On the edge-graceful (n, kn)-multigraphs conjecture

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Abstract. Lee conjectures that for any $k > 1$, a (n, nk)-multigraph decomposable into k Hamiltonian cycles is edge-graceful if n is odd. We investigate the edge-gracefulness of a special class of regular multigraphs and show that the conjecture is true for this class of multigraphs.

1. Introduction.

The study of edge-graceful simple graphs was initiated in 1985 by Lo [13]. A simple graph $G = (V, E)$ is said to be edge-graceful if there exists a bijection $f: E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $f^{+}: V \to \{0, 1, \ldots, |V| - 1\}$, defined by $f^{+}(v) = \sum\{f(u, v): (u, v) \in E(G)\} \pmod{|V|}$, is a bijection.

The concept of edge-graceful graphs can be viewed as the dual concept of graceful graphs. A graph $G = (V, E)$ is graceful if there exists an injection $g: V(G) \to \{0, 1, \ldots, |E|\}$ such that the induced mapping $g^{*}: E \to \{1, 2, \ldots, |E|\}$, defined by $g^{*}(a, b) = |g(a) - g(b)|$ for all (a, b) in E is a bijection. Graceful graphs were considered by Rosa [14] in the early 60's and popularized by Golomb [3]. For further details on graceful graphs and their applications, we refer the readers to [1], [3], and [4].

In this paper, we extend the concept of edge-gracefulness for simple graphs to multigraphs. All the multigraphs considered herein have no loops. A multigraph G is said to be *regular* if for any u, v in $V(G)$, we have $d_G(u) = d_G(v)$.

The second author has conjectured that for any $k > 1$, a (n, nk)-multigraph decomposable into k Hamiltonian cycles is edge-graceful if n is odd in 1988 Southeastern International Conference on Combinatorics, Graph Theory, and Computing. The conjecture is true for several families of (n, nk)-graphs, such as kth power cycles [9], complete graphs [6], and regular complete k-partite graphs [10].

For arbitrary (n, nk)-multigraphs decomposable into k Hamiltonian cycles, this conjecture appears to be difficult.

We investigate the edge-gracefulness of a class of regular (n, nk)-multigraphs, which are called *step-multigraphs* and which we denote by $S(n; a_1, a_2, \ldots, a_k)$. Here, n is odd, and $a_1, a_2, \ldots, a_k$ are integers such that $1 \leq a_1 \leq a_2 \leq \ldots \leq a_k \leq (n-1)/2$. The vertices of $S(n; a_1, a_2, \ldots, a_k)$ are numbered by $0, 1, \ldots, n-1$, and the edge-set is given by $\{(u,v): u, v \in V(G), u \neq v, v - u \equiv a_i \pmod{n}$ for $i = 1, 2, \ldots, k\}$.

Note that every node i has nodes $i \pm a_1, i \pm a_2, \ldots, i \pm a_k \pmod{n}$ adjacent to it. Thus the step-multigraph $S(n; a_1, a_2, \ldots, a_k)$ is $2k$-regular. Note also that the a_i's may not be distinct. If $a_1, a_2 \ldots, a_k$ are all distinct, then $S(n; a_1, a_2, \ldots, a_k)$ is a simple graph (perhaps disconnected), which is referred to as a *circulant* in [2]. It is well-known that a circulant is connected if and only if $GCD(n, a_1, a_2, \ldots, a_k) = 1$. Figure 1 depicts a disconnected circulant, $S(8; 2, 4)$.

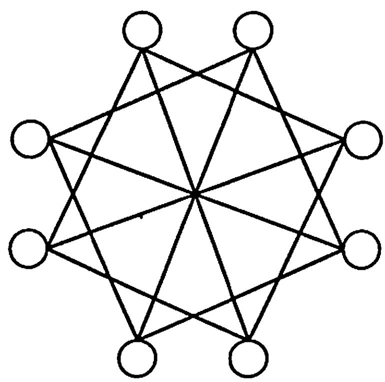

Figure 1

The study of step-mutigraphs $S(n; a_1, a_2, \ldots, a_k)$ is motivated by the fact that this family of graphs contains many other families of graphs as special cases. For examples, $S(n; 1)$ is a cycle on n vertices C_n, and $S(n; 1, 2, \ldots, (n-1)/2)$ is a complete graph K_n when n is odd (see also Section 4). Furthermore, it is easy to see that if $GCD(n, a_i) = 1$ for some i, then the set of n edges, $\{(u,v): u, v \in V(G), u \neq v, v - u \equiv a_i \pmod{n}\}$, forms a Hamiltonian cycle. Thus $S(n; a_1, a_2, \ldots, a_k)$ is decomposable into k Hamiltonian cycles if $GCD(n, a_i) = 1$ for $i = 1, 2, \ldots, k$. Consequently, it would be interesting to see if the conjecture holds here. In fact, the answer is in the affirmative, and we have the following main result.

Theorem 1. *The step-multigraph $S(n; a_1, a_2, \ldots, a_k)$ with n odd is edge-graceful.*

We see immediately from Theorem 1 that the conjecture above is true for step-multigraphs $S(n; a_1, a_2, \ldots, a_k)$ with n odd and decomposable into k Hamiltonian cycles.

2. Preliminaries.

Lo [13] shows that if G is a simple (p, q)-graph with $p = |V(G)|$ and $q = |E(G)|$ and G is edge-graceful, then p divides $q^2 + q - [p(p-1)]/2$. This result can be extended to multigraphs.

Theorem 2. *If G is a (p, q)-multigraph with $p = |V(G)|$ and $q = |E(G)|$ and G is edge-graceful, then p divides $q^2 + q - [p(p-1)]/2$.*

The first author, Li-Min Lee and G. Murty [6] showed that any simple graph G with $p \equiv 2 \pmod 4$ is not edge-graceful. We also have the similar result for multigraphs.

Theorem 3. *A multigraph G with $p \equiv 2 \pmod 4$ is not edge-graceful.*

For other results on edge-graceful graphs, we refer the readers to [5 - 13].

It is easy to see that the step-multigraph $S(n; a_1, a_2, \ldots, a_k)$ with n odd satisfies the necessary condition as stated in Theorem 2.

For an edge (u, v) in the step-multigraph $S(n; a_1, a_2, \ldots, a_k)$ we define the vertex difference of this edge to be $(v - u) \pmod n$, which must be an element of the set $\{a_1, a_2, \ldots, a_k\}$. For a given vertex u, the set of vertex differences for all edges incident with u is a union of two identical copies of $\{a_1, a_2, \ldots, a_k\}$.

We partition the edges of $S(n; a_1, a_2, \ldots, a_k)$ into k classes such that the edges in the ith class, A_i, has the same vertex difference a_i, where $i = 1, 2, \ldots, k$. That is,

$$A_i = \{(u, v) : u, v \in V(G), u \neq v, v - u \equiv a_i \pmod n\}.$$

Note that each class has exactly n edges. Note also that if we have $a_i = a_j$ for $i \neq j$, we still end up with two distinct classes A_i and A_j, although the edges in the two classes are the same. It is not difficult to see that the edges in each of these classes form either a Hamiltonian cycle, a union of disjoint cycles, or a union of disjoint edges.

Since the vertex labels are calculated by using modulo n arithmetic, we may replace the set of edge lablels $\{1, 2, \ldots, nk\}$ by the set $\{1 \bmod n, 2 \bmod n, \ldots, nk \bmod n\}$. That is, we can use k copies of $B = \{0, 1, \ldots, n-1\}$ as the edge labels.

3. Labeling techniques.

In this section, we introduce two labeling techniques, which we use to label the edges of $S(n; a_1, a_2, \ldots, a_k)$. The edges in a class A_i will be labeled by one of the techniques.

Technique 1:

For a class A_i, we label edge $(u, u + a_i)$ by label u, where $u = 0, 1, \ldots, n-1$. Notice that the collection of the edge labels form the set $B = \{0, 1, \ldots, n-1\}$. Now consider any vertex u. It has two incident edges $(u, u + a_i)$ and $(u - a_i, u)$, and thus the sum of the two edge labels is $u + (u - a_i) \pmod{n}$, or $2u - a_i \pmod{n}$.

Technique 2:

For a class A_i, we label edge $(u, u + a_i)$ by label $(n - u - 1)$, where $u = 0, 1, \ldots, n-1$. Note again that the collection of the edge labels form the set B. A vertex u has two incident edges $(u, u + a_i)$ and $(u - a_i, u)$, and thus the sum of the two edge labels is $[n - u - 1] + [n - (u - a_i) - 1]$, or $2n - 2u + a_i - 2 \pmod{n}$, or $-2u + a_i - 2 \pmod{n}$.

Observe that if we label two classes A_i and $A_j (i \neq j)$ by Technique 1 and Technique 2, respectively, the sum of edge labels for all vertices are identical and equal to a constant, $-a_i + a_j - 2 \pmod{n}$.

4. Edge labelings for $S(n; a_1, a_2, \ldots, a_k)$.

To prove Theorem 1, we label the edges of $S(n; a_1, a_2, \ldots, a_k)$ using the two Techniques discussed in the previous section. There are two cases, depending on whether k is odd or even.

(1) When k is odd:
We label each of the first $(k + 1)/2$ classes $(A_1, A_2, \ldots, A_{(k+1)/2})$ by Technique 1, and each of the remaining $(k - 1)/2$ classes $(A_{(k+1)/2+1}, \ldots, A_k)$ by Technique 2. Consider any vertex u. The vertex label of u is given by $[((k + 1)/2)(2u)] + [((k - 1)/2)(-2u)] - f_1 \pmod{n}$, or $2u - f_1 \pmod{n}$, where f_1 is some integer independent of u. It is easy to see that the vertex labels are distinct. (An alternative way of obtaining the vertex labels is to make use of the observation in the previous section. We can form exactly $(k - 1)/2$ pairs of classes such that each pair is labeled by both Technique 1 and Technique 2. The edge labels of these $(k - 1)/2$ pairs result in identical vertex labels. We are then left with one class labeled by Technique 1. Thus for a vertex u, its vertex label must necessarily be of the form $2u - f_1 \pmod{n}$.)

(2) When k is even:
We label each of the first $k/2 + 1$ classes $(A_1, A_2, \ldots, A_{k/2+1})$ by Technique 1, and each of the remaining $k/2 - 1$ classes $(A_{k/2+2}, \ldots, A_k)$ by Technique 2. Again, the vertex label of u is given by $[(k/2 + 1)(2u)] + [(k/2 - 1)(-2u)] - f_2 \pmod{n}$, or $4u - f_2 \pmod{n}$, where f_2 is some integer independent of u. We see immediately that the vertex labels are again distinct.

We have thus proved Theorem 1.

We illustrate this result by the following examples.

Example 1: $n = 5, a_1 = a_2 = a_3 = a_4 = 1$ (see Figure 2).

Example 2: $n = 9, a_1 = 1, a_2 = 1, a_3 = 3$ (see Figure 3).

We note the following corollaries, most of which have been proven elsewhere.

Corollary 4. *The complete graph, $K_n (= S(n; 1, 2, \ldots, (n-1)/2))$, is edge-graceful when n is odd* [6].

Corollary 5. *The kth power cycle, $C_n^k (= S(n; 1, 2, \ldots, k))$, where $1 \leq k \leq (n-1)/2$, is edge-graceful when $k < n/2$ (see* [9, 11, 13]*).*

Corollary 6. *The multicycle kC_n, where there are k multiple edges for each pair of adjacent vertices, is edge-graceful if n is odd.*

Proof: We note that kC_n is $S(n; a_1, a_2, \ldots, a_k)$, where $a_1 = a_2 = \ldots = a_k = 1$. ∎

Corollary 7. *The regular complete k-partite graphs, $K_{n,n,\ldots,n}$, is edge-graceful if both n and k are odd* [10].

Proof: It is not difficult to see that $K_{n,n,\ldots,n}$ can be expressed as the step-multigraph $S(nk; 1, 2, \ldots, k-1, k+1, \ldots, 2k-1, 2k+1, \ldots, jk-1, jk+1, \ldots, (nk-1)/2)$; see also [10]. ∎

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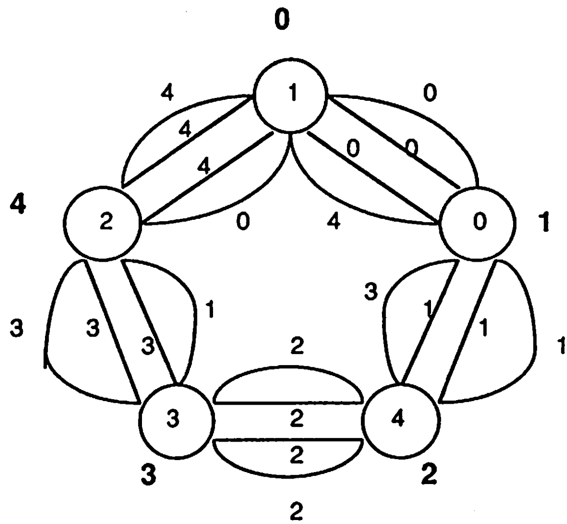

Figure 2. $S(5; 1, 1, 1, 1)$

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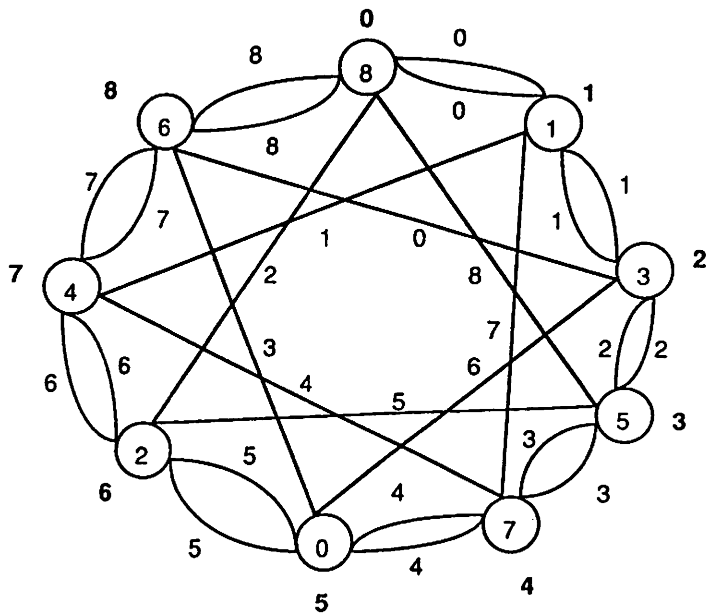

Figure 3. $S(9;1,1,3)$

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