of k, except for (perhaps) one fixed point. (When k is even each nontrivial cycle must have a length that is a multiple of $2k$.) The following algorithm can then be used.

```
Mark all edge uncolored.
While there are uncolored edges:
  pick an uncolored edge x = uv;
  pick a random starting color i;
  let c = i;
  while x is uncolored;
    color x with color c;
    let u = σ(u), v = σ(v), and x = uv;
    let c = c + 1 (mod k).
```

When this procedure terminates and all edges are colored, the three color graphs will be isomorphic, in fact σ is an isomorphism. Some of the constructions described in this paper were produced by combining this procedure with the Metropolis algorithm. That algorithm was first presented in [10] and was used in a combinatorial setting in [8].

2. Ramsey Numbers

The problem of determining the Ramsey numbers of complete bipartite graphs was studied in some detail in [2], where it was determined that

$$k^2 - k + 2 \leq r(C_4; k) \leq k^2 + k + 2 \qquad (1)$$

held whenever $k-1$ is a prime power (the upper bound holds for all k). It is known that $r(C_4;2) = 6$ [3] and $r(C_4;3) = 11$ [1]. In both cases the lower bound can be established by an isomorphic factorization. (This fact is noted in [1] where Clapham is credited with the 3-color isomorphic coloring.) In [4] we noted that $r(C_4;4) \geq 18$.

Theorem 1.

$$\begin{aligned} r(C_4;2) &= 6 \\ r(C_4;3) &= 11 \\ r(C_4;4) &\geq 18 \\ r(C_4;5) &\geq 26 \end{aligned}$$

Proof: Only the final case is new, and this follows from the coloring given by the following adjacency matrix.

04454	11413	53312	54325	15223
40551	52252	41442	31543	12133
45011	21331	35255	34215	42324
55102	23244	24131	14532	15343
41120	33435	53524	22514	32145

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12133	40551	52252	41442	31543
42324	45011	21331	35255	34215
15343	55102	23244	24131	14532
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To make it easier to understand the graph, we also give the adjacency list for color 1. Note that the vertices are labeled from 0 to 24.

0:	5	6	8	13	20
1:	4	11	16	20	22
2:	3	4	6	9	18
3:	2	12	14	15	20
4:	1	2	18	22	
5:	0	10	11	13	18
6:	0	2	9	16	21
7:	8	9	11	14	23
8:	0	7	17	19	20
9:	2	6	7	23	
10:	5	15	16	18	23
11:	1	5	7	14	21
12:	3	13	14	16	19
13:	0	5	12	22	24
14:	3	7	11	12	
15:	3	10	20	21	23
16:	1	6	10	12	19
17:	8	18	19	21	24
18:	2	4	5	10	17
19:	8	12	16	17	
20:	0	1	3	8	15
21:	6	11	15	17	24
22:	1	4	13	23	24
23:	7	9	10	15	22
24:	13	17	21	22	

In this example the isomorphisms are given by the permutation $(0\ 1 \ldots 24)$.

The problem of determining two-color Ramsey numbers for cycles in graphs was completely settled by Faudree and Schelp [5]. However, no exact values or $r(C_n; k)$ are known for $n > 4$ and $k > 2$. Our best efforts for C_6 and C_8 are recorded in the following theorem. The value for $r(C_6; 3)$ is not from an isomorphic factorization, but rather by the techniques described in [4]. An isomorphic factorization by 3 of K_{10}, free of monochromatic C_6's, was found. Of course, such a factorization of K_{11} does not exist.

Theorem 2.

$$r(C_6; 3) \geq 12$$
$$r(C_8; 3) \geq 16$$
$$r(C_6; 4) \geq 18$$

Proof: The coloring for $r(C_6;3)$ is given by the following matrix.

0	3	3	2	2	2	1	3	2	1	1
3	0	3	1	3	1	2	2	1	2	2
3	3	0	3	1	3	1	3	2	3	2
2	1	3	0	2	2	3	1	2	3	3
2	3	1	2	0	2	1	3	2	1	1
2	1	3	2	2	0	3	1	1	3	3
1	2	1	3	1	3	0	2	3	2	2
3	2	3	1	3	1	2	0	1	2	2
2	1	2	2	2	1	3	1	0	1	2
1	2	3	3	1	3	2	2	1	0	2
1	2	2	3	1	3	2	2	2	2	0

The coloring for $r(C_8;3)$ follows. A drawing of the graph is provided in Figure 1. The isomorphism is derived from the permutation (0 1 2)(3 4 5)(6 7 8)(9 10 11)(12 13 14). Note that in the drawing vertex 3 is not shown. The neighbors of vertex 3 are those vertices represented as circles. The vertices represented as squares are not adjacent to vertex 3.

0	1	3	3	2	3	2	1	3	3	2	1	2	2	2
1	0	2	1	1	3	1	3	2	2	1	3	3	3	3
3	2	0	1	2	2	3	2	1	1	3	2	1	1	1
3	1	1	0	1	3	3	1	1	1	2	1	1	1	1
2	1	2	1	0	2	2	1	2	2	2	3	2	2	2
3	3	2	3	2	0	3	3	2	1	3	3	3	3	3
2	1	3	3	2	3	0	1	3	3	2	1	2	2	2
1	3	2	1	1	3	1	0	2	2	1	3	3	3	3
3	2	1	1	2	2	3	2	0	1	3	2	1	1	1
3	2	1	1	2	1	3	2	1	0	3	2	1	1	1
2	1	3	2	2	3	2	1	3	3	0	1	2	2	2
1	3	2	1	3	3	1	3	2	2	1	0	3	3	3
2	3	1	1	2	3	2	3	1	1	2	3	0	1	3
2	3	1	1	2	3	2	3	1	1	2	3	1	0	2
2	3	1	1	2	3	2	3	1	1	2	3	3	2	0

Finally we give the color for $r(C_6;4)$. Figure 2 shows the graph. The complementing permutation is (0 1 2 3 4 5 6 7)(8 9 10 11 12 13 14 15)(16).

0	1	4	1	2	2	2	4	4	1	3	2	3	3	2	2	2
1	0	2	1	2	3	3	3	3	1	2	4	3	4	4	3	3
4	2	0	3	2	3	4	4	4	4	2	3	1	4	1	1	4
1	1	3	0	4	3	4	1	2	1	1	3	4	2	1	2	1
2	2	2	4	0	1	4	1	3	3	2	2	4	1	3	2	2
2	3	3	3	1	0	2	1	3	4	4	3	3	1	2	4	3
2	3	4	4	4	2	0	3	1	4	1	1	4	4	2	3	4
4	3	4	1	1	1	3	0	4	2	1	2	2	1	1	3	1
4	3	4	2	3	3	1	4	0	3	1	1	3	2	3	2	3
1	1	4	1	3	4	4	2	3	0	4	2	2	4	3	4	4
3	2	2	1	2	4	1	1	1	4	0	1	3	3	1	4	1
2	4	3	3	2	3	1	2	1	2	1	0	2	4	4	2	2
3	3	1	4	4	3	4	2	3	2	3	2	0	3	1	1	3
3	4	4	2	1	1	4	1	2	4	3	4	3	0	4	2	4
2	4	1	1	3	2	2	1	3	3	1	4	1	4	0	1	1
2	3	1	2	2	4	3	3	2	4	4	2	1	2	1	0	2
2	3	4	1	2	3	4	1	3	4	1	2	3	4	1	2	0

Next we show a construction that gives a lower bound for the Ramsey number of the octahedral graph, $K_{2,2,2}$.

Theorem 3.

$$r(K_{2,2,2}) \geq 26.$$

Proof: The theorem follows from the construction given below. We show one of the color graphs, presented as an adjacency list. The isomorphism that shows the graph to be self complementary is the permutation

(0 1 2 3 4 5 6 7)(8 9 10 11 12 13 14 15)(16 17 18 19 20 21 22 23)(24).

Note that the graph is regular of degree 12.

0:	1	2	3	5	10	12	13	16	17	18	19	22
1:	0	5	7	8	9	10	12	15	16	21	22	24
2:	0	3	4	5	7	12	14	15	18	19	20	21
3:	0	2	7	9	10	11	12	14	17	18	23	24
4:	2	5	6	7	9	14	16	17	20	21	22	23
5:	0	1	2	4	9	11	13	14	16	19	20	24
6:	4	7	8	9	11	12	13	16	18	19	22	23
7:	1	2	3	4	6	11	15	16	18	21	22	24
8:	1	6	9	10	11	12	13	14	15	18	20	21
9:	1	3	4	5	6	8	12	17	18	20	23	24
10:	0	1	3	8	11	14	15	16	17	20	22	23
11:	3	5	6	7	8	10	13	14	19	20	22	24
12:	0	1	2	3	6	8	9	14	19	21	22	23
13:	0	5	6	8	11	14	16	17	18	19	21	24
14:	2	3	4	5	8	10	11	12	13	16	21	23
15:	1	2	7	8	10	16	18	19	20	21	23	24
16:	0	1	4	5	6	7	10	13	14	15	18	23
17:	0	3	4	9	10	13	18	20	21	22	23	24
18:	0	2	3	6	7	8	9	13	15	16	17	20
19:	0	2	5	6	11	12	13	15	20	22	23	24
20:	2	4	5	8	9	10	11	15	17	18	19	22
21:	1	2	4	7	8	12	13	14	15	17	22	24
22:	0	1	4	6	7	10	11	12	17	19	20	21
23:	3	4	6	9	10	12	14	15	16	17	19	24
24:	1	3	5	7	9	11	13	15	17	19	21	23

References

1. A. Bialostocki and J. Schönheim, *On Some Turán and Ramsey Numbers for C_4*, in "Graph Theory and Combinatorics", (B. Bollobás, ed.), Academic Press, London, 1984, pp. 29–33.
2. F.R.K. Chung and R.L. Graham, *On Multicolor Ramsey Numbers for Complete Bipartite Graphs*, J. Combin. Theory-Ser. B. **18** (1975), 164–169.
3. V. Chvátal and F. Harary, *Generalized Ramsey Theory for Graphs II: Small Diagonal Numbers*, Proc. Amer. Math. Soc. 32 (1972), 389–394.
4. G. Exoo, *Constructing Ramsey Graphs with a Computer*, Congr. Number. 59 (1987), 31–36.
5. R.J. Faudree and R.H. Schelp, *All Ramsey Numbers for Cycles in Graphs*, Discrete Math **8** (1974), 161–173.
6. F. Harary, "Graph Theory", Addison-Wesley, Reading, Mass, 1969.
7. F. Harary and R.W. Robinson, *Isomorphic Factorizations X : Unsolved Problems*, Jour. Graph Theory 9 (1985), 67–86.

8. S. Kirkpatrick, C.D. Gelatt Jr., and M.P. Vecchi, *Optimization by Simulated Annealing*, Science **220** (1983), 671–680.
9. R. Mathon, *On Self-Complementary Strongly Regular Graphs*, Discrete Math **69** (1988), 263–281.
10. N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller, *Equation of State Calculations for Fast Computing Machines*, Journal of Chemical Physics **6** (1953), 1087.
11. G. Ringel, *Selbstkomplementäre Graphen*, Arch. Math **24** (1963), 354–358.
12. H. Sachs, *Über selbstkomplementäre Graphen*, Publ. Math. Debrecen. **9** (1962), 276–288.