A Class of Arbitrarily Graceful Planar Bipartite Graphs

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Abstract. We exhibit here an infinite family of planar bipartite graphs which admit a k-graceful labeling for all $k \geq 1$.

1. Introduction

In this paper all graphs are finite, undirected, and have no loops or multiple edges. A graph $G = (V, E)$ with vertex set V and edge set E is said to be a *k-graceful graph* for a fixed integer $k \geq 1$ if there exists an injective mapping $\beta : V \to \{0, 1, \ldots, k+|E|-1\}$ such that the induced mapping $\beta^* : E \to \{k, k+1, \ldots, k+|E|-1\}$ defined by $\beta^*(e) = |\beta(a) - \beta(b)|$ for each edge e joining vertices a and b is bijective. When $k = 1$, we have the usual definition of a graceful graph. For more information on graceful graphs, the reader should consult the papers [2,3,4,7,8]. A graph G is *arbitrarily graceful* if it is k-graceful for all integers $k \geq 1$. The notion of graceful graphs was introduced by Rosa[14] and was popularized by Golomb[6]. Slater[15], Maheo and Thuillier[13] independently generalized the concept of graceful graphs to that of k-graceful graphs.

Acharya[1] showed that every convex Eulerian polynomino is k-graceful for all $k \geq 1$. More generally, Lee and Ng[11] showed that every Young tableau graph is k-graceful. Various classes of graphs, such as Mongolian tents, diamonds, pyramids and lotus have been shown to be k-graceful[9], [12]. In this paper, we shall exhibit another class of planar bipartite graphs and show that they are arbitrary graceful.

2. Planar Block Graphs

We shall call a graph $G = (V, E)$ of the form below (Fig. 1) a *planar block graph.*

If n copies of planar block graphs are given, then we can construct the graph $B(n)$ by juxtaposing these planar block graphs together as shown in Fig. 2. We shall call the graph $B(n)$ a *chain planar block graph.*

Note that $B(n)$ contains $5n+2$ vertices and we shall denote these vertices in a systematic way by $u_1, u_2, \ldots u_n$; $v_1, v_2, \ldots v_n$; $w_1, w_2, \ldots w_n$; $P_1, P_2, \ldots, P_{n+1}$; and $q_1, q_2, \ldots, q_{n+1}$ according to their horizontal positions of the graph. Similarly,

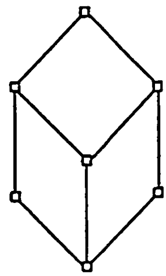


Fig. 1. A planar Block graph

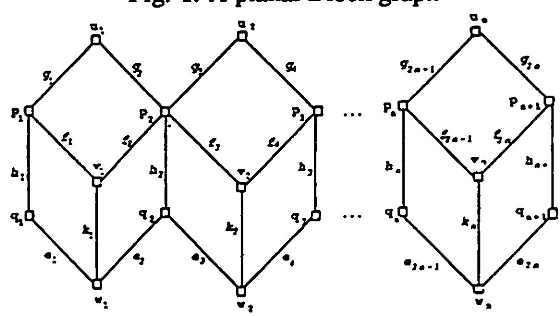


Fig 2. A chain planar block graph $B(n)$

$B(n)$ contains $8n+1$ edges and we shall denote these edges by $e_1, e_2, \ldots, e_{2n}$; $f_1, f_2, \ldots, f_{2n}$; $g_1, g_2, \ldots, g_{2n}$; $h_1, h_2, \ldots, h_{n+1}$; and $k_1, k_2, \ldots, k_{n+1}$ as shown in Fig. 2.

Lemma 1. *The chain planar block graph $B(n)$ is graceful for any integer $n \geq 1$.*

Proof: Let V and E denote the sets of vertices and edges of $B(n)$ respectivley. Using the above notations for the vertices and edges of $B(n)$, we shall exhibit two different graceful labelings of $B(n)$.

Graceful labeling Method A.

Edge Labeling: To show that $B(n)$ is graceful, we first construct an edge labeling of $B(n)$ by assigning for each edge e of $B(n)$, a value $\alpha(e)$ by the following formulae:

$$
\begin{aligned}
\alpha(e_i) &= i, & i &= 1,2,\ldots,2n \\
\alpha(f_i) &= 4n+1+i, & i &= 1,2,\ldots,2n \\
\alpha(g_i) &= 6n+1+i, & i &= 1,2,\ldots,2n \\
\alpha(h_i) &= 2n-1+2i, & i &= 1,2,\ldots,n+1 \\
\alpha(k_i) &= 2n+2, & i &= 1,2,\ldots,n
\end{aligned}
$$

Note that α is in fact a one-to-one mapping from the edge set E of $B(n)$ onto the set of integers $\{1,2,\ldots,8n+1\}$.

Vertex Labeling: Once the edges of $B(n)$ have been labeled by the mapping α, there is a natural way to label the vertices of $B(n)$ by the following algorithm. For each vertex v of $B(n)$ we shall label v by the value $\beta(v)$. Firstly, assign the value 0 to the vertex v_n that is , set $\beta(v_n) = 0$. Now if e is an edge joining the vertices a and b of $B(n)$ and if the value $\beta(a)$ has been assigned, then the value $\beta(b)$ can be obtained by the following ways:

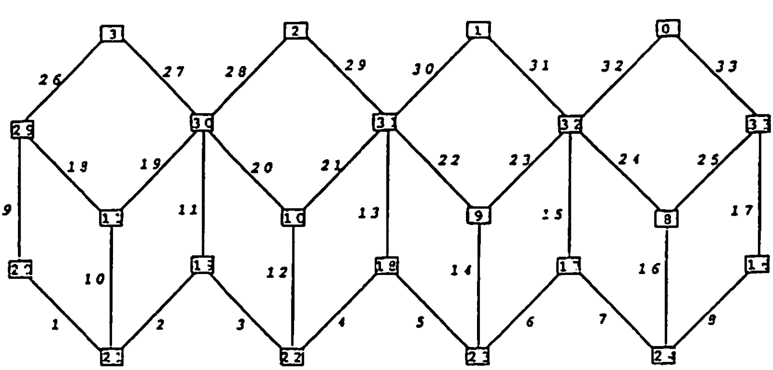
$$\beta(b) = \begin{cases} \beta(a) + \alpha(e), & \text{if } \beta(a) + \beta(b) \leq |E|; \\ \beta(a) - \alpha(e), & \text{otherwise} \end{cases}$$

This algorithm should be carried out in the sequence of assigning firstly, the values to the vertices on the top part of the graph and then to those at a lower level. Using the above algorithm, we obtain a vertex labeling of $B(n)$ given by the following formulae:

$$\begin{array}{ll} \beta(u_i) = n - i, & i = 1, 2, \ldots, n \\ \beta(v_i) = 3n - i, & i = 1, 2, \ldots, n \\ \beta(w_i) = 5n + i, & i = 1, 2, \ldots, n \\ \beta(p_i) = 7n + i, & i = 1, 2, \ldots, n + 1 \\ \beta(q_i) = 5n + 1 - i, & i = 1, 2, \ldots, n + 1 \end{array}$$

Observe that β is in fact a one-to one mapping from the vertex set V of $B(n)$ into the set of integers $\{0, 1, 2, \ldots, 8n + 1\}$. Moreover, the induced mapping $\beta^* : E \rightarrow \{1, 2, \ldots, 8n + 1\}$ defined by $\beta^*(e) = |\beta(a) - \beta(b)|$ for any edge e jointing the vertices a and b, is precisely equal to the edge labeling $\alpha : E \rightarrow \{1, 2, \ldots, 8n + 1\}$ given above. Since α is a bijective mapping, we have shown that the chain planar block graph is indeed a graceful graph.

Example 1: Graceful labeling of $B(4)$ by Method A:



Graceful labeling Method B

Using the same notation for the vertices and edges of the graph $B(n)$, we now present another graceful labeling for $B(n)$:

Edge labeling

The edges of $B(n)$ can be labeled by the mapping

$$\alpha : E \rightarrow \{1,2,\dots,8n+1\}$$

defined by the following formulae:

$$\alpha(e_i) = \begin{cases} 1+3(\frac{i-1}{2}), & i=1,3,5,\dots,2n-1 \\ 3(\frac{i}{2}), & i=2,4,6,\dots,2n \end{cases}$$

$$\alpha(f_i) = \begin{cases} 3n+2+3(\frac{i-1}{2}), & i=1,3,5,\dots,2n-1 \\ 3n+3(\frac{i}{2}), & i=2,4,6,\dots,2n \end{cases}$$

$$\alpha(g_i) = 6n+1+i, \quad i=1,2,\dots,2n$$

$$\alpha(h_i) = 3n-2+3i, \quad i=1,2,\dots,n+1$$

$$\alpha(k_i) = 3i-1, \quad i=1,2,\dots,n$$

Note that α is a bijective mapping from E onto the set $\{1,2,\dots,8n+1\}$.

Vertex labeling

Once we have labeled the edges of $B(n)$ by the mapping α and assigned the value 0 to the vertex v_n of $B(n)$, then the vertex labeling of $B(n)$ is determined and is governed by the following mapping

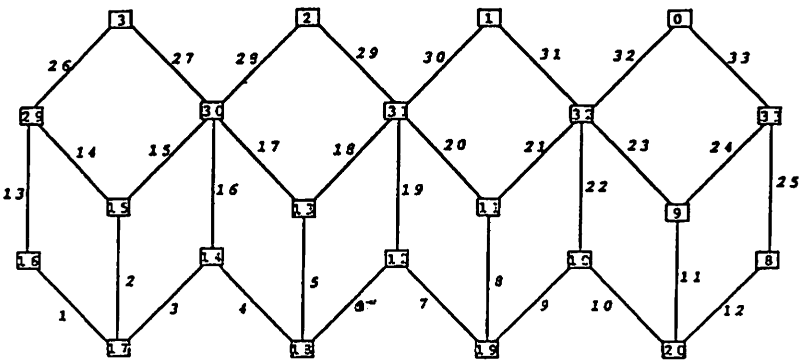
$$\beta : V \rightarrow \{0,1,2,\dots,8n+1\}$$

where

$$\begin{aligned} \beta(u_i) &= n-i, & i&=1,2,\dots,n \\ \beta(\nu_i) &= 4n+1-2i, & i&=1,2,\dots,n \\ \beta(w_i) &= 4n+i, & i&=1,2,\dots,n \\ \beta(p_i) &= 7n+i, & i&=1,2,\dots,n+1 \\ \beta(q_i) &= 4n-2(i-1), & i&=1,2,\dots,n+1 \end{aligned}$$

Again, β is an injective mapping from the vertex set V of $B(n)$ into the set $\{0,1,2,\dots,8n+1\}$. Moreover, the induced mapping $\beta^\star$ of β, defined by $\beta^\star(e) = |\beta(a) - \beta(b)|$ for any edge e joining the vertices a and b in $B(n)$, is precisely equal to α. As α is bijective, we have shown that β is a graceful labeling of $B(n)$.

Example 2: Graceful labeling of $B(4)$ by Method B.



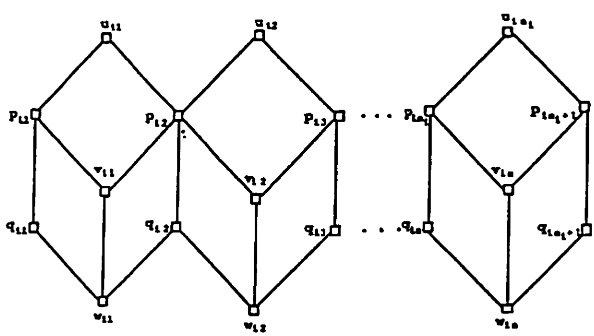


Fig. 3 Graph of $B(n_i), i = 1, 2, \ldots, s$.

3. Block Graphs

Suppose that k chain planar block graphs $B(n_1), B(n_2), \ldots, B(n_s)$ are given, where $(n_1, n_2, \ldots, n_s)$ is a sequence of s positive integers. We can construct a graph $B(n_1, n_2, \ldots, n_s)$ by amalgamating all these s chain planar block graphs together in the following way.

Let us denote the vertices of $B(n_i), i = 1, 2, \ldots, s$ in a systematic way by u_{i1}, $u_{i2}, \ldots, u_{in_i}$; $v_{i1}, v_{i2}, \ldots, v_{in_i}$; $w_{i1}, w_{i2}, \ldots, w_{in_i}$; $p_{i1}, p_{i2}, \ldots, p_{in_i+1}$ and q_{i1}, $q_{i2}, \ldots, q_{in_i+1}$ as shown below:

For $s = 2$, the graph $B(m_1 n_2)$ is obtained by placing the graph $B(n_2)$ on the top of the graph $B(n_1)$ and identifying the vertices (and hence the corresponding edges) occurring at the lowest level of $B(n_2)$ with those on the top level of $N(n_1)$. That is, when $n_1 \leq n_2$, we identify the vertices q_{21}, w_{21}, $q_{22}, w_{22}, \ldots, q_{2n_i}, \ldots, w_{2n_i}$ of $B(n_2)$ with the vertices $u_{11}, p_{11}, u_{12}, p_{12}, \ldots$, u_{1n_1}, p_{1n_1} of $B(n_1)$ respectively, and in the case when $n_1 > n_2$, we identify the vertices $q_{21}, w_{21}, q_{22}, w_{22}, \ldots, q_{2n_2}, w_{2p_2}, q_{2n_2+1}$ of $B(n_2)$ with the vertices $u_{11}, p_{11}, u_{12}, p_{12}, \ldots, u_{1n_2}, p_{1n_2}, u_{1n_2+1}$ of $B(n_1)$ respectively. In general, if the graph $B(n_1, n_2, \ldots, n_i)$ is constructed and $B(n_{i+1})$ is given, then we can form the graph $B(n_1, n_2, \ldots, n_i, n_{i+1})$ by placing $B(n_{i+1})$ on the top of $B(n_1, n_2, \ldots, n_i)$ and identifying vertices $q_{i+11}, w_{i+11}, q_{i+12}, w_{i+12}, \ldots, q_{i+1j}, w_{i+1j}$ of $B(n_{i+1})$ respectively with the vertices $u_{i1}, p_{i1}, u_{i2}, p_{i2}, \ldots, u_{ij}, p_{ij}$ of $B(n_1 n_2, \ldots, n_i)$ where $j = \min(n_i, n_{i+1})$. We shall call $B(n_1, n_2, \ldots, n_s)$ the *block graph* determined by the sequence $(n_1, n_2, \ldots, n_s)$. The following diagrams depict the block graphs $B(1,2,3)$ and $B(2,4,3)$.

Our main result in this section is:

Theorem. *For any integer $k \geq 1$, and any sequence $(n_1, n_2, \ldots, n_s)$ of s positive integers $n_1, n_2, \ldots, n_s$, the block graph $B(n_1, n_2, \ldots, n_s)$ is k-graceful.*

Lemma 2. *The block graph $B(n_1, n_2, \ldots, n_s)$ is graceful.*

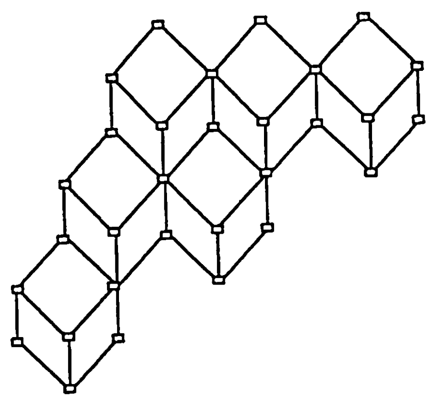


Fig 4.a The Block Graph $B(1,2,3)$.

Proof: Basically, we shall apply the graceful labeling method A of Lemma 1 to label the edges of $B(n_1)$ and then apply the graceful labeling method B to label the edges of $B(n_2), \ldots, B(n_s)$, beginning from the edges occurring at the lowest level of the graph until we have labeled those on the top of graph. Then, we assign the value 0 to the vertex v_{sn_s} of the graph $B(n_1, n_2, \ldots, n_s)$ and this will propogate a vertex labeling of $B(n_1, n_2, \ldots, n_s)$ which can be shown to be graceful.

Instead of proving the above lemma rigorously, we shall illustrate a graceful labeling of $B(n_1, n_2, \ldots, n_s)$ by the following examples.

Maheo and Thuillier[13] showed that if (A, B) is a bipartition of a graceful bipartite graph G such that the labeling of the vertices of G has the property that $\beta(a) > \beta(b)$ for all $a \in A$ and $b \in B$, then we can induce a k-graceful labeling

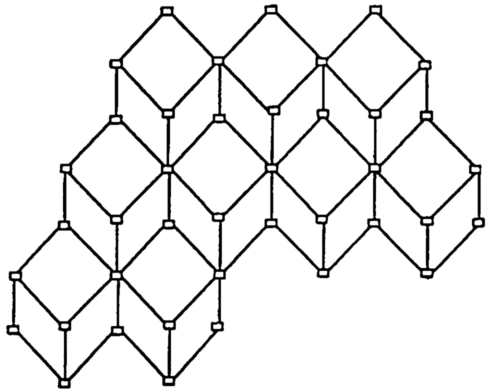
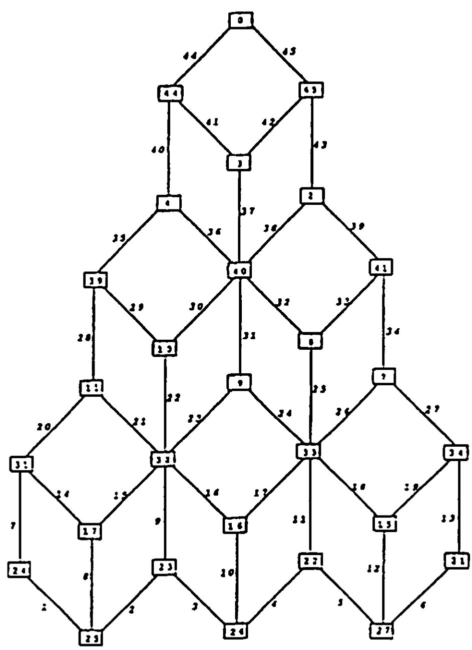


Fig 4.b The Block Graph $B(2,4,3)$

Example 3: Graceful labeling of $B(3,2,1)$:

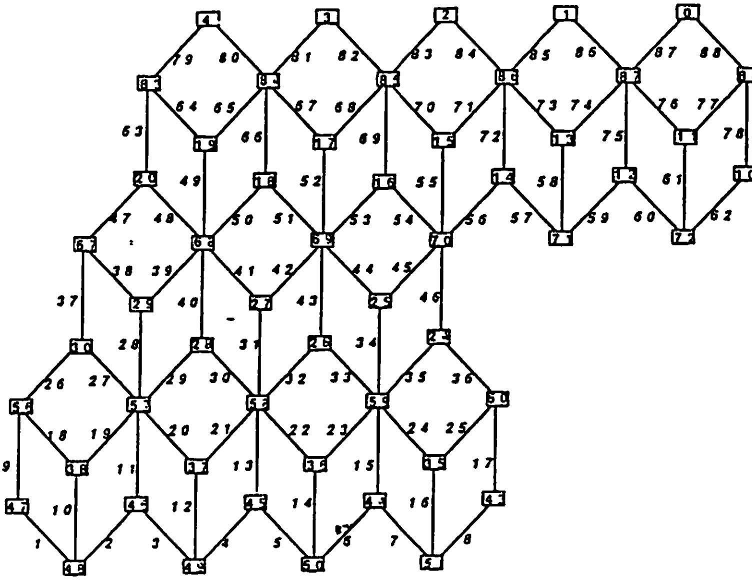


β^k on G as follows:

$$\beta^k(\nu) = \begin{cases} \beta(\nu) + k, & \text{if } \nu \in A \\ \beta(\nu), & \text{if } \nu \in B \end{cases}$$

Now, if $B(n_1, n_2, \ldots, n_s)$ is given, then we can bipartition the vertices of $B(n_1, n_2, \ldots, n_s)$ in the following way:

Example 4: Graceful labeling of $B(4,3,5)$



Let $\{w_1, w_2, \ldots, w_{n_1}\}$ be the set of vertices at the bottom of $B(n_1, n_2, \ldots, n_s)$. A vertex v is said to be of distance d from the set $(w_1, w_2, \ldots, w_{n_1})$ if d is the minimum length of any path from v to any vertex in the set $(w_1, w_2, \ldots, w_n)$. Now, we let A and B be the sets of vertices of $B(n_1, n_2, \ldots, n_s)$ with even and odd distances from $(w_1, w_2, \ldots, w_{n_1})$ respectively. We observe that A, B and β satisfy the condition that $\beta(a) > \beta(b)$ for any $a \in A$ and $b \in B$. Hence, by the result of Maheo and Thuillier, we conclude that $B(n_1, n_2, \ldots, n_s)$ is k-graceful for any positive integer k.

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