Some Interesting Sextuple Coverings

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1. Introduction.

In a Balanced Incomplete Block Design with parameter set (v,b,r,k,λ), there are v varieties arranged in b blocks of k elements each; each variety occurs r times and each variety pair occurs λ times. According to the Fisher Inequality, b ≥ v; if b = v, we have a Symmetric Balanced Incomplete Block Design.

More generally, in a covering we demand that there be b blocks of k elements and that each pair occur at least λ times; it is also usual to demand that b should be minimal, in which case it is denoted by the symbol Cλ(2,k,v). In this paper we shall mainly be concerned with the case k = 6 and λ = 1.

The analogue of a Symmetric BIBD is a covering in which b = v. Since the covering numbers for small values of k are known, we can easily record such covering designs for k = 3, 4, and 5. For example, the Fano Geometry shows that C(2,3,7) = 7; also, C(2,3,6) = 6. In both cases, the covering design is unique and is found by cycling an initial block (1,2,4), modulo 7 or 6, as the case may be.

For k = 4, C(2,4,13) = 13, C(2,4,12) = 12, and C(2,4,11) = 11. Again, the designs may be obtained by cycling the initial blocks (1,2,4,10), (1,2,4,8), and (1,2,4,9), modulo 13, modulo 12, and modulo 11, respectively.

When we consider the case k = 5, the situation changes slightly. Again C(2,5,21) = 21, because of the finite geometry on 21 points, but (cf. [1]), C(2,5,20) > 20. However, C(2,5,19) = 19 and C(2,5,18) = 18; we merely need to cycle the initial blocks (1,2,4,8,13) and (1,2,4,10,15), modulo 19 and modulo 18, respectively.

In this paper, we shall discuss C(2,6,v) for v in the "b = v" range, that is, for v between 26 and 31. Since there is a finite geometry on 31 points, we have C(2,6,31) = 31; this is the last case in which b = v. For v > 31, the value of C(2,6,v) climbs rapidly. We note that C(2,6,32) = 38 (just add 7 blocks containing element 32, with all other elements, to the finite geometry). Also, C (2,6,33) = 39 (add 8 blocks containg the pair (32,33), combined with all the other 31 elements, to the blocks of the finite geometry).

In looking at the values of C(2,6,v) for v < 31, we shall make use of the method of block weights described in [4] and the concept of geometric points (cf. [2]).

2. Three Solutions and One Question.

One easily sees that C(2,6,27) = 27; one may cycle either (1,2,4,8,13,21) or (1,2,4,8,12,17), modulo 27. It is also easy to obtain a cyclic solution for v = 28 by cycling, modulo 7, on the four initial blocks ($1_a2_a4_a0_b0_c0_d$), where a,b,c,d, are the subscripts 1,2,3,4, and are all different (a takes on the values 1,2,3,4).

We now discuss C(2,6,30), and start by supposing that it is equal to 30.

First we recall the concept of the *excess graph* of a covering. If the pairs of a covering are used to form a graph, then the resulting graph will consist of the complete graph K_n and some extra edges that form an excess graph E. For example, C(2,3,5) = 4 and the covering is given by (1,2,3), (1,4,5), (2,4,5), (3,4,5). The graph of the covering design is then given by K_5 + E, where E consists of three isolated points 1,2,3, and two points 4 and 5 that are joined by an edge of multiplicity 2.

In [4], it was shown that one could associate with each block of a covering a non-negative function called the weight; this weight can be defined in several ways, but we shall use the weight introduced in [4]. It can be calculated as

$$w(B) = (b-1) - \Sigma(r_i-1) + \Sigma f_{ij}(E),$$

where the summation is taken over all elements, or pairs, of the block B and where $f_{ij}(E)$ denotes the multiplicity of the edge (ij) in the excess graph.

Now every element in a covering design on 30 varieties must occur at least 6 times; since there are only 30 blocks, $r = 6$ constantly for all elements in the design. So the weight of any block is given by $29 - 30 + \Sigma f_{ij}(E)$; this proves that any block in the design must contain at least one of the "repeating pairs" that appear in E. However, the number of extra pairs in the covering design is just $30(15) - 30(29)/2 = 15$; each of these pairs occurs twice and consequently there are only 30 "repeating pairs". We have thus shown that exactly one pair in each block is repeated, and thus every block in the design has weight exactly zero.

Now write down the block (1,2,3,4,5,6), and suppose that (1,2) is the repeated pair in this block. The design then consists of this initial block, one block containing the pair (1,2), four more blocks containing 1, four more blocks containing 2, five more blocks containing 3, five more blocks containing 4, five more blocks containing 5, and five more blocks containing 6. Now form a new design by deleting the elements 1,2,3,4,5,6. We are left with only 29 blocks; extend the single quadruple to a quintuple, and we have the result that $C(2,5,24) \leq 29$; but this is not possible, as shown in [3]. Hence we must have $C(2,6,30) > 30$, and therefore $C(2,6,30) = 31$.

I do not know the value of C(2,6,29), although I conjecture that it is greater than 29.

3. The Covering Number C(2,6,26).

The usual Fisher-Yates type of counting argument (merely a variation on the usual $bk = rv$ argument) for the total number of elements in the covering array establishes that

$$6\, C(2,6,26) \geq 26\, C(1,5,25) = 130.$$

If $C(2,6,26) = 22$, which is the lowest possible value satisfying this inequality, we see that there are at least 24 *geometric points* in the covering; these are points that have frequency exactly 5 and so are isolated points in the excess graph. The excess graph thus contains only isolated points with either a single point of valence 10 (impossible, since this forces loops in the graph), or two points of valence 5 each. Both these cases are obviously impossible, but we prefer to rule them out by noting that, in either case, there must be a block in the design made up totally of geometric points. Then this block must have weight $21 - 6(4) + 0$, and this is not possible.

Now suppose that C(2,6,26) = 23. Then the number of geometric points is at least 18. Again, we calculate w(gggggg) = 22 - 6(4), and so there can not be a block consisting only of geometric points. Suppose there is a block with 5 geometric points and a point a of frequency $5+\alpha$. Then the weight of this block is $22 - (24+\alpha)$, and so this possibility is also excluded. Hence we have shown that there are at most 4 geometric points in any block.

Since there are 23 blocks, the number of pairs of geometric points is at most 23(6) = 138. However, there are at least 18 geometric points and so the number of pairs of geometric points is at least 18(17)/2 = 153. This is a contradiction, and so C(2,6,26) > 23.

Finally, we suppose that the value of C(2,6,26) = 24. Then there are at least 12 geometric points. By exactly the same weight argument as in the case of 23 blocks, we find that there can be at most 4 geometric points in any block. Now let there be b_i blocks that contain exactly i geometric points, and suppose that there are exactly N geometric points (we have already pointed out that $N \geq 12$). We have the equations

$$b_0 + b_1 + b_2 + b_3 + b_4 = 24,$$

$$b_1 + 2b_2 + 3b_3 + 4b_4 = 5N,$$

$$b_2 + 3b_3 + 6b_4 = N(N-1)/2.$$

Subtract twice the third equation from thrice the second equation to give

$$3b_1 + 4b_2 + 3b_3 = N(16-N).$$

It follows that $N \leq 16$.

It is interesting to derive the complete set of solutions for C(2,6,26) = 24, and that will be done elsewhere. However, here we merely need the value of C(2,6,26), and so we shall give a solution for N = 16. Clearly, if N = 16, the last equations show that $b_1 = b_2 = b_3 = 0$, $b_4 = 20$, $b_0 = 4$. It is well known that the design made up of 20 quadruples on 16 elements is unique and is given by the 20 blocks of an affine geometry, divided into 5 resolution classes. The four blocks of any resolution classes must all be associated with the same two abnormal elements of the design, and so we can write the 20 blocks as abR_1, cdR_2, efR_3, ghR_4, ijR_5, where we use the notation abR_1 to designate the four blocks based on R_1, namely, (a,b,1,2,3,4), (a,b,5,6,7,8), (a,b,9,10,11,12), (a,b,13,14,15,16); a similar convention holds for the other blocks. The design can be completed to a C(2,6,26) cover if and only if all the missing pairs from the ten abnormal elements can be put into 4 blocks. But this is easily done simply by writing (abcdef), (abghij), (cdghij), (efghij); actually, this is just the C(2,3,5) = 4 covering on the five pairs ab, cd, ef, gh, and ij. Thus we have established that C(2,6,26) = 24.

4. Conclusion.

These results are related to those given in [2] and are also connected with a more extended discussion of small sextuple coverings that will appear elsewhere.

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