A note on the amida number of a regular graph

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Abstract. For any integers r and n, $2 < r < n-1$, it is proved that there exists an order n regular graph of degree r whose amida number is $r+1$.

We use the terminology that a graph has neither loops nor multiple edges. We consider only finite graphs. A connected graph G is called an *amida* graph of type n, as defined in [1], if there exist distinct vertices s and t, a matching M (a set of edges with no vertices in common), and a collection of n distinct paths between s and t such that: (i) the edges of each path are alternately in and out of M, (ii) if an edge is in two of the paths, then it is in M, and (iii) an edge in M is in at most two of the n paths. The paths are called *amida* paths. A graph is said to be *n-amida* if it is an amida graph of type n. The *amida number* of a graph G is the largest n for which G is n-amida and is denoted by $am(G)$.

The amida numbers for some families of graphs have been determined in [1]. For example, the amida number of all cycles is 2 and the amida number of the complete graph K_n is n when n is even and at least 4, and is $n-1$ when n is odd and at least 3. Since cycles and complete graphs are extreme cases of regular graphs, the known results suggest that a graph of odd order which is regular of odd degree r has amida number at most r. In fact, this is claimed as remark 2 in [1]. However, this claim is not true as the following result indicates.

Theorem. *There is a connected r-regular graph on n vertices with amida number $r+1$ whenever n is even and $2 < r < n-1$ or n is odd and $2 < r \leq n-2$.*

Proof: When $r = 3$, the graph of Figure 1 works for all $n \geq 4$. The dark edges are the members of M and the four amida paths are obvious. When $n = 2m = 6$, then v_1 is adjacent to v_3 and u_1 is adjacent to u_3. When $n = 2m = 4$, then the edge v_2u_2 is not in M, and v_1 is adjacent to u_2 and v_2 is adjacent to u_1.

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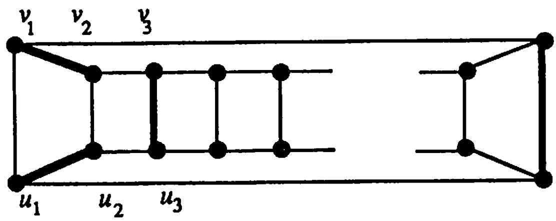

Figure 1

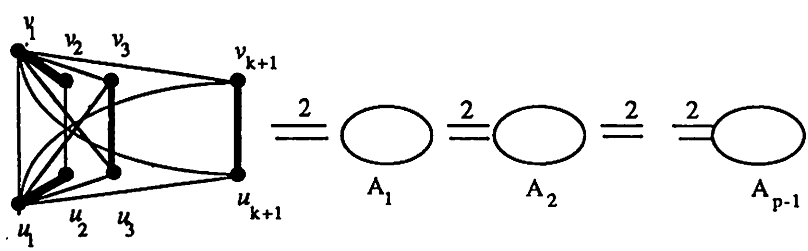

Figure 2

Now let $r = 2k + 1$ be odd, $k \geq 2$, and let $n = 2 + p(2k) + 2d, p \geq 1$ and $0 \leq 2d < 2k$. Consider the graph H shown in Figure 2. Let $A_1, A_2, \ldots, A_{p-1}$ denote the last $p-1$ groupings of vertices each containing $2k$ vertices. We shall add $2d$ vertices at the end of the discussion. The dark edges amongst $v_1, v_2, \ldots, v_{k+1}, u_1, u_2, \ldots, u_{k+1}$ are the elements of M. These edges together with the other edges incident with v_1 and u_1 give us $2k + 2$ amida paths. We now add edges to H to produce a graph G that is regular of degree r.

Let A_0 denote the vertices $v_2, v_3, \ldots, v_{k+1}, u_2, u_3, \ldots, u_{k+1}$. It is well known that the complete graph K_{2t} has a 1-factorization for every positive integer t. Each double marked line between the groupings $A_0, A_1, \ldots, A_{p-1}$ represents two perfect matchings between the two corresponding groupings. Add $2k - 4$ 1-factors inside A_0, add $2k - 3$ 1-factors inside each of $A_1, \ldots, A_{p-2}$, and add $2k - 1$ 1-factors inside A_{p-1}. Call the resulting graph G. Clearly, it has amida number $2r + 2$, is regular of degree r, and is connected.

We now add $2d$ vertices $w_1, w_2, \ldots, w_{2d}$ to G as follows. Take one of the 1-factors in A_{p-1} and replace each edge xy of the 1-factor by the edges w_1x and w_2y. This leaves the degrees of the vertices of A_{p-1} unaltered and makes the degrees of w_1 and w_2 equal to k. Now do the same for another 1-factor in A_{p-1} and add the edge w_1w_2. The degrees of both w_1 and w_2 are now r. Repeat this for the remaining pairs of vertices w_3 and w_4, and so on. There are enough 1-factors in A_{p-1} because $2d < 2k$. The resulting graph G' works and is connected.

Now suppose $r = 2k$ is even, $k \geq 2$, and n is even, $n \geq 2k + 2$. Let G be the graph described above for $r + 1 = 2k + 1$ and the same n. Now delete the edge v_1u_1, delete a perfect matching between A_0 and A_1, delete a perfect matching from inside each of $A_2, A_3, \ldots, A_{p-1}$, and delete each of the edges $w_{2i-1}w_{2i}$. The resulting graph G'' has amida number $r + 1$, is regular of degree r and is still connected.

This leaves us only with the case of $r = 2k$ and n odd. By the hypotheses of the theorem, we know $n \geq r + 2$. So start with the graph G'' of the preceding paragraph with $n-1$ vertices and regular of degree r. All we need to do is add an additional vertex preserving the degree. There is still a perfect matching between A_0 and A_1. Add a new vertex z by replacing each edge xy of the perfect matching by the 2-path xzy. The resulting graph has the desired properties.

The preceding argument depends on p being at least 2, but only a small adjustment is needed in the case that n falls between $2k + 2$ and $4k$. Namely, at the part of the proof where $2k - 4$ 1-factors are added to A_0 to produce G, add $2k - 2$ 1-factors instead. Then the $2d$ vertices are added to G to produce G' using 1-factors in A_0. If $2d < 2k - 2$, there are enough 1-factors in A_0 to achieve G'' and the graph that arises from it. This leaves the case that $2d = 2k - 2$. If $2d \geq 4$, obtain G'' from G' by deleting a matching between the $2d$ vertices and $v_3, v_4, \ldots, v_{k+1}, u_3, u_4, \ldots, u_{k+1}$, the edge v_1u_1, and an edge from each of v_2 and u_2 to a vertex x and y of the $2d$ vertices, where x is not adjacent to y. Then add an edge joining x and y. To get the last graph, adjoin a new vertex and use the 1-factor in the last $2d$ vertices together with edges from v_2 and u_2 to non-adjacent vertices amongst the $2d$ vertices as above. Finally, when $2d = 2k - 2 = 2$, using a modification of the graph of Figure 2, it is easy to find graphs on eight and nine vertices, respectively, regular of degree 4 and with amida number equal to 5. ∎

Let $f(n, r)$ denote the maximum amida number over all graphs on n vertices which are regular of degree r. We can combine the known results and Theorem 1 to determine $f(n, r)$. We have

$$f(n,r) = \begin{cases} r & \text{if } r=1,\ r=2, \text{ or } r=n-1 \text{ when } r \text{ is even} \\ r+1 & \text{otherwise.} \end{cases}$$

References

1. L. M. Orton and R. D. Ringeisen, *The amida number of a graph*, Congressus Num. 44 (1984), 315–320.