# Maximally Nonhamiltonian Graphs

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## 1. Introduction

The problem of characterizing hamiltonian graphs is one of the most challenging and exciting open problems in graph theory. As a consequence an extensive literature exists on the subject. It is well-known that to decide whether a graph is hamiltonian is an $NP$-complete problem. So, it is very likely that there does not exist an "efficient" characterization. Therefore, the investigation of hamiltonian graphs goes in various directions, the problem is tackled from different points of view. As with other problems, the graphs extremal with respect to given property are studied. In the case of hamiltonian graphs, basically two such classes of graphs are taken into account; hypohamiltonian and maximally nonhamiltonian graphs.

A graph $G$ is said to be hypohamiltonian if $G$ is not hamiltonian but removing an arbitrary vertex results in a hamiltonian graph. For instance, the Petersen graph is hypohamiltonian and, in addition, it is the hypohamiltonian graph with the minimum number of vertices. Not too much is known about hypohamiltonian graphs and even the question for which numbers $n$ there is a hypohamiltonian graph on $n$ vertices, was open for a long time. (For more details on this problem see [1]). However, in what follows we will concentrate on maximally nonhamiltonian graphs and hypohamiltonian graphs will be mentioned only marginally.

A graph $G$ is called maximally nonhamiltonian, in short an MNH-graph, if $G$ is nonhamiltonian and becomes hamiltonian with the addition of any new edge. The Petersen graph is an instance also of an MNH graph. Unlike hypohamiltonian graphs, it is easy to see that, for any natural $n \geq 3$, there is an MNH graph on $n$ vertices because each nonhamiltonian graph has a MNH supergraph on the same set of vertices.

In this paper we deal with the question of how sparse and how dense an MNH graph can be. We were attracted to this area by two problems of P. Erdös. They are discussed in section 3. Section 4 is devoted to graphs which are sparse with respect to the number of edges, section 5 to graphs sparse with respect to their clique number. This question arises in connection with another problem of Erdös on dense MNH graphs. A partial solution to this problem is presented.

## 2. Preliminaries

Let us first introduce some further notions and symbols. Unless stated otherwise we make use of the standard terminology of graph theory. Throughout, $G$ denotes a simple graph with vertex set $V(G)$ and edge set $E(G)$, $\bar{G}$ denotes the complement of $G$, and $e(G)$ stands for the number of edges of $G$. The complete graph on $n$ vertices is denoted by $K_n$ and $K_{n_1,n_2}$, denotes the complete bipartite graph.

If $G_1$ and $G_2$ are disjoint graphs we write $G_1 \cup G_2$ and $G_1 + G_2$ to denote their union and join, respectively.

The clique number of a graph is the number of vertices in its largest complete subgraph. We finish this section with a lemma which can be found in [17].

**Lemma.** *Let $n_i > 0, i = 1, \ldots, r+1$ be natural numbers. Then the graph $K_r + (\bigcup_{i=1}^{r+1} K_{n_i})$ is a MNH graph.*

## 3. Two problems of P. Erdös

We start with MNH graphs dense with respect to the number of edges. O. Ore [15] answered the question of how dense MNH graphs can be.

**Theorem 1 [15].** *If $G$ is an MNH graph on $n \geq 3$ vertices, then $e(G) \leq \binom{n}{2} - (n-2)$ and the equality holds only for the graph $(K_1 \cup K_{n-2}) + K_1$ and $3K_1 + K_2$ in the singular exceptional case $n = 5$.*

Denote by $\mathcal{H}_n$ the family of all graphs on $n$ vertices which have nonhamiltonian complement, i.e. if $G$ is a subgraph of the complete graph $K_n$, then $G \in \mathcal{H}_n$ if and only if every Hamiltonian cycle of $K_n$ has a common edge with $G$. Thus, if $G$ is minimal graph in $\mathcal{H}_n$ then its complement is an MNH graph. The two following problems of P. Erdös [9] are, in a way, a generalization of a problem solved by Ore.

- P1. Let $g(n)$ be the maximum number $r$ such that there are $r$ graphs $G_i \in \mathcal{H}_n$, $1 \leq i \leq r$ which can be packed into $K_n$. Determine $g(n)$.
- P2. Determine $f(n, r) = \min \sum_{i=1}^{r} e(G_i)$, where minimum is taken over all $r$-tuples of graphs $G_i \in \mathcal{H}_n$, $1 \leq i \leq r$ which can be packed into $K_n$.

As a direct consequence of Theorem 1 we get, $e(G) \geq n-2$ for $G \in \mathcal{H}_n$ and the equality holds, in the case $n \neq 5$, only for the star $K_{1,n-2}$. As it is possible to pack up to 3 stars $K_{1,n-2}$ into $K_n$, $f(n,r) = r(n-2)$ for $r = 1, 2, 3$ and $n \geq 4$.

For $r \geq 4$ P. Erdös conjectured $f(n,r) < 2^r.n$. As regards the number $g(n)$ he proved $\log_2(n+1) \leq g(n) < c\sqrt{n}$ and believed that the value of $g(n)$ is close to the lower bound. In the rest of this section we present a complete solution to both P1 and P2 given in [12].

Let $n \geq 3.2^{r-3} + 1$ and let $v_1, v_2, \ldots, v_n$ be vertices of the graph $K_n$. We shall define $r$ graphs $G_1, G_2, \ldots, G_r$ by writing down their edge sets:

$$E(G_1) = \{v_1 v_2\} \bigcup \{v_i v_j;\ 4 \leq j \leq n\}$$

$$E(G_2) = \{v_2 v_3\} \bigcup \{v_2 v_j;\ 4 \leq j \leq n\}$$

$$E(G_3) = \{v_3 v_1\} \bigcup \{v_3 v_j;\ 4 \leq j \leq n\}$$

$$E(G_i) = \{v_j v_k;\ 3.2^{i-4} + 1 \leq j \leq 3.2^{i-3}, 3.2^{i-4} + 1 \leq k \leq n,\ j \neq k\}$$
$$\text{for } i = 4, 5, \ldots, r.$$

It is a matter of routine to check that $G_1, G_2, \ldots, G_r$ are $r$ graphs packed into $K_n$. $G_1, G_2$ and $G_3$ are stars $K_{1,n-2}$, $G_i = (K_{a_i} + \bar{K}_{n-2a_i}) \cup \bar{K}_{a_i}$, where $a_i = 3.2^{i-4}$ for $i = 4, \ldots, r$, $a_1 = a_2 = a_3 = 1$. The graphs $\bar{G}_i$, $1 \leq i \leq r$ are well known as MNH graphs (see e.g. [8]), which implies $G_i \in \mathcal{H}_n$. Thus $g(n) \geq 3 + \lfloor \log_2 \frac{n-1}{3} \rfloor$ for $n \geq 4$. This construction was known to several people. In [12] the reverse inequality was proved which has given the answer to P1.

**Theorem 2 [12].** $g(n) = 3 + \lfloor \log_2 \frac{n-1}{3} \rfloor$ *for* $n \geq 4$.

In [12] it is also proved, using Posa's sufficient condition for a graph to be hamiltonian (see [16]), that $\sum_{i=1}^{r} e(G_i) \leq \sum_{i=1}^{r} e(H_i)$ for all $r < g(n)$, where $H_i$ is an $r$-tuple of graphs from $\mathcal{H}_n$ which can be packed into $K_n$. In the case $r = g(n)$ in order to obtain a solution to P2 it is necessary to take, instead of $G_r$ the graph $G'_r$ for $n$ odd, $G''_r$ for $n$ even, where

$$E(G'_r) = \left\{ v_j v_k;\ 3.2^{r-4} + 1 \leq j < k \leq 3.2^{r-4} + \frac{n+1}{2} \right\}$$

$$E(G''_r) = \left\{ v_j v_k;\ 3.2^{r-4} + 1 \leq j < k \leq 3.2^{r-4} + \frac{n+2}{2} \right\} - \{v_{j_0} v_{k_0}\}$$

$$j_0 = 3.2^{r-4} + 1, \quad k_0 = 3.2^{r-4} + 2.$$

Thus to get $r$ graphs which are optimal to P2 it is sufficient to take a set of graph which are optimal to P2 for $r - 1$ and then to add one more graph in "optimal fashion".

Now a routine calculation gives:

**Theorem 3 [12].** *Let* $r \geq 4$, $n \geq 3.2^{r-3} + 1$. *Then* $f(n,r) = w(n,r)$ *for* $r \leq 3 + \log_2 \frac{n+b_n}{9}$, *and* $f(n,r) = w(n, r-1) + c_n$ *for* $3 + \log_2 \frac{n+b_n}{9} < r \leq g(n)$, *where* $w(n,r) = 3.2^{r-4}(2n - 3.2^{r-3} - 1)$, $b_n = 1$, $c_n = \frac{n^2-1}{8}$ *for n odd,* $b_n = 4$, $c_n = \frac{n^2+2n-8}{8}$ *for n even.*

## 4. MNH graphs sparse with respect to the number of edges

A nonhamiltonian graph is an MNH graph if and only if every two non-adjacent vertices are joined by a hamiltonian path. Thus, if we have a sparse graph there are many pairs of nonadjacent vertices, that is, there are many hamiltonian paths but we have only few edges. So, how sparse can MNH graph be? This question was raised by Bollobas [2]. He posed the problem of finding the minimum number of edges, $f(n)$, in an MNH graph on $n$ vertices. For $n \geq 7$, Bondy [3] proved that such a graph with $m$ vertices of degree 2, has at least $\frac{3n+m}{2}$ edges. Therefore, $f(n) \geq \lceil \frac{3n}{2} \rceil$ for $n \geq 7$. So, in order to attain this lower bound, we need a cubic MNH graph for $n$ even and, for $n$ odd, an MNH graph all of whose vertices but one are of degree 3 and one vertex is of degree 4.

The values of $f(n)$ for $n \leq 10$ can be read of in the paper [14] where the list of all MNH graphs on up to 10 vertices is presented. For $n > 10$ an almost complete answer to the question of Bollobas is given in [5], [6] and [7]. If a cubic graph is hamiltonian then it is 3-edge colorable. Therefore snarks, 4-edge chromatic cubic graphs are candidates for smallest MNH graphs. First, Clark and Entringer [5] showed that the famous Isaacs' snarks [13] and small modifications of them are MNH graphs which implied $f(n) = \frac{3}{2}n$ for all even $n \geq 36$ and various small $n$ even. Later, together with Crane and Shapiro [6], [7] they showed, again by suitable modification of Isaacs' snarks, that $f(n) = \frac{3n+1}{2}$ for all odd $n \geq 53$ and various small odd $n$.

Combining results from [5], [6], and [7] $f(n)$ remains undetermined only for $n = 13$–$19$, $23$–$27$, $31$, $33$–$35$, $43$ and $51$.

In [10] it has been shown that sparse MNH graphs are not "rare" by presenting a class $\mathcal{T}_n$ of MNH graphs which are sparse with respect to the number of edges. In fact, for $G \in \mathcal{T}_n$, $e(G) \leq \frac{9}{5}r$, where $r = (n+1)5$, is the number of vertices of $G$. Further, $|\mathcal{T}_n| \to \infty$ for $n \to \infty$. Actually, $|\mathcal{T}_n|$ grows exponentially.

In order to be able to describe the class $\mathcal{T}_n$ we need to introduce a construction of Thomassen [18] which creates new hypohamiltonian graphs from old ones. Let $G_1$ and $G_2$ be graphs and suppose $z_1 \in G_1$, $z_2 \in G_2$ are vertices of degree 3. Let $u_i, v_i, w_i$ are neighbours of $Z_i$ in $G_1$ for $i = 1, 2$. Denote by $G = G_1 \circ G_2$ a graph obtained from $G_1$ and $G_2$ by deleting the vertices $z_1$ and $z_2$ and identifying the pairs of vertices $u_1$ and $u_2$, $v_1$ and $v_2$, and $w_1$ and $w_2$. Thomassen proved

**Theorem 4 [18].** *If $G_1$ and $G_2$ are hypohamiltonian graphs then also $G_1 \circ G_2$ is hypohamiltonian.*

Let $P$ be a Petersen graph and let $\mathcal{T}_n = \{P\}$ and $\mathcal{T}_n = \{G \circ P, \text{where } G \in \mathcal{T}_{n-1}\}$, $n > 1$. Thus, the graphs from $\mathcal{T}_n$ are obtained by pasting together $n$ copies of Petersen graph by Thomassen's construction. In [10] it was proved that in this case (and some other cases as well) the construction preserves not only the property of being hypohamiltonian but also the property of being maximally non-hamiltonian. Thus, each graph of $\mathcal{T}_n$ is an MNH graph.

To show that $|\mathcal{T}_n|$ grows exponentially with $n \to \infty$ it is sufficient to realize that to each graph $G \in \mathcal{T}_n$ we can assign a tree on $n$ vertices that "shows" which pairs of $n$ copies of Petersen graph where pasted together and vice versa to each tree on $n$ vertices with maximum degree four we can assign graph from with the "structure" of the given tree. It is well-known that the number of these trees grows exponentially, so $\mathcal{T}_n$ also does.

## 5. MNH Graphs Sparse with Respect to the Clique Number

In the previous section the sparseness of a graph was measured by the number of its edges. Another possible way to measure the sparseness is by means of its clique number. We were led to this concept by the following problem of Erdös (personal communication) concerning dense MNH graphs.

P3. Let $n \geq 3$, $k \geq 2$ be natural numbers.
Determine $g(k,n) = \max\{e(G); G$ is an MNH graph on $n$ vertices with clique number $k\}$. The weaker version of P3, also posed by Erdös reads as follows: Does there exist an absolute constant $C$ such that $g(k,n) \leq \binom{n}{2} - C(n-k)n$? So, P3 is another generalization of the problem solved by Ore [15]. Thus, for $k = n-1$, by Theorem 1, $g(n-1,n) = \binom{n}{2} - (n-2)$.

Before giving values of $g(k,n)$ for $\frac{n}{2} < k$ we introduce one more notation. Denote by $F(k,n)$ the graph $K_{n-k} + (K_{2k-n} \cup \bar{K}_{n-k})$. This graph plays an important role in the theory of hamiltonian graphs. It is a nonhamiltonian graph of given minimum degree $k$ having maximum possible number of edges (see Erdös [8]). At the same time, the graph is an example which shows that the sufficient conditions, in terms of degrees of vertices, for a graph to be hamiltonian, due to Chvatal [4] cannot be, in a sense, improved.

**Theorem 5 [11].** *For $\frac{n}{2} < k < n$, $g(k,n) = \binom{k}{2} + (n-k)^2$. Moreover, $F(k,n)$ is the unique graph, up to isolated vertices, on which this minimum is attained.*

The graphs are constructible only when $n > k > \frac{n}{2}$. Thus, it is of some interest to know what are extremal graphs like for other values of $k$.

**Theorem 6 [11].** *Let $n \geq 4$. Then $g(n,2n) = \binom{n+2}{2} - 3$ and the unique extremal graph, up to isolated vertices, is $(3K_2 \cup \bar{K}_{n-4}) + K_{n-2}$.*

Denote by $\mathcal{Y}(k,n)$ the family of graphs $G = K_r + \left(\bigcup_{i=1}^{r+1} K_{n_i}\right)$, where $n_1 \geq n_2 \geq \cdots \geq n_{r+1} > 0$, $n = r + \sum_{i=1}^{r+1} n_i$ and $k = r + n_1$. Clearly, clique number of $G \in \mathcal{Y}(k,n)$ equals $k$ and by Lemma $G$ is an MNH graph.

Hence, all extremal graphs from both Theorem 1 and Theorem 2 are from $\mathcal{Y}(k,n)$. This and other evidence leads us to conjecture that $g(k,n) = \min\{e(G), G \in \mathcal{Y}(k,n)\}$ for $k > 2\sqrt{n+1} - 1$. It is easy to show that $\mathcal{Y}(k,n)$ is an empty family for $k \leq 2\sqrt{n+1} - 1$. To our knowledge the values of $g(k,n)$ mentioned above in Theorems 1 and 2 are the only one known so far. Now we proceed to bounds on $g(r,n)$.

The following theorem answers in the affirmative the weaker version of P3.

**Theorem 7 [11].** *For any pair $(k,n)$ from the domain of $g$ there are absolute constants $c_1, c_2$ such that*

$$\binom{n}{2} - c_2(n-k)\cdot n \leq g(k,n) \leq \binom{n}{2} - c_1(n-k)\cdot n.$$

*Moreover, one can take $c_1 = \frac{1}{4}$, $c_2 = 1$ and these constants cannot be improved.*

So, P3 includes in itself the question for which values $(k, n)$ there is an MNH graph on $n$ vertices with clique number $k$. In particular, how sparse can an MNH graph be with respect to its clique number?

For $k > 2\sqrt{n+1} - 1$, $(k, n)$ belongs to the domain of $G$ because $\mathcal{Y}(k, n) \neq \emptyset$. For almost all values $n$, and $k \leq 2\sqrt{n+1} - 1$ an MNM graph on $n$ vertices with clique number $k$ can be made up by pasting together copies of the Petersen graph together with a copy of Isaacs' snark by the construction of Thomassen and then replacing a vertex of degree three by a suitable complete graph (for more details see [10], and [11]). Combining these results we get

**Theorem 8 [11].** *For any $n \geq 60$, $2 \leq k \leq n-1$, there exists an MNH graph on $n$ vertices with clique number $k$.*

The sparsest graphs from this point of view are those of clique number 2, that is, triangle free graphs. If we want to classify graphs of clique number 2 more finely, probably the most natural way would be to do it by means of girth of graph, the length of the shortest cycle. Now the question is how sparse an MNH graph can be with respect to its girth. In [10] the following question is posed.

**Problem:** Is it true that for each natural number $n$ there exists an MNH graph of girth at least $n$?

There are infinitely many MNH graphs of girth 5, because any graph of $\mathcal{T}_n$ has the property. Isaacs' snarks $I_r$, $r \geq 7$ form an infinite class of MNH graphs of girth 6. Coxeter graph on 28 vertices is an MNH graph of girth 7 and by applying Thomassen construction to this graph we get an infinite class of MNH graphs of girth 7. As far as we know no MNH graph of girth $> 7$ has been found yet.

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