Matching, Rook and Chromatic Polynomials and Chromatically Vector Equivalent Graphs

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Abstract. It is shown that under certain conditions, the embeddings of chessboards in square boards, yield non-isomorphic associated graphs which have the same chromatic polynomials. In some cases, sets of non-isomorphic graphs with this property are formed.

1. Introduction.

The graphs considered here are finite, undirected, and contain no loops and no multiple edges. Let G be such a graph with p nodes. A *matching* in G is a spanning subgraph of G, whose components are nodes and edges only. A *k-matching* is a matching with k edges. The *matching polynomial* of G is

$$M(G;\underline{w}) = \sum_{k=0}^{\lfloor p/2 \rfloor} a_k w_1^{p-k} w_2^k,$$

where a_k is the number of k-matchings in G and w_1 and w_2 are *weights* (or indeterminates) associated with each node and edge, respectively, in G. If we put $w_1 = w_2 = w$, then the resulting polynomial in w is called the *simple matching polynomial* of G. We refer the reader to Farrell [1] for the basic properties of matching polynomials.

A *chessboard* (also called a *board*) is an array of cells in rows and columns — the rows and columns being determined by the cells. The *rook polynomial* of a chessboard B, is the polynomial

$$R(B;x) = \sum_{k=0} r_k x^k,$$

where r_k is the number of ways of placing k non-taking rooks on B. Let N denote the positive integers. The board B can also be regarded as a finite subset of $N \times N$, in which case we refer to B as an *embedded board*. An introduction to rook polynomials can be found in Riordan [7].

The *chromatic polynomial* of a graph G, is the polynomial $P(G; \lambda)$ which represents the number of ways of coloring the nodes of G with λ colors, in such a way that adjacent nodes receive different colors. Throughout this paper, we will assume that $P(G; \lambda)$ is expressed in the *falling factorial basis*, that is,

$$P(G; \lambda) = \sum_{k=0}^{p-1} c_k(\lambda)_{p-k},$$

where c_k is the number of color partitions of the nodes of G into $n-k$ non-empty indistinguishable classes and for $r > 0$, $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-r+1)$. We refer the reader to Read [6] for the basic properties of $P(G; \lambda)$.

For the three polynomials defined above, we refer to the vectors of non-zero coefficients as the *matching vector* (denoted by $m(G)$), the *rook vector* (denoted by $r(B)$), and the *chromatic vector* (denoted by $c(G)$), respectively. Let G and H be graphs. We call G and H *matching equivalent* if and only if $m(G) = m(H)$. We call G and H *chromatically vector equivalent* (*cv-equivalent*) if and only if $c(G) = c(H)$. If in addition, G and H have the same number of nodes, then we call them *co-matching* or *co-chromatic*, respectively. Two chessboards A and B are called *rook equivalent* if and only if $r(A) = r(B)$. (In Korfhage [5], define the σ-polynomial of a graph G, denoted $\sigma(G)$. It is easy to show that two graphs G and H are cv-equivalent if and only if $\sigma(G) = \sigma(H)$.)

In this paper, we show that various graphs associated with the different embeddings of a chessboard are chromatically vector equivalent, and in some cases, co-chromatic. We also extend some of the results given in Goldman, Joichi and White [4]. This paper is central to our investigation. The main thrust of our paper has been mentioned in [4] as a topic "worthy of further study".

We denote the node and edge sets of G by $V(G)$ and $E(G)$, respectively. The complete graph with p nodes is denoted by K_p. The complement of G is denoted by $\overline{G}$. If G and H are chromatically vector equivalent, then we write $G \sim H$. When denoting the position of a cell in a chessboard, the column index precedes the row index. Thus, cell (i, j) is the cell in *column* i and *row* j.

2. Some relevant definitions and results.

In this paper, we essentially bring together some of the main ideas developed in the articles: Farrell [2], Farrell and Whitehead [3], and Goldman *et al* [4]. We will therefore try, as far as possible, to maintain consistency of definitions. We will also quote, without proofs, the relevant results that can be found in these articles.

The following definition is given in [2].

Definition: Let B be a chessboard with m rows and n columns. (Throughout this paper, when we say that a chessboard has m rows and n columns, we mean that each of these rows and each of these columns contain at least one cell.) With B, we can associate a graph G_B, constructed as follows. The node set of G_B is the union of the two disjoint sets $\{r_1, r_2, \ldots, r_m\}$ and $\{c_1, c_2, \ldots, c_n\}$ representing the row and column labels, respectively. Nodes c_i and r_j are joined by an edge if and only if cell (c_i, r_j) belongs to B. G_B is called the *associated graph*.

It is clear from the definition that G_B is bipartite. Dually, given any bipartite graph G, we can find a chessboard B_G such that G is the graph associated with B_G.

The following result is proven in [2].

Theorem 1. $R(B;x) = M(G_B;(1,x))$; *that is, the rook polynomial of the board* B *can be obtained from the matching polynomial of* G_B, *by replacing* w_1 *and* w_2 *with* 1 *and* x, *respectively.*

Hence, we have the following corollary.

Corollary 1.1. *For any chessboard* B, $r(B) = m(G_B)$.

The following definitions are taken from [4].

Definitions: An embedded board B is called an *n-board* if $B \subseteq [n] \times [n]$ where $[n] = \{1, 2, \ldots, n\}$. A n-board B is called *proper* if and only if (i) $(i,j) \in B$ implies $i > j$ and (ii) $(i,j) \in B$ and $(j,k) \in B$ implies that $(i,k) \in B$. Associated with a proper n-board B is a graph $\Gamma_n(B)$ defined as follows: The node set of $\Gamma_n(B)$ is $V(\Gamma_n(B)) = \{1, 2, \ldots, n\}$ and nodes i and j $(i > j)$ are adjacent if and only if $(i,j) \notin B$.

Let G be a graph with n nodes. G is called a *board-graph* if and only if there is a board $B \subseteq [n] \times [n]$ such that $r(B) = c(G)$. Also, G is called a Γ*-graph* if and only if there is a proper n-board B such that $G \cong \Gamma_n(B)$.

The following result is established in [4].

Theorem 2. *Let* B *be a proper* n*-board. Then* $r(B) = c(\Gamma_n(B))$.

The following result is taken from [3].

Theorem 3. $m(G) = c(\overline{G})$ *if and only if* G *is* Δ*-free (triangle free), that is,* G *does not contain a subgraph isomorphic to* K_3.

From Theorem 1, we see that every rook vector is a matching vector of a bipartite graph. Since every bipartite graph is Δ-free, we have (from Theorem 3) the following result.

Corollary 3.1. *Every rook vector is a chromatic vector.*

It is clear that Corollary 3.1 is also immediate from Theorem 2; in [4], Corollary 3.1 is given as a corollary of this theorem. Therefore, we have given an independent derivation of this interesting result.

3. Chromatically vector equivalent graphs associated with a chessboard.

The following theorem yields a technique for constructing chromatically equivalent graphs.

Theorem 4. *For any proper chessboard B, the graphs $\Gamma_n(B)$ and $\overline{G}_B$ are chromatically vector equivalent. that is, $\Gamma_n(B) \sim \overline{G}_B$.*

Proof: From Corollary 1.1, $r(B) = m(G_B)$. From Theorem 2, $r(B) = c(\Gamma_n(B))$. Therefore,

$$m(G_B) = c(\Gamma_n(B)). \tag{1}$$

But, G_B is a bipartite graph and is therefore Δ-free. It follows from Theorem 3, that $m(G_B) = c(\overline{G}_B)$. Hence, Equation (1) yields $c(\overline{G}_B) = c(\Gamma_n(B))$. Therefore, $\Gamma_n(B) \sim \overline{G}_B$. ∎

We illustrate Theorem 4 using the board B of Figure 1 in [4]. The board, together with the associated graphs are shown in Figure 1.

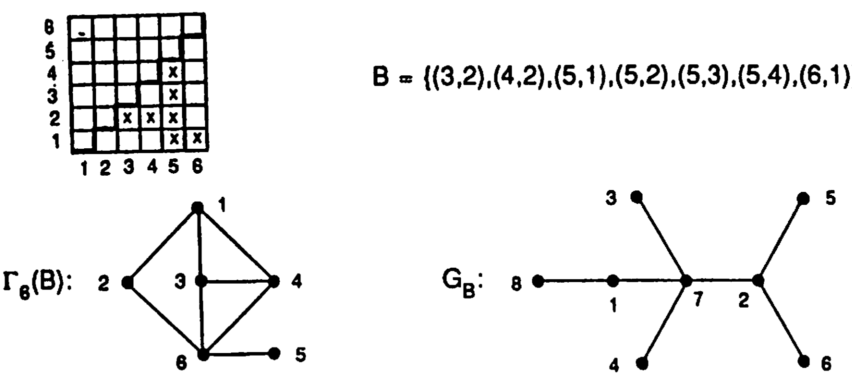


Figure 1

The following polynomials can be easily verfied.

$$R(B; x) = 1 + 7x + 11x^2 + 4x^3$$
$$M(G_B; \underline{w}) = w_1^8 + 7 w_1 6 w_2 + 11 w_1^4 w_2^2 + 4 w_1^2 w_2^3$$
$$P(\Gamma_6(B); \lambda) = (\lambda)_6 + 7(\lambda)_5 + 11(\lambda)_4 + 4(\lambda)_3.$$

The graph $\overline{G}_B$ is shown in Figure 2. The chromatic polynomial of $\overline{G}_B$ is

$$P(\overline{G}_B; \lambda) = (\lambda)_8 + 7(\lambda)_7 + 11(\lambda)_6 + 4(\lambda)_5.$$

Hence, we have

$$c(\overline{G}_B) = c(\Gamma_6(B)) = (1,7,11,4).$$

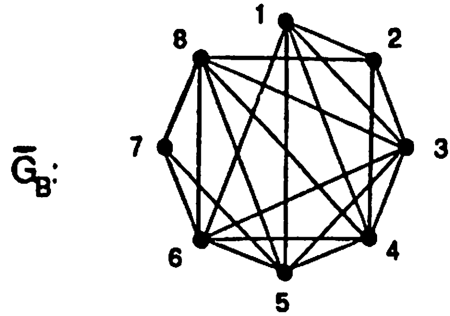


Figure 2

Note that in this case, $\overline{G}_B \sim \Gamma_6(B)$. However, they are not co-chromatic. Trivially, we can obtain a co-chromatic pair by adding a complete graph on two nodes (K_2) to the graph $\Gamma_6(B)$ and joining the nodes of K_2 to all the nodes of $\Gamma_6(B)$. The resulting graph can be denoted by $\Gamma_6(B) \odot K_2$, where $\odot$ denotes the *Zykov product* (also known as the join) of two graphs. Trivially, $G \odot K_0 \cong G$.

It is clear that the number of cells below the main diagonal of a $n \times n$ square array of cells $\binom{n}{2}$. This square array defines the complete graph K_n with n nodes, when we define the edges to be (i,j) if and only if $i > j$. The graph $\Gamma_n(B)$ is a subgraph of this graph. The following lemmas give information about the parameters of the graphs defined so far. They are easy to derive.

Lemma 1. *Let B be a board with r rows, c columns, and k cells. Then*

(i) $|V(G_B)| = r + c$,
(ii) $|E(G_B)| = k$,
(iii) $|E(\overline{G}_B)| = \binom{r+c}{2} - k$.

Lemma 2. *Let B be a proper n-board with k cells. Then*

(i) $|V(\Gamma_n(B))| = n$,
(ii) $|E(\overline{\Gamma_n(B)})| = k$,
(iii) $|E(\Gamma_n(B))| = \binom{n}{2} - k$.

Definition: Let G and H be chromatically vector equivalent graphs. G and H are called *trivially (vector) equivalent* (written as $G_{\sim T}H$) if and only if either $G \cong H \odot K_m$ or $H \cong G \odot K_s$, for some non-negative integer m or s. Otherwise, G and H are *non-trivially (vector) equivalent* (written as $G_{\sim NT}H$).

We note that when the chromatic polynomials of G and H are written in the usual form (as powers of λ) the analogous definition of trivially equivalent is that either $G \cong H \cup \overline{K}_m$ or $H \cong G \cup \overline{K}_s$, for some non-negative integer m or s. In this case, the definition exactly parallels the definitions of trivially equivalent given in [2] for matching and rook polynomials.

Theorem 5. *Lt B be a proper n-board with r rows and c columns. If $n < r + c$, then $\overline{G}_{B \sim NT} \Gamma_n(B)$.*

Proof: We have already shown (in Theorem 4) that $\overline{G}_{B \sim} \Gamma_n(B)$. Since $n < r + c$, from the above lemmas, $\overline{G}_B$ has more nodes than $\Gamma_n(B)$. Let us assume that $\overline{G}_B \cong \Gamma_n(B) \odot K_m$, for some non-negative integer m. Since $n < r + c$, $m = r + c - n > 0$. It follows that $\overline{G}_B$ contains a node of valency $r + c - 1$, which implies that G_B contains an isolated node. This is impossible by the construction of G_B. Our assumption is false. Hence, $\overline{G}_{B \sim NT} \Gamma_n(B)$. ∎

Theorem 5 assures us that the technique implied by Theorem 4 always yields non-trivially equivalent graphs, *provided that* $n < r + c$. We note that there are chessboards for which there are no proper embeddings in square arrays of side less than $r + c$. An example of such a board is the square chessboard containing r^2 $(= c^2)$ cells.

4. Vector equivalent graphs associated with translated boards.

We now consider the effects of embedding a given board properly in square boards of different sizes. These embeddings can be obtained by translations of B to the right. The translations can be partitioned into three classes, according to the sizes of the resulting square boards. We consider the cases: (i) $n < r + c$, (ii) $n = r + c$, and (iii) $n > r + c$.

Case (i): $n < r + c$.

Theorem 5 settles the relationship between G_B and $\Gamma_n(B)$ when $n < r + c$.

Definitions: Let m be the side of the smallest square board in which the board B can be properly embedded. Then the proper m-board is called the *minimum board* for B; the graph $\Gamma_m(B)$ is called the *minimum graph* for B.

The following lemma is immediate from these definitions.

Lemma 3. *Let B be a board with r rows and c columns. Let m be the size of the minimum board for B. Then, $m \geq \max(r, c) + 1$.*

Suppose that we start with the minimum m-board and that $m < r + c$. Then we can translate B one cell at a time to the right, thereby obtaining $s = r + c - m$ embeddings in n-boards for which $n < r + c$. By Theorem 5, each of the graphs $\Gamma_i(B)$, $m \leq i < r + c$, are non-trivially equivalent to $\overline{G}_B$. Hence, we obtain the following result.

Theorem 6. *Let m be the size of the minimum board for a board B with r rows and c columns. Then the graphs $\Gamma_m(B)$, $\Gamma_{m+1}(B)$, ... and $\Gamma_{r+c-1}(B)$ are all chromatically vector equivalent. Furthermore, the equivalence is non-trivial.*

Proof: The first part of the theorem is clear from the discussion preceeding it. For some integers i and j where $j > i$, assume that $\Gamma_i(B)$ and $\Gamma_j(B)$ are trivially equivalent, that is, $\Gamma_j(B) \cong \Gamma_i(B) \odot K_m$ where $m = j - i$. Thus, $\overline{\Gamma_j(B)}$ has $m(> 0)$ isolated nodes which implies that there is an integer $t(1 \leq t \leq j)$ such that no cell of B is in row t and column t. Since no cell of B is in row t, $t > r$. Also, all the columns containing cells of B must occur after column t. This implies that $j \geq t + c$ which implies that $j > r + c$. This is a contradiction. Therefore, our assumption is false. The equivalence is non-trivial. ∎

The above theorem yields another method for constructing non-trivially chromatically vector equivalent graphs. This method depends on the condition that $m < r + c$. It is possible that for a given board, the only proper embeddings are those for which $n \geq r + c$. We now consider these cases.

Case (ii): $n = r + c$.

The following theorem settles the case when $n = r + c$.

Theorem 7. $\overline{\Gamma_n(B)} \cong G_B$ *if and only if* $n = r + c$.

Proof: Suppose that $\Gamma_n(B) \cong \overline{G}_B$. Then from Lemma 1 and Lemma 2, $n = r + c$.

Conversely, suppose that $n = r + c$. We define a mapping θ: $V(G_B) \rightarrow V(\overline{\Gamma_n(B)})$ as follows:

$$\theta(r_i) = r - (i - 1),$$
$$\theta(c_j) = r + j.$$

Under θ, the cells of B define the edges of G_B and $\overline{\Gamma_n(B)}$ in the same manner, that is, θ preserves adjacencies in G_B and $\overline{\Gamma_n(B)}$. Thus, $G_B \cong \overline{\Gamma_n(B)}$. ∎

The following corollary is immediate.

Corollary 7.1. $\Gamma_n(B) \cong \overline{G}_B$ *if and only if* $n = r + c$.

This corollary suggests that in a search for chromatically vector equivalent graphs, no new graphs can be obtained by translating the board to the point where $n = r + c$. We now investigate translations beyond that point.

Theorem 6 and Corollary 7.1 yield the following result.

Theorem 8. *The graphs* $\Gamma_i(B)$, *where* $m \leq i < r + c$, *are all mutually non-trivially equivalent.*

Case (iii): $n > r + c$.

The following theorem settles the case when $n > r + c$.

Theorem 9. $\Gamma_n(B) \cong \overline{G}_B \odot K_s$, *if and only if* $n > r + c$, *where* $s = n - (r + c) > 0$.

Proof: Suppose that $\Gamma_n(B) \cong \overline{G}_B \odot K_s$ where $s > 0$. Then it follows from Lemma 1 and Lemma 2 that $n > r + c$.

Conversely, suppose that $n > r + c$. This situation is illustrated in Figure 3.

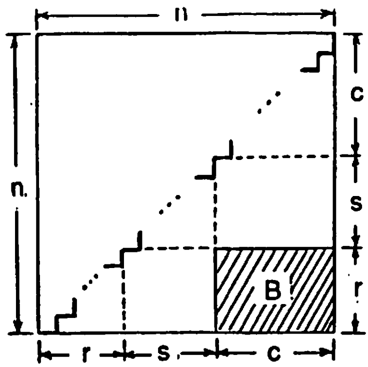


Figure 3

In this case, no cell of B can be in a row and a column with index in the set $\{r+1, r+2, \ldots, r+s\}$. It follows that $\overline{\Gamma_n(B)}$ contains s isolated nodes. Furthermore, the row and column indices of the cells in B are disjoint. By using a mapping similar to θ in the proof of Theorem 7, it can be easily shown that the non-trivial component of $\overline{\Gamma_n(B)}$ is isomorphic to G_B. Therefore, $\overline{\Gamma_n(B)} \cong G_B \cup \overline{K}_s$. The result follows by taking complements. ∎

Theorem 7 and Theorem 9 yield the following important result.

Theorem 10. $\Gamma_n(B) \sim_T \overline{G}_B$ *if and only if* $n \geq r + c$.

The message conveyed by Theorem 10 is that when using the embedding technique, in order to construct chromatically vector equivalent graphs, nothing is gained by considering embeddings in square boards of side greater than $r + c - 1$.

The following theorem gives an interesting property of the graph $\overline{\Gamma_n(B)}$.

Theorem 11. $\overline{\Gamma_n(B)}$ *is bipartite if and only if* $n \geq r + c$.

Proof: Suppose that $n \geq r + c$. Then it is clear from the proofs of Theorem 7 and Theorem 9 that $\overline{\Gamma_n(B)}$ is bipartite.

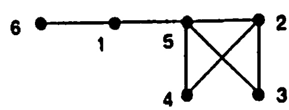
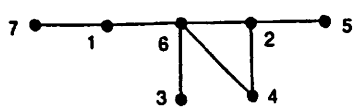
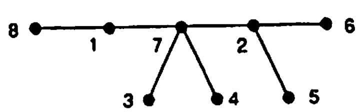
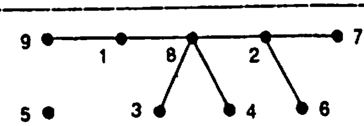
Conversely, suppose that $\overline{\Gamma_n(B)}$ is bipartite. Then its row and column indices must be disjoint. Label the rows $1, 2, 3, \ldots, r$. Then the column labels will be $a, a+1, \ldots, a+c-1$ where $a > r$. Thus, $n \geq a + c - 1$ which implies that $n \geq r + c$. ∎

As an illustration, we tabulate (in Table 1) the graphs $\overline{\Gamma_n(B)}$ associated with the chessboard B given in Figure 1, for $n = 6, 7, 8$ and 9. We have already determined the rook vector of B and the chromatic vector of $\Gamma_n(B)$. These vectors are as follows:

$$r(B) = c(\Gamma_n(B)) = (1, 7, 11, 4).$$

According to Theorem 6, all the graphs formed by translations have the same chromatic vector. Therefore, we give the graphs and their matching vectors. By Lemma 3, m must be at least 5; in this case, $m = 6$.

Table 1

n	$\overline{\Gamma_n(B)}$	$m(\overline{\Gamma_n(B)})$
6		(1,7,9,2)
7		(1,7,10,3)
8		(1,7,11,4)
9		(1,7,11,4)

Theorem 12. *Let B be a chessboard. Let B_1 and B_2 be two proper n-boards obtained by embedding B. Let R_i and C_i be the set of row labels and the set of column labels of B_i, respectively. Then $\Gamma_n(B_1)$ and $\Gamma_n(B_2)$ are co-chromatic. Furthermore, if $|R_1 \cap C_1| \neq |R_2 \cap C_2|$ then $\Gamma_n(B_1)$ is not isomorphic to $\Gamma_n(B_2)$.*

Proof: For $i \in \{1,2\}$, $\Gamma_n(B_1)$ is a board-graph because $r(B_i) = c(\Gamma_n(B_i))$. $\Gamma_n(B_1)$ and $\Gamma_n(B_2)$ are cv-equivalent because $r(B_1) = r(B) = r(B_2)$. These board-graphs are co-chromatic because they both have n nodes. For $i \in \{1,2\}$, $\Gamma_n(B_i)$ has $n-(|R_i|+|C_i|-|R_i \cap C_i|)$ nodes of valency $n-1$. Since $|R_1| = |R_2|$, $|C_1| = |C_2|$, and $|R_1 \cap C_1| \neq |R_2 \cap C_2|$, $\Gamma_n(B_1)$ is not isomorphic to $\Gamma_n(B_2)$. ∎

As an illustration of Theorem 12, we consider three embeddings of chessboard B shown in Figure 4. These embeddings are denoted as B_1, B_2, and B_3. For proper board B_1, $R_1 = \{1,2,3\}$, and $C_1 = \{4,5,6,7\}$ where $|R_1 \cap C_1| = 0$. For proper board B_2, $R_2 = \{1,2,3\}$, and $C_2 = \{3,4,5,6\}$ where $|R_2 \cap C_2| = 1$. For proper board B_3, $R_3 = \{1,2,3\}$, and $C_3 = \{2,3,4,5\}$ where $|R_3 \cap$

$C_3| = 2$. The graphs $\Gamma_7(B_1)$, $\Gamma_7(B_2)$, and $\Gamma_7(B_3)$ are pairwise co-chromatic and pairwise non-isomorphic.

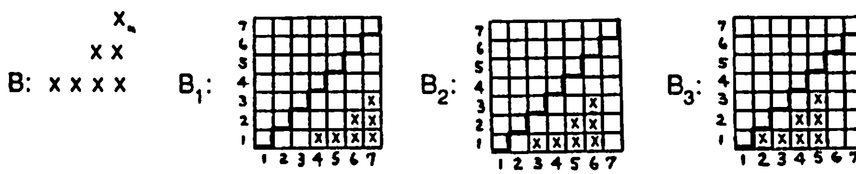


Figure 4

5. Some extensions using results on matching and rook equivalence.

Suppose that the boards B_1 and B_2 are rook equivalent. Then clearly, the graphs $\Gamma_n(B_1)$ and $\Gamma_n(B_2)$ will be cv-equivalent (chromatically vector equivalent) for all values of n for which proper boards exist. Therefore, rook equivalent boards provide a useful means of extending the number of cv-equivalent graphs associated with a given chessboard B. All rook equivalent boards will yield graphs which are cv-equivalent to the graphs $\Gamma_n(B)$. However, we note that there is no guarantee that the graphs obtained by using rook equivalent boards will be non-isomorphic to the graphs $\Gamma_n(B)$.

The following definition is taken from Farrell [2].

Definition: Let G be a (m, n)-bipartite graph with node bipartition sets $\{r_1, r_2, \ldots, r_m\}$ and $\{c_1, c_2, \ldots, c_n\}$. We can associate with G, a chessboard B_G, constructed as follows. B_G will have m rows and n columns. Cell $(i, j) \in B_G$ if and only if $c_i r_j \in E(G)$.

In the following lemma, $\sim_M$ and $\sim_R$ denote matching equivalence and rook equivalence, respectively. Complements of chessboards are taken relative to any fixed rectangular chessboard that contains the boards as subboards. G^B denotes the complement of G in a complete bipartite graph in which G is a subgraph. A proof of the lemma is given in [2].

Lemma 4. *Let G and H be bipartite graphs. Then*

$$G \sim_M H \Leftrightarrow B_G \sim_R B_H \Leftrightarrow \overline{B}_G \sim_R \overline{B}_H \Leftrightarrow B_{\overline{G}^B} \sim_R B_{\overline{H}^B} \Leftrightarrow \overline{G}^B \sim_M \overline{H}^B.$$

It is not difficult to deduce the following result.

Theorem 13. *Let G and H be matching equivalent bipartite graphs. Let G^* and H^* be the complements of G^B and H^B, respectively. Then, for suitable positive integers i and j, and for $k \in \{1, 2\}$, the elements of S_k are pairwise cv-equivalent where*

$$S_1 = \{\Gamma_i(B_G), \Gamma_i(B_H), \overline{G}, \overline{H}\} \text{ and } S_2 = \left\{\Gamma_j(B_{\overline{G}^B}), \Gamma_j(B_{\overline{H}^B}), G^*, H^*\right\}.$$

Definitions:

(i) A graph is called a *chromatic board graph* if and only if there exists a board B such that $c(G) = r(B)$. ;

(ii) A graph G Is called a *matching board graph* if and only if there exists a board B such that $m(G) = r(B)$.

The following theorem gives a characterization for matching board graphs.

Theorom 14. *A graph G is a matching board graph if and only if G is matching equivalent to a bipartite graph.*

Proof: Suppose that G is a matching board graph. Then there exists a board B such that $m(G) = r(B)$. Associated with the board B, there is a graph G_B such that $r(B) = m(G_B)$. Therefore, $m(G) = m(G_B)$. Since G_B is bipartite, G is matching equivalent to a bipartite graph.

Conversely, suppose that G is matching equivalent to a bipartite graph H, that is, $m(G) = m(H)$. Since H is bipartite, we can construct a board B_H, such that $r(B_H) = m(H)$. Thus, $m(G) = r(B_H)$. It follows that G is a matching board graph. ∎

The following observations can be verified.

(1) If a graph G is matching equivalent to a Δ-free graph H and $\overline{H}$ is a chromatic board graph, with board B ,then G is a matching board graph with board B.

(2) Triangle-free graphs whose complements are chromatic board graphs are matching equivalent to bipartite graphs.

(3) If G is a Δ-free graph, then $\overline{G}$ is a chromatic board graph if and only if G is matching equivalent to a bipartite graph. Alternatively, if $\overline{G}$ is a Δ-free graph, then G is a chromatic board graph if and only if $\overline{G}$ is matching equivalent to a bipartite graph.

(4) If G be a bipartite graph, then $\overline{G}$ is a chromatic board graph.

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