DOUBLY NESTED TRIPLE SYSTEMS AND NESTED $B[4, 3\lambda; v]$

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Abstract. A balanced incomplete Block design $B[k, \alpha; v]$ is said to be a nested design if one can add a point to each block in the design and so obtain a block design $B[k+1, \beta; v]$. Stinson(1985) and Colbourn and Colbourn (1983) proved that the necessary condition for the existence of a nested $B[3, \alpha; v]$ is also sufficient. In this paper, we investigate the case $k = 4$ and show that the necessary condition for the existence of a nested $B[4, \alpha; v]$, namely $\alpha = 3\lambda$, $\lambda(v-1) \equiv 0 \pmod 4$ and $v \geq 5$, is also sufficient. To do this, we need the concept of a doubly nested design. A $B[k, \alpha; v]$ is said to be doubly nested if the above $B[k+1, \beta; v]$ is also a nested design. When $k = 3$, such a design is called a doubly nested triple system. We prove that the necessary condition for the existence of a doubly nested triple system $B[3, \alpha; v]$, namely $\alpha = 3\lambda$, $\lambda(v-1) \equiv 0 \pmod 2$ and $v \geq 5$, is also sufficient with the four possible exceptions $v = 39$ and $\alpha = 3, 9, 15, 21$.

1. Introduction.

A *pairwise balanced design* (or,PBD) $B[K, \lambda; v]$ is a pair $(X, \mathcal{B})$, where X is a finite set of points, $\mathcal{B}$ is a set of subsets of X, v and λ are positive integers and K is some set of positive integers, if $|X| = v$, $|B| \in K$ for every $B \in \mathcal{B}$ and every unordered pair of points is contained in exactly λ blocks of $\mathcal{B}$.

If $K = \{k\}$, the $B[K, \lambda; v]$ is called a balanced incomplete block design and denoted by (v, k, λ)-BIBD. When $k = 3$, a $B[3, \lambda; v]$ is called a triple system. It is well known [3] that a necessary condition for the existence of a $B[k, \lambda; v]$ is that $\lambda(v-1) \equiv 0 \pmod{k-1}$ and $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

Let $D = (X, \mathcal{B})$ be a $B[k, \alpha; v]$. A *nesting* of the design D is a mapping $f: \mathcal{B} \to X$ such that $(X, \mathcal{B}_1)$, where $\mathcal{B}_1 = \{B \cup f(B) \mid B \in \mathcal{B}\}$, is a $B[k+1, \beta; v]$. Such a D is called a nested $B[k, \alpha; v]$. A *double nesting* of the design D is a pair of mappings (f, g), $f: \mathcal{B} \to X$, $g: \mathcal{B}_1 \to X$, such that $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$, where $\mathcal{B}_1 = \{B \cup f(B) \mid B \in \mathcal{B}\}$, $\mathcal{B}_2 = \{B \cup g(B) \mid B \in \mathcal{B}_1\}$, are respectively a $B[k+1, \beta; v]$ and a $B[k+2, \gamma; v]$. Such a D is called a doubly nested $B[k, \alpha; v]$. When $k = 3$, the design is called a doubly nested triple system.

In fact, the above definition is a generalization of a nested triple system. Stinson [10] and Colbourn and Colbourn [2] proved that the necessary and sufficient condition for the existence of a nested $B[3, \alpha; v]$ is that $\alpha(v-1) \equiv 0 \pmod 6$ and $v \geq 4$. In this paper we shall investigate the existence of nested $B[4, \alpha; v]$ and doubly nested $B[3, \alpha; v]$. By some simple counting, we have the following necessary conditions.

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Lemma 1.1. *A necessary condition for the existence of a nested* $B[4,\alpha;v]$ *is* $\alpha = 3\lambda, \lambda(v-1) \equiv 0 \pmod 4$ *and* $v \geq 5$.

Lemma 1.2. *A necessary condition for the existence of a doubly nested* $B[3,\alpha;v]$ *is* $\alpha = 3\lambda, \lambda(v-1) \equiv 0 \pmod 2$ *and* $v \geq 5$.

In Section 2, we use perpendicular arrays (PA), orthogonal idempotent Latin squares, PBD closure and some direct constructions to show that the above necessary condition for the existence of a doubly nested $B[3,\alpha;v]$ is also sufficient with the four possible exceptions $v = 39$ and $\alpha = 3,9,15,21$. In Section 3, we use almost resolvable designs and self-orthogonal Latin squares with symmetric orthogonal mates to show that the above necessary condition for the existence of a nested $B[4,\alpha;v]$ is also sufficient.

2. Doubly nested $B[3,3\lambda;v]$.

2.1 A construction from PA

A *perpendicular array* ($PA(v,k)$) is an $\binom{v}{2} \times k$ array, such that each cell is occupied with one of the points in X, $|X| = v$, and such that any two columns of the array contain each 2-subset of X exactly once. The number v is called the *order* and the number k the *depth* of the array. It is clear that if we delete any k' $(0 \leq k' < k)$ columns of a $PA(v,k)$, we obtain a $PA(v,k-k')$. If we denote rows of a $PA(v,k)$ by

$$(a_{i1},a_{i2},\ldots,a_{ik}),\ 1 \leq i \leq \binom{v}{2} \text{ and } \mathbf{B} = \left\{\{a_{i1},a_{i2},\ldots,a_{ik}\} \mid 1 \leq i \leq \frac{v(v-1)}{2}\right\},$$

then $(X,\mathbf{B})$ is a $B\left[k, \frac{k(k-1)}{2}; v\right]$. We employ this to show

Theorem 2.1. *If there exists a* $PA(v,k+2)$, *then there exists a doubly nested* $B\left[k,\frac{k(k-1)}{2};v\right]$.

Proof: We denote the rows of a $PA(v,k+2)$ by $(a_{i1},a_{i2},\ldots,a_{i,k+2})$, $1 \leq i \leq \frac{v(v-1)}{2}$. Then the two arrays which consist of rows $(a_{i1},a_{i2},\ldots,a_{i,k+1})$ and $(a_{i1},a_{i2},\ldots,a_{ik})$, $1 \leq i \leq \frac{v(v-1)}{2}$, are respectively a $PA(v,k+1)$ and a $PA(v,k)$. Let

$$\mathbf{B}_j = \left\{\{a_{i1},a_{i2},\ldots,a_{i,k+j}\} \mid 1 \leq i \leq \frac{v(v-1)}{2}\right\},\quad j = 0,1,2.$$

Then $(X,\mathbf{B}_j)$ is a $B\left[k+j,\frac{(k+j)(k+j-1)}{2};v\right]$ for $j = 0,1,2$. Now define a pair of mappings (f,g) as follows:

$$f{:}\mathbf{B}_0 \to X,\ f(B) = a_{i,k+1} \text{ where } B = \{a_{i1},a_{i2},\ldots,a_{ik}\},$$
$$g{:}\mathbf{B}_1 \to X,\ g(B) = a_{i,k+2} \text{ where } B = \{a_{i1},a_{i2},\ldots,a_{i,k+1}\},$$

Therefore, $\{B \cup f(B) \mid B \in \boldsymbol{B}_0\} = \boldsymbol{B}_1$ and $\{B \cup g(B) \mid B \in \boldsymbol{B}_1\} = \boldsymbol{B}_2$. Then (f,g) is a double nesting of $(X,\boldsymbol{B}_0)$ and $(X,\boldsymbol{B}_0)$ is a doubly nested $B[k, \frac{k(k-1)}{2}; v]$. ∎

Corollary 2.2. *There exists a doubly nested* $B[3, 3\lambda; v]$ *for all odd* $v \geq 5$, $v \neq 39$ *and all positive integers* λ.

Proof: From [6,9], there exists a $PA(v,5)$ for all odd $v \geq 5$ and $v \neq 39$, so we have a doubly nested $B[3,3;v]$ from Theorem 2.1. Taking each block λ times we obtain the required doubly nested design.

We will discuss the case of even v in the following sections and the case $v = 39$ in Section 2.4.

2.2 A construction from idempotent MOLS A *Latin square* of order v based on a v-set X is an $v \times v$ array such that each row and each column contains each element in X exactly once. Two Latin squares, $A = (a_{ij})$ and $B = (b_{ij})$, on X, are said to be *orthogonal* if $\{(a_{ij}, b_{ij}) \mid 1 \leq i,j \leq v\} = X \times X$. Without loss of generality we let $X = \{1,2,\ldots,v\}$. A Latin square on X is *idempotent* if the (i,i) entry is i for $1 \leq i \leq v$. k idempotent Latin squares $A_1, A_2, \ldots, A_k$ of order v are said to be k mutually orthogonal idempotent Latin squares if A_i and A_j are orthogonal for all $1 \leq i < j \leq k$, and are denoted by k idempotent MOLS(v). Suppose $A_1 = \left(a_{ij}^{(1)}\right), A_2 = \left(a_{ij}^{(2)}\right), \ldots, A_k = \left(a_{ij}^{(k)}\right)$ are k idempotent MOLS(v) on X; then $(X,\boldsymbol{B})$ is a $B[k+2, \lambda; v]$, where

$$\boldsymbol{B} = \left\{\{i,j,a_{ij}^{(1)},\ldots,a_{ij}^{(k)}\} \mid i \neq j,\ 1 \leq i,j \leq v\right\} \text{ and } \lambda = 2\binom{k+2}{2} = (k+2)(k+1).$$

We employ this to show

Theorem 2.3. *If there exist* $k(\geq 3)$ *idempotent MOLS*(v), *then there exists a doubly nested* $B[k, k(k-1); v]$.

Proof: Let $A_t = \left(a_{ij}^{(t)}\right)$, $1 \leq t \leq k$, be k idempotent MOLS(v) on X, and let

$$\boldsymbol{B}_r = \left\{\{i,j,a_{ij}^{(1)},\ldots,a_{ij}^{(k-2+r)}\} \mid i \neq j,\ 1 \leq i,j \leq v\right\},\quad r = 0,1,2.$$

Then $(X,\boldsymbol{B}_r)$ is a $B[k+r, (k+r)(k+r-1); v]$ for $r = 0,1,2$. Now define a pair of mappings (f,g) as follows:

$$f{:}\boldsymbol{B}_0 \to X,\quad f(B) = a_{ij}^{(k-1)} \text{ where } B = \{i,j,a_{ij}^{(1)},\ldots,a_{ij}^{(k-2)}\},$$
$$g{:}\boldsymbol{B}_1 \to X,\quad g(B) = a_{ij}^{(k)} \text{ where } B = \{i,j,a_{ij}^{(1)},\ldots,a_{ij}^{(k-1)}\}.$$

Obviously, $\{B \cup f(B) \mid B \in \boldsymbol{B}_0\} = \boldsymbol{B}_1$ and $\{B \cup g(B) \mid B \in \boldsymbol{B}_1\} = \boldsymbol{B}_2$, so (f,g) is a double nesting of the $(X,\boldsymbol{B}_0)$, and $(X,\boldsymbol{B}_0)$ is a doubly nested $B[k, k(k-1); v]$. ∎

Let $E = \{2,3,4,6,10,18,22,26,28\}$. The following lemma is taken from [1, Theorem 4.5].

Lemma 2.4. *There exists a set of 3 idempotent MOLS(v) for any positive integer $v \notin E$.*

Corollary 2.5. *There exists a doubly nested $B[3,3\lambda;v]$ for any positive integer $v \notin E$ and any positive even integer λ.*

Proof: From Lemma 2.4 and Theorem 2.3 there exists a doubly nested $B[3,6;v]$ for any $v \notin E$. Taking each block $\lambda/2$ times, we obtain a doubly nested $B[3,3\lambda;v]$. ∎

2.3 A direct construction

An incomplete BIBD$[k,\lambda;v,v_1]$ (or, $IB[k,\lambda;v,v_1]$) is a triple $(X,Y,\mathcal{B})$, where X is a v-set of points, $Y \subseteq X$, $|Y| = v_1$ and $\mathcal{B}$ is a set of blocks which satisfies the following properties:

1) for any $B \in \mathcal{B}$, $|B| = k$ and $|B \cap Y| \leq 1$;
2) any two points x, y, not both in Y, occur in exactly λ blocks of $\mathcal{B}$.

Suppose $(X,Y,\mathcal{B})$ is an $IB[k,\lambda;v,v_1]$ and $(Y,\mathcal{B}_1)$ is a $B[k,\lambda;v_1]$. Then $(X,\mathcal{B}\cup\mathcal{B}_1)$ is a $B[k,\lambda;v]$.

Let $G = Z_{v-n} = \{0,1,\ldots,v-n-1\}$ be the cyclic group of integers modulo $v-n$. Y consists of n infinite elements $\infty_1,\infty_2,\ldots,\infty_n$. We extend the group G by adding all elements in Y, with the property that $g+y = y+g = y$ for all $g \in G$ and $y \in Y$.

Direct Construction Theorem. *Let B_i $(1 \leq i \leq v-1)$ and D_j $(v-n \leq j \leq v-1)$ be 3-subsets of $X = G \cup Y$ which satisfy the following conditions:*

(1) *$B_1, B_2, \ldots, B_{v-1}, D_{v-n}, \ldots, D_{v-1}$ are base blocks of an $IB[3,6;v,n]$ $(X,Y,\mathcal{B}')$, where $\mathcal{B}'$ consists of all B_{ij}, as follows:*

$$B_{\overline{ij}} = \begin{cases} B_{j-i}+i, & \text{if } i \neq j, \text{ and } i,j \in G; \\ B_j + i, & \text{if } i \in G, \text{ and } v-n \leq j \leq v-1; \\ D_i + j, & \text{if } v-n \leq i \leq v-1, \text{ and } j \in G. \end{cases}$$

(2) *every element in $G\backslash\{0\}$ occurs exactly 3 times in the blocks $B_1, B_2, \ldots, B_{v-1}$, and every element in Y occurs exactly 3 times in the blocks $B_1, B_2, \ldots, B_{v-n-1}$;*

(3) *every element in Y occurs exactly 3 times in the blocks $D_1, D_2, \ldots, D_{v-n-1}$, and for every $i \in G\backslash\{0\}$, if i occurs τ times in the blocks $D_1, D_2, \ldots, D_{v-1}$, then $-i$ occurs exactly $6-\tau$ times in them, where $D_i = B_i - i$, $i \in G\backslash\{0\}$;*

(4) *there exists a doubly nested $B[3,6;n]$.*

Then there exists a doubly nested $B[3,6;v]$.

Proof: Suppose $(Y,\mathcal{B}'')$ is a doubly nested $B[3,6;n]$. Then there exists a double nesting (f'',g''). such that $(Y,\mathcal{B}_1'')$ and $(Y,\mathcal{B}_2'')$, where $\mathcal{B}_1'' = \{B \cup f''(B) \mid B \in$

$\mathcal{B}''\}$ and $\mathcal{B}_2'' = \{B \cup g''(B) \mid B \in \mathcal{B}_1''\}$ are respectively a $B[4,12;n]$ and a $B[5,20;n]$. Let $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$; then $(X,\mathcal{B})$ is a $B[3,6;v]$. We will prove that $(X,\mathcal{B})$ is a doubly nested design.

If we put the block B_{ij} in the cell (i,j) of a $v \times v$ square T, then the diagonal cells in T are empty and the $n \times n$ sub-square in the lower right corner is empty. Further, we know that T is cyclically generated modulo $v-n$. This is because

$$\begin{cases} B_{i+1,j+1} = B_{ij} + 1, & \text{if } i \neq j \text{ and } i,j \in G; \\ B_{i+1,j} = B_{ij} + 1, & \text{if } i \in G \text{ and } v-n \leq j \leq v-1; \\ B_{i,j+1} = B_{ij} + 1, & \text{if } v-n \leq i \leq v-1 \text{ and } j \in G. \end{cases}$$

It is obvious that the ith row (column) of T is indexed by i for $i \in G$. When $v-n \leq i \leq v-1$, the ith row (column) is indexed by $\infty_{i-v+n+1}$. By the cyclic property of T and conditions (2) and (3), T has the following two properties:

(i) every element in $(G \cup Y)\backslash\{i\}$ occurs exactly 3 times in the ith row, for $i \in G$, and every element in G occurs exactly 3 times in the ith row, for $v-n \leq i \leq v-1$;

(ii) for $i \in G$, every element in Y occurs exactly 3 times in the ith column and any $j \in G\backslash\{i\}$ occurs exactly τ times in the ith column if $j-i$ occurs exactly τ times in the 0th column. Every element in G occurs exactly 3 times in the ith column for $v-n \leq i \leq v-1$.

Now define a mapping $f: \mathcal{B} \to X$ as follows:

$$f(B) = \begin{cases} i, & \text{if } B = B_{ij},\ i \neq j,\ i \in G \text{ and } 0 \leq j \leq v-1; \\ \infty_{i-v+n+1}, & \text{if } B = B_{ij},\ v-n \leq i \leq v-1 \text{ and } j \in G; \\ f''(B), & \text{if } B \in \mathcal{B}''. \end{cases}$$

It can be shown that $(X,\mathcal{B}_1)$ is a $B[4,12;v]$ where $\mathcal{B}_1 = \{B \cup f(B) \mid B \in \mathcal{B}\}$. In fact, f maps B_{ij} to its row index. Let x and y be any two distinct elements in X. If $\{x,y\} \subset Y$, then $\{x,y\}$ is contained in exactly 12 blocks of $\mathcal{B}_1''$. If x and y are not both in Y, then by property (i), y occurs 3 times in the row indexed by x while x does not occur in that row. Therefore, $\{x,y\}$ is contained in exactly 3 of the corresponding blocks of $\mathcal{B}_1$. For y, we have a same result. Since $\{x,y\}$ is contained in exactly 6 blocks of the other rows of T, it is contained in exactly 6 of the corresponding blocks of $\mathcal{B}_1$. Therefore, $\{x,y\}$ is contained in exactly 12 blocks of $\mathcal{B}_1$. This guarantees that $(X,\mathcal{B}_1)$ is a $B[4,12;v]$.

Analogously, we use the column index to define a mapping $g: \mathcal{B}_1 \to X$ as follows:

$$g(B) = \begin{cases} j, & \text{if } B = B_{ij} \cup f(B_{ij}),\ i \neq j,\ 0 \leq i \leq v-1 \text{ and } j \in G; \\ \infty_{j-v+n+1}, & \text{if } B = B_{ij} \cup f(B_{ij}),\ i \in G, \text{ and } v-n \leq j \leq v-1; \\ g''(B), & \text{if } B \in \mathcal{B}_1''. \end{cases}$$

We shall show that $(X,\mathbf{B}_2)$ is a $B[5,20;v]$ where $\mathbf{B}_2 = \{B \cup g(B) \mid B \in \mathbf{B}_1\}$. Let $\{x,y\}$ be any 2-subset of X. If $\{x,y\} \subset Y$, then $\{x,y\}$ is contained in exactly 20 blocks of $\mathbf{B}_2''$. If $x \in G$ and $y \in Y$, then by property (ii), y occurs exactly 3 times in the column indexed by x while x does not occur in that column. Then y occurs in exactly 4 of the corresponding blocks of $\mathbf{B}_1$. Hence $\{x,y\}$ is contained in 4 of the corresponding blocks of $\mathbf{B}_2$. For y we have the same result. Since $\{x,y\}$ is contained in exactly 12 of the other blocks of $\mathbf{B}_1$, $\{x,y\}$ is contained in exactly 12 of the corresponding blocks of $\mathbf{B}_2$. Therefore, $\{x,y\}$ is contained in exactly 20 blocks of $\mathbf{B}_2$. Now let $\{x,y\} \subset G$. If $y-x$ occurs τ times in the 0th column, then, by condition (3), $x-y$ occurs $6-\tau$ times in the 0th column. By property (ii), y occurs τ times in the xth column while x does not. Hence y occurs in $\tau+1$ of the corresponding blocks of $\mathbf{B}_1$ but x does not. Analogously, x occurs exactly $(6-\tau+1)$ times in the yth column while y does not occur in that column. So $\{x,y\}$ is contained in exactly $(\tau+1)+(6-\tau+1) = 8$ of the corresponding blocks of $\mathbf{B}_2$. Since $\{x,y\}$ is contained in exactly 12 blocks of $\mathbf{B}_1$, $\{x,y\}$ is contained in 12 of the corresponding blocks of $\mathbf{B}_2$. Therefore, $\{x,y\}$ is contained in exactly 20 blocks of $\mathbf{B}_2$. This shows that $(X,\mathbf{B}_2)$ is a $B[5,20;v]$.

As above, (f,g) is a double nesting of $(X,\mathbf{B})$, so $(X,\mathbf{B})$ is a doubly nested $B[3,6;v]$. This completes the proof. ∎

Remark: If $n=1$, the conditions needed in the Theorem are only (1), (2) and (3). In this case, condition (1) says that $B_1, B_2, \ldots, B_{v-1}, D_{v-n}, \ldots, D_{v-1}$ are base blocks of a $B[3,6;v]$.

Lemma 2.6. *There exists a doubly nested* $B[3,6;v]$ *for* $v = 6, 10, 18, 22$.

Proof: Applying the Direct Construction Theorem, we get the required designs with base blocks shown as follows.

(1) $v=6$, $X = Z_5 \cup \{\infty\}$

$$B_1 = \{2,3,\infty\} \qquad D_1 = \{1,2,\infty\}$$
$$B_2 = \{1,4,\infty\} \qquad D_2 = \{4,2,\infty\}$$
$$B_3 = \{1,4,\infty\} \qquad D_3 = \{3,1,\infty\}$$
$$B_4 = \{1,2,3,\} \qquad D_4 = \{2,3,4\}$$
$$B_5 = \{2,3,4\} \qquad D_5 = \{1,3,4\}$$

(2) $v = 10$, $X = Z_9 \cup \{\infty\}$

$B_1 = \{2,7,\infty\}$	$D_1 = \{1,6,\infty\}$
$B_2 = \{4,6,8\}$	$D_2 = \{2,4,6\}$
$B_3 = \{4,6,7\}$	$D_3 = \{1,3,4\}$
$B_4 = \{2,3,\infty\}$	$D_4 = \{7,8,\infty\}$
$B_5 = \{1,8,\infty\}$	$D_5 = \{5,3,\infty\}$
$B_6 = \{2,5,8\}$	$D_6 = \{5,8,2\}$
$B_7 = \{1,5,6\}$	$D_7 = \{3,7,8\}$
$B_8 = \{3,4,5\}$	$D_8 = \{4,5,6\}$
$B_9 = \{1,3,7\}$	$D_9 = \{1,2,7\}$

(3) $v = 18, X = Z_{17} \cup \{\infty\}$

$B_1 = \{4,7,12\}$	$D_1 = \{3,6,11\}$
$B_2 = \{10,12,15\}$	$D_2 = \{8,10,13\}$
$B_3 = \{1,2,4\}$	$D_3 = \{1,15,16\}$
$B_4 = \{8,9,16\}$	$D_4 = \{4,5,12\}$
$B_5 = \{7,13,14\}$	$D_5 = \{2,8,9\}$
$B_6 = \{3,13,\infty\}$	$D_6 = \{7,14,\infty\}$
$B_7 = \{8,14,\infty\}$	$D_7 = \{1,7,\infty\}$
$B_8 = \{3,5,6\}$	$D_8 = \{12,14,15\}$
$B_9 = \{2,15,\infty\}$	$D_9 = \{6,10,\infty\}$
$B_{10} = \{11,12,16\}$	$D_{10} = \{1,2,6\}$
$B_{11} = \{6,8,14\}$	$D_{11} = \{3,12,14\}$
$B_{12} = \{4,9,16\}$	$D_{12} = \{4,9,14\}$
$B_{13} = \{1,5,11\}$	$D_{13} = \{5,9,15\}$
$B_{14} = \{9,10,13\}$	$D_{14} = \{12,13,16\}$
$B_{15} = \{2,5,11\}$	$D_{15} = \{4,7,13\}$
$B_{16} = \{6,10,15\}$	$D_{16} = \{7,11,16\}$
$B_{17} = \{1,3,7\}$	$D_{17} = \{2,9,11\}$

(4) $v = 22$, $X = Z_{17} \cup \{\infty_i \mid 1 \le i \le 5\}$

$B_1 = \{3, 13, \infty_1\}$	$D_1 = \{2, 12, \infty_1\}$
$B_2 = \{9, 13, \infty_1\}$	$D_2 = \{7, 11, \infty_1\}$
$B_3 = \{11, 12, \infty_1\}$	$D_3 = \{8, 9, \infty_1\}$
$B_4 = \{3, 5, 6\}$	$D_4 = \{16, 1, 2\}$
$B_5 = \{1, 3, \infty_2\}$	$D_5 = \{13, 15, \infty_2\}$
$B_6 = \{10, 12, \infty_2\}$	$D_6 = \{4, 6, \infty_2\}$
$B_7 = \{4, 15, \infty_2\}$	$D_7 = \{14, 8, \infty_2\}$
$B_8 = \{13, 14, \infty_3\}$	$D_8 = \{5, 6, \infty_3\}$
$B_9 = \{1, 2, \infty_3\}$	$D_9 = \{9, 10, \infty_3\}$
$B_{10} = \{8, 14, \infty_3\}$	$D_{10} = \{15, 4, \infty_3\}$
$B_{11} = \{4, 7, \infty_4\}$	$D_{11} = \{10, 13, \infty_4\}$
$B_{12} = \{9, 16, \infty_4\}$	$D_{12} = \{14, 4, \infty_4\}$
$B_{13} = \{7, 10, \infty_4\}$	$D_{13} = \{11, 14, \infty_4\}$
$B_{14} = \{8, 16, \infty_5\}$	$D_{14} = \{11, 2, \infty_5\}$
$B_{15} = \{6, 7, \infty_5\}$	$D_{15} = \{8, 9, \infty_5\}$
$B_{16} = \{2, 15, \infty_5\}$	$D_{16} = \{3, 16, \infty_5\}$
$B_{17} = \{10, 12, 15\}$	$D_{17} = \{1, 5, 12\}$
$B_{18} = \{6, 8, 14\}$	$D_{18} = \{11, 15, 16\}$
$B_{19} = \{2, 5, 11\}$	$D_{19} = \{13, 10, 5\}$
$B_{20} = \{1, 5, 11\}$	$D_{20} = \{3, 7, 12\}$
$B_{21} = \{4, 9, 16\}$	$D_{21} = \{1, 3, 10\}$

∎

2.4 The PBD construction

Lemma 2.7. *If there exists a $B[K', \lambda_1; v]$ and a $B[K, \lambda_2; k]$ for every $k \in K'$, then there exists a $B[K, \lambda_1\lambda_2; v]$.*

Proof: The conclusion follows from [3, Lemma 2.5]. ∎

Theorem 2.8. *If there exists a* $B[K,\lambda_1;v]$ *and a doubly nested* $B[3,3\lambda_2;k]$ *for every* $k \in K$, *then there exists a doubly nested* $B[3,3\lambda_1\lambda_2;v]$.

Proof: Let $(X,\mathbf{A})$ be a $B[K,\lambda_1;v]$ and let $(A,\mathbf{B}_A)$ be a doubly nested $B[3,3\lambda_2;|A|]$ for every $A \in \mathbf{A}$. Let (f_A,g_A) be a double nesting of $(A,\mathbf{B}_A)$. Then $(A,\mathbf{B}_A^{(1)})$ and $(A,\mathbf{B}_A^{(2)})$, where $\mathbf{B}_A^{(1)} = \{B \cup f_A(B) \mid B \in \mathbf{B}_A\}$ and $\mathbf{B}_A^{(2)} = \{B \cup g_A(B) \mid B \in \mathbf{B}_A^{(1)}\}$ are respectively a $B[4,6\lambda_2;|A|]$ and a $B[5,10\lambda_2;|A|]$. From Lemma 2.7, $(X,\mathbf{B})$, $(X,\mathbf{B}_1)$ and $(X,\mathbf{B}_2)$, where $\mathbf{B} = \cup_{A\in\mathbf{A}}\mathbf{B}_A$, $\mathbf{B}_1 = \cup_{A\in\mathbf{A}}\mathbf{B}_A^{(1)}$ and $\mathbf{B}_2 = \cup_{A\in\mathbf{A}}\mathbf{B}_A^{(2)}$, are respectively a $B[3,3\lambda_1\lambda_2;v]$ a $B[4,6\lambda_1\lambda_2;v]$ and a $B[5,10\lambda_1\lambda_2;v]$. Now we define a pair of mappings (f,g) as follows:

$$f{:}\mathbf{B} \to X,\quad f(B) = f_A(B), \text{ if and only if } B \in \mathbf{B}_A;$$
$$g{:}\mathbf{B}_1 \to X,\quad g(B) = g_A(B), \text{ if and only if } B \in \mathbf{B}_A^{(1)}.$$

Obviously, $\{B \cup f(B) \mid B \in \mathbf{B}\} = \mathbf{B}_1$ and $\{B \cup g(B) \mid B \in \mathbf{B}_1\} = \mathbf{B}_2$. Then (f,g) is a double nesting of $(X,\mathbf{B})$ and $(X,\mathbf{B})$ is a doubly nested $B[3,3\lambda_1\lambda_2;v]$. ∎

We assume that the reader is familiar with the terminology of group divisible designs (GDDs) and transveral designs (TDs) (see, for example, [3]).

Lemma 2.9. *There exist doubly nested* $B[3,6;26]$ *and* $B[3,6;28]$.

Proof: Deleting five points in a block from a $TD[6,1,5]$, we obtain a $GDD[\{5,6\},1,\{4,5\};25]$ of type $4^5 5^1$. Adding a new point ∞ to each group, we obtain a $B[\{5,6\},1;26]$. By Corollary 2.2 and Lemma 2.6, there exists a doubly nested $B[3,6;5]$ and a doubly nested $B[3,6;6]$. We apply Theorem 2.8 to get a doubly nested $B[3,6;26]$. From [3], we know that a $B[7,2;28]$ exists, and by Corollary 2.2, we have a doubly nested $B[3,3;7]$. We apply Theorem 2.8 to get a doubly nested $B[3,6;28]$. ∎

Lemma 2.10. *There exists a doubly nested* $B[3,3\lambda;39]$ *for any positive integer* $\lambda \geq 9$.

Proof: Since there exists a $B[19,9;39]$ (see [7]), by Corollary 2.2 there exists a doubly nested $B[3,3;19]$. Apply Theorem 2.8 to get a doubly nested $B[3,27;39]$. On the other hand, there exists a doubly nested $B[3,3\lambda;39]$ for any positive even integer λ from Corollary 2.5. If we combine all blocks of the two designs, we obtain the required doubly nested design.

Combining all results in this section, we obtain the main result of this section.

Theorem 2.11. *The necessary condition for the existence of a doubly nested* $B[3,\alpha;v]$, *namely* $\alpha = 3\lambda$, $\lambda(v-1) \equiv 0 \pmod 2$ *and* $v \geq 5$, *is also sufficient with the 4 possible exceptions of* $v = 39$ *and* $\alpha = 3,9,15,21$.

Proof: If λ is odd, then we use Corollary 2.2 and Lemma 2.10. If λ is even, then we use Corollary 2.5, Lemma 2.6 and Lemma 2.9. ∎

3. Nested $B[4,3\lambda;v]$.

3.1 A construction from almost resolvable designs

An *almost resolvable design* $ARB[k,k-1;v]$ is a triple $(X,\boldsymbol{B},\{\boldsymbol{D}_1,\boldsymbol{D}_2,\ldots,\boldsymbol{D}_v\})$ which satisfies:

(i) $(X,\boldsymbol{B})$ is a $B[k,k-1;v]$,
(ii) $\boldsymbol{D}_1,\boldsymbol{D}_2,\ldots,\boldsymbol{D}_v$ is a partition of $\boldsymbol{B}$, and
(iii) for every $\boldsymbol{D}_i$ $(1 \le i \le v)$, there exists a point $x \in X$, such that $\boldsymbol{D}_i$ is a partition of $X\backslash\{x\}$, where $\boldsymbol{D}_i$ is called an almost parallel class and x the singleton of $\boldsymbol{D}_i$.

From [4, 8] we have the following lemmas.

Lemma 3.1. *There exists an* $ARB[4,3;v]$ *for every positive integer* $v \equiv 1 \pmod 4$.

Lemma 3.2. *Suppose* $(X,\boldsymbol{A})$ *is an* $ARB[k,k-1;v]$. *Then any* $x \in X$ *has a unique almost parallel class such that* x *is its singleton.*

Theorem 3.3. *If there exists an* $ARB[k,k-1;v]$, *then there exists a nested* $B[k,k-1;v]$.

Proof: Let $(X,\boldsymbol{A})$ be an $ARB[k,k-1;v]$. From Lemma 3.2, we can label the parallel class by $\boldsymbol{D}_x$ such that x is the singleton of $\boldsymbol{D}_x$. Now define a mapping $f:\boldsymbol{A} \rightarrow X$ as follows:

$$f(A) = x \text{ if and only if } A \in \boldsymbol{D}_x.$$

Then $(X,\boldsymbol{A}_1)$, where $\boldsymbol{A}_1 = \{A \cup f(A) \mid A \in \boldsymbol{A}\}$ is a $B[k+1,k+1;v]$. Hence, f is a nesting of $(X,\boldsymbol{A})$, and $(X,\boldsymbol{A})$ is a nested $B[k,k-1;v]$. ∎

Corollary 3.4. *There exists a nested* $B[4,3\lambda;v]$ *for all positive integers* λ *and* v, *where* $v \equiv 1 \pmod 4$.

Proof: From Lemma 3.1 and Theorem 3.3, we obtain a nested $B[4,3;v]$. Taking each block λ times, we obtain a nested $B[4,3\lambda;v]$. ∎

We will discuss the cases $v \equiv 3 \pmod 4$ and $v \equiv 0 \pmod 2$, in the next section.

3.2 A construction from SOLSSOMs

A Latin square of order v is said to be *self-orthogonal*, and is denoted by SOLS(v), if the square and its transpose are orthogonal. If a self-orthogonal Latin square A of order v and a symmetric Latin square B are orthogonal, then A is called a *self-orthogonal Latin square with a symmetric orthogonal mate*, and denoted by SOLSSOM(v).

From [5, 11, 12], we have the following lemma.

Lemma 3.5. *There exists a SOLSSOM(v) for any odd $v \geq 5$.*

Theorem 3.6. *If there exists a SOLSSOM(v), then there exists a nested $B[4,6;v]$.*

Proof: Let $A = (a_{ij})$ be a SOLS(v) based on $X = \{1,2,\ldots,v\}$ and let $B = (b_{ij})$ be a symmetric orthogonal mate of A. Without loss of generality, we assume that A is idempotent. Let

$$\mathbf{A} = \{\{i,j,a_{ij},a_{ji}\} \mid i \neq j, 1 \leq i,j \leq v\} \text{ and}$$
$$\mathbf{A}_1 = \{\{i,j,a_{ij},a_{ji},b_{ij}\} \mid i \neq j, 1 \leq i,j \leq v\}.$$

Then $(X,\mathbf{A})$ and $(X,\mathbf{A}_1)$ are respectively a $B[4,12;v]$ and a $B[5,20;v]$. So if the $B[4,12;v]$ $(X,\mathbf{A})$ contains a block $\{i,j,a_{ij},a_{ji}\}$ then it also contains the block $\{j,i,a_{ji},a_{ij}\}$. Suppose $\mathbf{B} = \{\{i,j,a_{ij},a_{ji}\} \mid 1 \leq i < j \leq v\}$; then $(X,\mathbf{B})$ is a $B[4,6;v]$. Let $\mathbf{B}_1 = \{\{i,j,a_{ij},a_{ji},b_{ij}) \mid 1 \leq i < j \leq v\}$. Since $b_{ji} = b_{ij}$, we know that $(X,\mathbf{B}_1)$ is a $B[5,10;v]$. Now define a mapping $f{:}\,\mathbf{B} \to X$ as follows:

$$f(B) = b_{ij} \text{ if and only if } B = \{i,j,a_{ij},a_{ji}\} \in \mathbf{B}.$$

Hence, $\{B \cup f(B) \mid B \in \mathbf{B}\} = \mathbf{B}_1$, f is a nesting of $(X,\mathbf{B})$, and $(X,\mathbf{B})$ is a nested $B[4,6;v]$. ∎

Corollary 3.7. *There exists a nested $B[4,3\lambda;v]$ for any odd $v \geq 5$ and any positive even integer λ.*

Proof: By Lemma 3.5 and Theorem 3.6, we have a nested $B[4,6;v]$. Taking each block $\lambda/2$ times, we obtain a nested $B[4,3\lambda;v]$. ∎

Lemma 3.8. *There exists a nested $B[4,3\lambda;v]$ for any even $v \geq 5$ and any positive integer $\lambda \equiv 0 \pmod 4$.*

Proof: Suppose $\lambda = 2\lambda_1$, where $\lambda_1 \equiv 0 \pmod 2$. By Theorem 2.11, we have a nested $B[4,6\lambda_1;v]$. This is the required nested design $B[4,3\lambda;v]$. ∎

Combining Corollaries 3.4, 3.7 and Lemma 3.8, we obtain the main theorem of this section.

Theorem 3.9. *The necessary condition for the existence of a nested $B[4,\alpha;v]$, namely $\alpha = 3\lambda, \lambda(v-1) \equiv 0 \pmod 4$ and $v \geq 5$, is also sufficient.*

Proof: The necessary condition can be divided into the following three cases:

(1) $v \equiv 1 \pmod 4$ and $\lambda \geq 1$;
(2) $v \equiv 3 \pmod 4$ and λ even;
(3) $v \equiv 0 \pmod 2$ and $\lambda \equiv 0 \pmod 4$.

The conclusion follows from Corollaries 3.4, 3.7 and Lemma 3.8. ∎

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