The Embeddings of $S_3(2,4,V)$

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1. Introduction.

A *pairwise balanced design* (or PBD) of index λ, denoted by $B[K,\lambda;v]$, is a pair $(X,\mathcal{A})$ where X is a set of v elements (called points) and $\mathcal{A}$ is a collection of subsets (called blocks) of X, such that every unordered pair of points is contained in exactly λ blocks of $\mathcal{A}$ and every block in $\mathcal{A}$ has its size in K. A PBD $B[\{k\},\lambda;v]$ is also denoted (v,k,λ)-BIBD or $S_\lambda(2,k,v)$. It is well known that a $S_\lambda(2,4,v)$ exists if and only if

$$\lambda(v-1)\equiv 0 \pmod 3 \text{ and } \lambda v(v-1)\equiv 0 \pmod{12},$$

or equivalently,

$$\begin{gathered}\lambda\equiv 1,5 \pmod 6 \text{ and } v\equiv 1,4 \pmod{12}\\ \lambda\equiv 2,4 \pmod 6 \text{ and } v\equiv 1 \pmod 3\\ \lambda\equiv 3 \pmod 6 \text{ and } v\equiv 0,1 \pmod 4 \text{ and}\\ \lambda\equiv 0 \pmod 6 \text{ and } v\geq 4.\end{gathered}$$

If $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ are two PBDs such that $X\subseteq Y$ and $\mathcal{A}\subseteq\mathcal{B}$, we say that $(X,\mathcal{A})$ is embedded in $(Y,\mathcal{B})$ and that $(Y,\mathcal{B})$ contains $(X,\mathcal{A})$ as a *subdesign*. If the $(X,\mathcal{A})$ is missing from $(Y,\mathcal{B})$, then we denote this design by $(Y,X,\mathcal{B}\backslash\mathcal{A})$ or $(v,u;\lambda;K)$-IPBD where $|Y|=v$, $|X|=u$ and the set of the block sizes is K. We also say that the PBD has a hole of size u. In fact, the missing subdesign need not exist. The necessary condition for a $S_\lambda(2,4,v)$ to contain a $S_\lambda(2,4,u)$ as a subdesign is that $v\geq 3u+1$. The problem which has attracted much interest in recent years (that is, [2], [8], [9], [10], [17], [18]) is that of determining whether the necessary condition is also sufficient. The embeddings of $S_\lambda(2,4,u)$ into $S_\lambda(2,4,v)$ of index $\lambda=1$ and $\lambda=2$ are completely solved. The following two theorems have been obtained.

Theorem 1.1. *([10]) Suppose* $v\equiv 1$ *or* $4 \pmod{12}$ *and* $u\equiv 1$ *or* $4 \pmod{12}$. *Then there exists a* $(v,4,1)$*-BIBD containing a* $(u,4,1)$*-BIBD as a subdesign if and only if* $v\geq 3u+1$.

Theorem 1.2. *([8]) Suppose* $v\equiv u\equiv 1 \pmod 3$. *Then there exists a* $S_2(2,4,u)$ *as a subdesign if and only if* $v\geq 3u+1$.

In this paper we shall consider the case $\lambda=3$. We shall prove the following theorem.

[1] Research supported by NSFC under Grant 1880451.

Theorem 1.3. *Suppose* $v \equiv 0$ *or* $1 \pmod 4$ *and* $u \equiv 0$ *or* $1 \pmod 4$. *Then there exists a* $S_3(2,4,v)$ *containing a* $S_3(2,4,u)$ *as a subdesign if and only if* $v \geq 3u+1$.

Let $u \equiv 0$ or $1 \pmod 4$, $v \equiv 0$ or $1 \pmod 4$ and $v \geq 3u+1$. Then we say that the ordered pair (v, u) is admissible. Let

$$A_4 = \{(v,u): (v,u) \text{ is admissible}\} \text{ and}$$
$$E_4 = \{(v,u): \text{ there is a } S_3(2,4,v) \text{ containing a } S_3(2,4,u)\}.$$

Then Theorem 1.3 says that $A_4 = E_4$.

The following two lemmas which were proved in [17] are useful in this paper.

Lemma 1.4. *Let* $K_4 = \{4,5,8,9,12\}$. *Suppose* $(v,u) \in A_4$ *and* $v \geq 4u-3$. *Then for* $u \geq 28$, *any* $B[K_4, 1; u]$ *can be embedded in some* $B[K_4, 1; v]$.

Lemma 1.5. *Let* $(v,u) \in A_4$. *If* $4u-3 \leq v \leq 5u$, *then any* $B[K_4, 1; u]$ *can be embedded in some* $B[K_4, 1; v]$.

By virtue of the above lemmas, there are two cases needed to be treated. One case is that $u > 28$ and $3u+1 \leq v \leq 4u-3$, and the other is that $u \leq 28$ and $3u+1 \leq v \leq 4u-3$, or $v > 5u$.

2. Preliminaries.

A *group divisible design* (GDD) denoted by GD$[K, \lambda, M; v]$ is a triple $(X, \mathcal{G}, \mathcal{A})$ where X is a v-set, $\mathcal{G}$ and $\mathcal{A}$ are collections of some subsets of X (called groups and blocks respectively) such that

(1) $|G| \in M$ for every $G \in \mathcal{G}$;
(2) $|B| \in K$ for every $B \in \mathcal{A}$;
(3) $|G \cap B| \leq 1$ for every $G \in \mathcal{G}$ and every $B \in \mathcal{A}$; and
(4) every pairset $\{x, y\}$, where x and y belong to distinct groups, is contained in exactly λ blocks of $\mathcal{A}$.

The *group type* of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G|: G \in \mathcal{G}\}$ and denoted by $1^i 2^j 3^k \ldots$, which means that in the multiset there are i occurrences of 1, j occurrences of 2, etc. A set of blocks is called a parallel class if the blocks partition X. When $\mathcal{A}$ can be partitioned into parallel classes, the GDD is called *resolvable*. For ease of notation we sometimes write a GD$[K, \lambda, M, v]$ as a GD$[K, \lambda]$ together with its group type or just as a GD$[K, \lambda]$.

A *sub*-GDD $(Y, \mathcal{G}', \mathcal{A}')$ of a GDD $(X, \mathcal{G}, \mathcal{A})$ is a GDD whose points and blocks are respectively points and blocks of the GDD $(X, \mathcal{G}, \mathcal{A})$ and whose every group is contained in some group of the latter. If the sub-GDD is missing, then it is called an *incomplete* GDD, or IGDD, and denoted by IGDD $(X, Y, \mathcal{G}, \mathcal{A} \backslash \mathcal{A}')$. Sometimes we denote it by an IGD$[K, \lambda]$ when $|A| \in K$ for every $A \in \mathcal{A} \backslash \mathcal{A}'$, and define its group type to be the multiset of ordered pairs $\{|G|, |G\cap Y| : G \in \mathcal{G}\}$. Also the missing sub-GDD need not exist.

A *transversal design* $T(k, v)$ is a GD$[\{k\}, 1, \{v\}; kv]$. It is well-known that the existence of a $T(k, v)$ is equivalent to the existence of $k-2$ mutually orthogonal Latin squares (MOLS) of order v. For the existence of $T(k, v)$ the reader is referred to [1] and [20]. A design which is obtained by deleting all blocks of a $T(k, u)$ from a $T(k, v)$ is called an *incomplete array* and denoted by $IA_{k-2}(v, u)$.

Now we generalize the concept of IPBD. If a $B[K, \lambda]$ has m holes, we denote it by $(X; Y_1, Y_2, \dots, Y_m, A)$ -$\Diamond_m$-IPBD where X is the point set of the design, Y_i is the point set of the ith hole of the design, $1 \le i \le m$, and A is the set of blocks. In this design, every pair of points $\{x, y\}$ occurs in λ blocks unless $\{x, y\} \subset Y_i$ for some i, $1 \le i \le m$, in which case the pair occurs in no block. When $\lambda = 1$, $m = 2$, $|X| = v$, $|Y_1| = w_1$, $|Y_2| = w_2$ and $|Y_1 \cap Y_2| = w_3$ it is denoted by $(v; w_1, w_2; w_3; K)$-$\Diamond$-IPBD. This notation is first used by Rees and Stinson [9].

The following known results will be used in this paper.

Lemma 2.1. *([10]) Let $v \equiv 7$ or $10 \pmod{12}$ and $u \equiv 7$ or $10 \pmod{12}$. Then there exists a $(v, u; \{4\})$-IPBD with index $\lambda = 1$ if and only if $v \ge 3u + 1$.*

The next lemma comes from Theorem 2.2 in [20].

Lemma 2.2. *If $0 \le a \le n$ and there exist $T(5, m)$, $T(5, m+1)$ and $T(6, n)$, then there exists an $IA_3(mn+a, a)$. If we further have $T(5, a)$, then there exist an $IA_3(mn+a, m+a)$ and an $IA_3(mn+a, m)$ provided $a < n$.*

A *resolvable* BIBD, denoted by (v, k, λ)-RBIBD, is a (v, k, λ)-BIBD such that its block set can be partitioned into parallel classes. The existence of $(v, 4, \lambda)$-RBIBD is determined in [4] and [12].

Lemma 2.3. *([4]) A $(v, 4, 1)$-RBIBD exists if and only if $v \equiv 4 \pmod{12}$.*

Lemma 2.4. *([12]) A $(v, 4, 3)$-RBIBD exists if and only if $v \equiv 0 \pmod{4}$.*

The proof of the following lemma is similar to that of Theorem 4.2 in [17], so it is omitted here.

Lemma 2.5. *Let $u \equiv 0 \pmod{4}$, $v \equiv 1$ or $4 \pmod{12}$ and $3u + 1 \le v \le 4u$. Then $(v, u) \in E_4$.*

Now we give the main construction of this paper. This is a generalization of the construction in, for example, [14], [9], [10], and [18].

Construction 2.6. *Suppose $(X, \mathcal{G}, \mathcal{A})$ is a GD$[K, \lambda]$ which has two missing sub-GDDs $(Y_1, \mathcal{G}_1, \mathcal{B}_1)$ and $(Y_2, \mathcal{G}_2, \mathcal{B}_2)$. For all $G_i \in \mathcal{G}$, $1 \le i \le n$, denote $G_i \cap Y_1$ and $G_i \cap Y_2$ by G_i' and G_i'' respectively. Let S_1 be a point set disjoint from*

X and $S_3 \subseteq S_2 \subseteq S_1$. Suppose the following designs exist (all these designs have their block sizes in K and index λ):

(1) *a $(G_i \cup S_1; S_1, G_i' \cup S_2, G_i'' \cup S_3; \mathcal{A}_i)$-$\diamondsuit_3$-IPBD, for $1 \le i \le n-1$;*
(2) *a $(G_n \cup S_1; G_n' \cup S_2, G_n'' \cup S_3\ ; \mathcal{A}_n)$-$\diamondsuit$-IPBD; and*
(3) *a $(Y_1 \cup S_2, (Y_1 \cap Y_2) \cup S_3, \mathcal{A}_y)$-IPBD.*

Then there is a IPBD $\left(X \cup S_1, Y_2 \cup S_3, (\mathcal{A}\backslash(\mathcal{B}_1 \cup \mathcal{B}_2)) \cup (\cup_{i=1}^{n} \mathcal{A}_i) \cup \mathcal{A}_y\right)$ which has block sizes in K and index λ.

When $Y_2 = \phi$ in the above construction, we obtain the following construction which was first presented by Stinson (see [14]).

Construction 2.7. *Suppose there is an IGD[K, λ] of type $\{(t_1, u_1), (t_2, u_2), \ldots, (t_n, u_n)\}$ and let $b \ge a \ge 0$. Suppose the following designs exist:*

(1) *a $(t_i + b; u_i + a, b; a; K)$-$\diamondsuit$-IPBD, with index λ, for $1 \le i \le n-1$, and*
(2) *a $(t_n + b, u_n + a; K)$-IPBD with index λ.*

Then there exists a $(t + b, u + a; K)$-IPBD with index λ, where $t = \Sigma t_i$ and $u = \Sigma u_i$.

To obtain a GDD with two missing sub-GDDs, we shall use the following construction. For simplicity, we only state the special case instead of the general form. To give a point x weight $t(x)$ means that the point x is replaced by the set $\{x\} \times I_{t(x)}$ where $I_{t(x)} = \{1, 2, \ldots, t(x)\}$.

Construction 2.8. *Suppose $(X, Y, \mathcal{G}, \mathcal{A})$ is an IGDD of index unity and $W \subseteq X$. Let $t: X \to Z^+ \cup \{0\}$ be a function (that is, to give the point x weight $t(x)$). Suppose for every $A \in \mathcal{A}$ there is an IGD[K, λ] with group type $\{t(x): x \in A\}$ such that its groups are $\{\{x\} \times I_{t(x)}: x \in A\}$ and the missing sub-GDD has its groups $\{\{x\} \times \{1\}: x \in A \cap W\}$. Then there is a GD[$K, \lambda$] constructed on $\cup_{x \in X}\{x\} \times I_{t(x)}$ which has two missing sub-GDDs constructed on $\cup_{x \in W} \{x\} \times \{1\}$ and $\cup_{x \in Y}\{x\} \times I_{t(x)}$ respectively.*

The main recursive constructions of this paper are the following Lemmas.

Lemma 2.9. *Let $(v, u) \in A_4$ and $u = 4s + t_1 + t_2$, where $t_1 = 0$ or 1, $1 < t_2 \le s$ and $s + t_1 \equiv 0$ or $1 \pmod 4$. Suppose there exist $T(5, s)$ and $(v - 12s, t_1 + t_2; 3; \{4\})$-IPBD. Then $(v, u) \in E_4$ for $3u + 1 \le v \le 3u + 4s - 2(t_1 + t_2)$.*

Proof: Make use of Construction 2.8 by starting from a $T(5, s)$ instead of the IGDD. Denote the groups of the $T(5, s)$ by G_i, $1 \le i \le 5$. Let U be a set of order t_2 which is contained in G_5. Let set W consist of all points of U and all points of G_i, $1 \le i \le 4$. Give the points of U weight 3 or 4, the points of $G_5 \backslash U$ weight 0 or 3 and the others weight 3. Fill in any block which contains a point of U with a GD[$\{4\}$, 3] of type $3^4\ 4^1$ (which is obtained by deleting four points from a block in a $T(5, 4)$) or type 3^5 (which is obtained by giving weight 3 to every point of a $S_3(2, 4, 5)$). Fill in other blocks a GD[$\{4\}$, 3] of type 3^5 (which is obtained by deleting 1 point of a $S(2,4,16)$) or type 3^4 (which is obtained by deleting 1 point of a $S(2,4,13)$). Thus we obtain a GD[$\{4\}$, 3] of type $(3s)^4 n^1$ which contains a sub-GDD of type $s^4(t_2)^1$. As $t_2 > 1$, here n can be any positive integer between $3t_2$ and $3s + t_2$. Now we use Construction 2.7 by starting from this IGDD. Let $a = t_1$. Let $b = 1$ or s when $t_1 = 0$, or $b = 4$ or $s+1$ when $t_1 = 1$. The required $(3s + b; s + a, b; a; \{4\})$-◇-IPBD can be easily obtained. Then we obtain a $(12s + n + b, u; 3; \{4\})$-IPBD. So the conclusion follows by selecting suitable values of n and b. ∎

The next lemma which is essentially Lemma 3.1 in [2] is also a corollary of Construction 2.7.

Lemma 2.10. *Suppose there exists an $IA_3(m+n,n)$. Suppose also there exist a $(m+n+a, n+a; 3; \{4\})$-IPBD and a $(m'+n'+a, n'+a; 3; \{4\})$-IPBD where $0 \le m' \le m$, $0 \le n' \le n$ and $a \ge 0$. Then there exists a $(v,u;3;\{4\})$-IPBD where $u = 4n + n' + a$ and $v = 4m + u + m'$.*

From Construction 2.7 we easily obtain the following:

Lemma 2.11. *Suppose there exist a $T(k,m)$, a $(m+n,n;3;\{4\})$-IPBD and a $B[\{4\},3;m'+n]$ where $k = 5$ or 9, $n \ge 0$ and $0 \le m' \le m$. Then there exists a $(v, m'+n; 3; \{4\})$-IPBD where $v = (k-1)m + m' + n$. If there also exists a $(m'+n, n; 3; \{4\})$-IPBD, then there exists a $(v, m+n; 3; \{4\})$-IPBD.*

Note that when $m' = 0$ in the above lemma, the existence of a $T(k,m)$ can be weakened to that of a $T(k-1,m)$.

Lemma 2.12. *Let $(v,u) \in A_4$. Suppose that $u \equiv 0 \pmod 4$, $v \ge 3.5u$ and $v - u \equiv 0$ or $5 \pmod{20}$ or that $u \equiv 1 \mod 4$, $v \ge 3.5u - 2.5$ and $v - u \equiv 0$ or $15 \pmod{20}$. If there exists a $T(6,(v-u)/5)$ then $(v,u) \in E_4$.*

Proof: When $u \equiv 1 \pmod 4$, delete some points from one group of a $T(6,(v-u)/5)$ such that this group is of size $(u-1)/2$. Give weight 2 to every point of this group and weight 1 to others. Now for every block of size 5, fill in it with a $S_3(2,4,5)$. For every block of size 6, fill in it with a GD[$\{4\}$, 3] of type $2^1\,1^5$ which is displayed below.

points: $Z_5 \cup \{\infty_1, \infty_2\}$
groups: $\{\infty_1, \infty_2\} \cup \{i\}$, $i = 0,1,2,3,4$
blocks: $\{\infty_1, 0, 1, 2\}$ $\{\infty_2, 0, 1, 3\}$ mod 5.

Finally add one new point to every group and fill in the groups with $S_3(2,4,(v-u)/5+1)$ or $S_3(2,4,u)$ to obtain the required design.

When $u \equiv 0 \pmod 4$, the proof is similar and the details are omitted here. ∎

Lemma 2.13. $\{(61,8), (32,9), (56,9), (65,12), (41,13), (44,13), (45,13), (53,16), (57,16), (60,17), (93,28), (96,28), (89,29), (93,29), (96,29)\} \subset E_4$.

Proof: These designs are listed in Appendix. Two of which are given by L. Zhu and H. Shen, and we put * and ** on them respectively. ∎

3. The embeddings of $S_3(2,4,u)$ for $u \equiv 0,1,4,5,8$ and $9 \pmod{16}$.

First we discuss the embeddings of $S_3(2,4,4)$ and $S_3(2,4,5)$. It is easy to know that if there exists a $(v,u;1;K_4)$-IPBD then there exists a $(v,u;3;\{4\})$-IPBD.

Lemma 3.1. *([2]) Let $v \equiv 0$ or 1 (mod 4). A $B[K_4,1;v]$ containing a block of size 4 or 5 exists whenever $v \geq 13$ or 17 respectively.*

The proof of the following lemma is simple and it is omitted.

Lemma 3.2. *Let $u \equiv 0$ or $1 \pmod 4$. Then $(3u+1,u) \in E_4$.*

Proposition 3.3. *$(v,4)$ or $(v,5) \in E_4$ if and only if $(v,4)$ or $(v,5) \in A_4$ respectively.*

Proof: The conclusion comes from Lemma 3.1 and 3.2 immediately. ∎

Lemma 3.4. *Let d be a non-negative integer, $d \equiv 0$ or $1 \pmod 4$. If $(u,d) \in E_4$ for any $(u,d) \in A_4$, then $(v,u) \in E_4$ where $(v,u) \in A_4$, $u \geq 3d+1$ and $4u-d+1 \leq v \leq 5u-4d$ or $v = 4u-3d$.*

Proof: When $(u,d) \neq (4,1)$ apply Lemma 2.11 by letting $k=5$, $m=u-d$ and $2d+1 \leq m' \leq u-d$ or $m'=0$ and $n=d$. ∎

Proposition 3.5. *Let $u \equiv 0,1,4,5,8$ or $9 \pmod{16}$ and $u \geq 20$. If $(v,u) \in A_4$ and $3u+1 \leq v \leq 4u-12$, then $(v,u) \in E_4$.*

Proof: Make use of Lemma 2.9 by taking s, t_1 and t_2 as following:

u	s	t_1	t_2
0, 4 (mod 16)	$(u-4)/4$	1	3
1, 5 (mod 16)	$(u-5)/4$	1	4
8 (mod 16)	$(u-4)/4$	0	4
9 (mod 16)	$(u-5)/4$	0	5.

Let $d = 4$ in Lemma 3.4. This shows $(4u-12,u) \in E_4$ for $u \geq 13$ and completes the proof. ∎

Now we shall use Lemma 3.4 to consider the case $4u-11 \leq v \leq 4u-4$. To do this, we need the embeddings of $S_3(2,4,u)$ for $u \in \{8,9,12\}$.

The method used in the proof of the next lemma is from [6].

Lemma 3.6. *Suppose there exist* $(w+t_1, t_1; 1; \{4\})$*-IPBD and* $(w+t_2, t_2; 2; \{4\})$*-IPBD, where* $t_1 > t_2$ *and* $t_1 - t_2 \equiv 0 \pmod 3$. *Then there exists a* $(w + t_2 + (t_1 - t_2)/3, t_2 + (t_1 - t_2)/3; 3; \{4\})$*-IPBD.*

Proof: Let Y, X and W be point sets such that $|Y| = w$, $|X| = t_1$, $|W| = t_2$ and $W \subset X$. Construct a $(w + t_1, t_1; 1; \{4\})$-IPBD $(Y \cup X, X, \mathcal{A}_1)$ and a $(w + t_2, t_2; 2; \{4\})$-IPBD $(Y \cup W, W, \mathcal{A}_2)$. Let U be a new point set of order $(t_1 - t_2)/3$. Now partition the points of $X \backslash W$ into triples and substitute every point of a triple by a point of U. At the same time, $\mathcal{A}_1$ becomes $\mathcal{A}_1'$ accordingly. It is easy to now that $(Y \cup W \cup U, W \cup U, \mathcal{A}_1' \cup \mathcal{A}_2)$ is the required design. ∎

Corollary 3.7. *There exists a* $(11, 2; 3; \{4\})$*-IPBD.*

Proof: There exist $S(2,4,13)$ and $S_2(2,4,10)$, equivalently, $(13,4;1;\{4\})$-IPBD and $(10,1;2;\{4\})$-IPBD. The conclusion follows from Lemma 3.6. ∎

Lemma 3.8. *If* $25 < v < 76$ *and* $(v,8) \in A_4$, *then* $(v,8) \in E_4$.

Proof: There exist three incomplete MOLS of order 10 which have 5 disjoint holes of order 2 (for the details see [7]). From this we obtain an incomplete $T(5,10)$ which has 5 holes of order 2. Delete 9 points of one group of this TD and fill in other groups with GD[$\{4\}$, 3] of type 2^5 (see [3]). Fill in blocks with $S_3(2,4,5)$ or $S_3(2,4,4)$. Fill in holes with four $S_3(2,4,8)$ and one $S_3(2,4,9)$. Thus $(41,8) \in E_4$. For $v = 45$ or 49, make use of Lemma 2.10 by taking $m = 9$, $n = 2$, $n' = a = 0$ and $m' = 1$ or 5. The required IPBD comes from Corollary 3.7 and $IA_3(11,2)$ comes from [16]. For $v = 48, 53, 68$ and 73, the conclusion follows from Lemma 2.12. For $v = 28$ and 61, see Lemma 2.5 and Lemma 2.13. For the other values of v, use Lemma 2.11 by letting $v = (k-1)m + m' + n$ as following:

$44 = 4 \times 9 + 8 + 0$ $\quad 52 = 4 \times 11 + 7 + 1$ $\quad 56 = 4 \times 12 + 7 + 1$ $\quad 57 = 8 \times 7 + 0 + 1$

$60 = 4 \times 13 + 8 + 0$ $\quad 64 = 8 \times 8 + 0 + 0$ $\quad 65 = 8 \times 8 + 1 + 0$ $\quad 69 = 8 \times 8 + 5 + 0$

$72 = 8 \times 9 + 0 + 0$.

For $29 \le v \le 40$ and $(v,8) \in A_4$, it follows from Lemma 1.5 that $(v,8) \in E_4$. The proof is completed. ∎

Lemma 3.9. *If* $28 < v < 85$ *and* $(v,9) \in A_4$, *then* $(v,9) \in E_4$.

Proof: For $v = 29$, give weight 4 to every point of a $GD[\{4\}, 3]$ of type $2^1\ 1^5$ and add a new point to every group of it. For $v = 48$ and 60, the conclusion comes from Lemma 3.6 and the existence of $(52, 13; 1; \{4\})$-IPBD, $(46, 7; 2; \{4\})$-IPBD, $(64, 13; 1; \{4\})$-IPBD and $(58, 7; 2; \{4\})$-IPBD. For $v = 64, 69$ and 84, the conclusion comes from Lemma 2.12. For $v = 32$ and 56, see Lemma 2.13. For $v = 49$, add 9 new points to a $(40, 4, 1)$-RBIBD such that every new point is added in the blocks of a parallel class. Thus we obtain a $B[\{4,5,9\},1;49]$ and then the required design. For $v = 52$, let $m = 9$, $n = 2$, $m' = 7$, $n' = 1$ and $a = 0$ in Lemma 2.10. The $IA_3(9,2)$ comes from [21] and the required IPBD comes from Corollary 3.7. For the other values of v, we use Lemma 2.11 by taking $v = (k-1)m + m' + n$ as following:

$53 = 4 \times 11 + 8 + 1$ $57 = 4 \times 12 + 9 + 0$ $61 = 4 \times 13 + 9 + 0$ $65 = 8 \times 8 + 0 + 1$

$68 = 8 \times 8 + 3 + 1$

for $72 \leq v \leq 81$, $v = 8 \times 9 + b + 0$, where $0 \leq b \leq 9$.

For $33 \leq v \leq 45$, we use Lemma 1.5. This completes the proof. ∎

Lemma 3.10. *If* $37 < v < 112$ *and* $(v,12) \in A_4$, *then* $(v,12) \in E_4$.

Proof: For $v \in \{64,72,76\}$ make use of Lemma 2.11 by letting $v = (k-1)m + m' + n$ as following:

$64 = 4 \times 13 + 12 + 0$ $72 = 4 \times 15 + 12 + 0$ $76 = 4 \times 16 + 12 + 0$.

For $v \in \{44,77,80,81,\ldots,112\}$, make use of Lemma 2.10 by letting $n = 3$, $n' = 0$, $a = 0$ and $v = 4m + u + m'$ as following:

$44 = 4 \times 8 + 12 + 0$

for $77 \leq v \leq 92$, $v = 4 \times 16 + 12 + b$ where $0 \leq b \leq 16$

for $93 \leq v \leq 112$, $v = 4 \times 20 + 12 + b$ where $0 \leq b \leq 20$.

The required $IA_2(11,3)$ see [5], the $IA_3(19,3)$ see [13] and the $IA_3(23,3)$ comes from Lemma 2.2 and the fact $23 = 4 \times 5 + 3$. For the required IPBDs, delete one point from a $T(5,4)$ to obtain a $(19,3;1;K_4)$-IPBD and then a $(19,3;3;\{4\})$-IPBD. Delete two points from one group of a $T(5,5)$ to obtain a $(23,3;1;k_4)$-IPBD and then a $(23,3;3;\{4\})$-IPBD. The required $(11,3;3;\{4\})$-IPBD is displayed below:

points: $Z_8 \cup \{\infty_i\colon\ 1 \leq i \leq 3\}$

blocks: $\{0,2,4,6\}$ $\{\infty_1,0,1,3\}$ $\{\infty_2,0,1,3\}$ $\{\infty_3,0,1,4\}$ mod 8.

For $v = 41$, add a point to every group of a $IA_2(10,3)$ and fill in groups with $(11,3;3;\{4\})$-IPBD.

For $v = 73$, the construction is similar to that of the case $v = 41$, but is from an $IA_2(18,3)$ instead of $IA_2(10,3)$. The required IPBD is obtained by deleting one point of a $T(5,4)$.

For $v = 61$, add a point to every group of a $T(4,15)$ which has 5 disjoint holes of order 3 (see [15]). Fill in groups with a GD[$\{4,5\}$, 1] of type $4^1\ 3^4$ which is obtained by deleting 3 points from a group and 1 point from another group of a

$T(5,4)$. Thus we obtain a $B[\{4,5,12,13\},1;61]$ and then a $(61,12;3;\{4\})$-IPBD.

For $v = 68$, add 4 points to every group of a $T(4,16)$ which has 5 disjoint holes, 4 of order 3 and 1 of order 4. This $T(4,16)$ can be easily obtained from a GD$[K_4,1]$ of type $4^1\ 3^4$. Then fill in groups with GD$[K_4,1]$ of type $4^2\ 3^4$ which is obtained by deleting 4 points from one group and 1 point from another group of a $T(5,5)$.

For $v = 40$, the conclusion comes from Lemma 2.5.

For $v = 65$ see Lemma 2.13.

For $v = 69$, the conclusion follows from Lemma 3.6 and the existence of $(73,16;1;\{4\})$-IPBD and $(67,10;2;\{4\})$-IPBD.

The proof is completed. ▮

Lemma 3.11. *Suppose for* $3w+1 \le v \le 9w+4$ *and* $(v,w) \in A_4, (v,w) \in E_4$, *and for* $u = 3w+1, 3u+1 \le v \le 4u$ *and* $(v,u) \in A_4$, $(v,u) \in E_4$. *Then* $(v,w) \in E_4$ *for every* $(v,w) \in A_4$.

Proof: For $9w+4 \le v \le 12w+4$, there is a $(v,u;3;\{4\})$-IPBD and a $(u,w;3;\{4\})$-IPBD. So there is a $(v,w;3;\{4\})$-IPBD.

For $v > 12w+4$, apply Lemma 2.11 by letting $k=5$, $n=0$, $m=3w+b$ and $0 \le m' \le 3w+b$, where $b = 1,4,5,8,\ldots$, when $w \equiv 0 \pmod 4$ or $b = 1,2,5,6,\ldots$, when $w \equiv 1 \pmod 4$. This completes the proof. ▮

To complete the proof of the embeddings of $S_3(2,4,u)$ for $u \in \{8,9,12\}$, we shall apply the above lemma and some results about the embeddings of $S_3(2,4,u)$ for $u \in \{25,28,37\}$.

Lemma 3.12. *If* $76 \le v \le 125$ *and* $(v,25) \in A_4$, *then* $(v,25) \in E_4$.

Proof: Use Lemma 2.10 by letting $m=16$, $n=6$, $m'=n'=0$ and $a=1$. The required $IA_2(22,6)$ comes from [5] and the $(23,7;3;\{4\})$-IPBD is displayed below:

points: $Z_{16} \cup \{\infty_i\colon\ 1 \le i \le 7\}$

blocks: $\{0,4,8,12\}\ \{\infty_1,0,1,7\}\ \{\infty_2,0,1,7\}\ \{\infty_3,0,1,7\}$
$\{\infty_4,0,2,5\}\ \{\infty_5,0,2,5\}\ \{\infty_6,0,2,5\}\ \{\infty_7,0,4,8\}$ mod 16.

This shows that $(89,25) \in E_4$.

Delete one point of a $T(6,5)$ to obtain a GD$[\{5,6\},1]$ of type $5^5\ 4^1$. Use Construction 2.8 by starting from this GDD. Let W consist of all points of the groups with size 5. Give every point of W weight 3 and the other points weight 4. Fill in blocks with size 6 of a GDD which is obtained by deleting 1 point from a $T(4,5)$. Fill in blocks with size 5 of a GDD with type 3^5 which is obtained by giving weight 3 to a $S_3(2,4,5)$. Thus we obtain a GD$[\{4\},3]$ of type 15^5 16^1 which has a sub-GDD constructed on W. Use Construction 2.6 for this GDD by letting $Y_1 = \phi$, $Y_2 = W$, $|S_1| = 1$ and $S_2 = \phi$. The required IPBDs are $(16,5;3;\{4\})$-IPBD and $S_3(2,4,17)$. So $(92,25) \in E_4$.

Let $k = 5$, $m = m' = 17$ and $n = 8$ in Lemma 2.11 to obtain a $(93,25;3;\{4\})$-IPBD.

For the other values of v, see Proposition 3.5. The proof is complete. ∎

Lemma 3.13. $(v,37) \in E_4$ *if and only if* $(v,37) \in A_4$.

Proof: From Proposition 3.5 and Lemma 1.4 we know that there are only 4 values of v which need to be treated. Let $k = 5$, $m = m' = 25$ and $n = 12$ in Lemma 2.11. This shows $(137,37) \in E_4$. Let $k = 5$, $m = 28$, $n = 9$ and $m' = 19$ or 20 in Lemma 2.11. This shows $(140,37)$ and $(141,37) \in E_4$. Finally, $(144,37) \in E_4$ by Lemma 3.4 with $d = 5$. This completes the proof. ∎

Proposition 3.14. $(v,28) \in E_4$ *if and only if* $(v,28) \in A_4$.

Proof: For $v \in \{88,97,100\}$, the conclusion comes from Lemma 2.5. For $v \in \{93,96\}$, see Lemma 2.13.

An $IA_2(23,7)$ exists by [5]. $(23,7;3;\{4\})$-IPBD is given in the proof of Lemma 3.12. Using Lemma 2.10 with $(m,n,m',n',a) = (16,7,0,0,0)$, we have $(92,28) \in E_4$. An $IA_2(22,7)$ exists by [5]. Using Construction 2.7 with $(b,a) = (1,0)$ and $(t_i,u_i) = (22,7)$ for $1 \leq i \leq 4$, we have $(89,28) \in E_4$.

For $v = 101$, add a new point to every group of an $IA_2(25,7)$ (see [5]) and fill in groups of $(26,7;3;\{4\})$-IPBD which is displayed below:

points: $Z_{19} \cup \{\infty_i\colon\ 1 \leq i \leq 7\}$

blocks: $\{0,1,2,4\}\{\infty_1,0,5,8\}\{\infty_2,0,3,9\}\{\infty_3,0,7,15\}$
$\{\infty_4,0,4,10\}\{\infty_5,0,5,12\}\{\infty_6,0,5,6\}\{\infty_7,0,2,10\}$ mod 19.

For $v = 104$, use Lemma 2.11 by letting $k = 5$, $m = m' = 19$ and $n = 9$.

For $v = 105$ or 108, use Lemma 2.11 by taking $k = 5$, $m = 20$, $n = 8$ and $m' = 17$ or 20.

For the other values of v, see Lemma 1.4. The proof is completed. ∎

Proposition 3.15. *Let* $u = 8,9$ *or* 12. *Then* $(v,u) \in E_4$ *if and only if* $(v,u) \in A_4$.

Proof: The conclusion follows from Lemma 3.8, 3.9, 3.10, 3.11, 3.12, and 3.13, and Proposition 3.14. ∎

Theorem 3.16. *Let* $u \equiv 0,1,4,5,8,$ *or* $9 \pmod{16}$ *and* $u > 25$. *Then* $(v,u) \in E_4$ *for every* $(v,u) \in A_4$.

Proof: Let $n = 4$, $b = 9$, $a = 1$, $t_i = 28$ and $u_i = 8$ for $1 \leq i \leq 4$ in Construction 2.7. The IGDD is an $IA_2(28,8)$. The $(37;9,9;1;K_4)$-◇-IPBD can be obtained by deleting 4 points from one group of a $T(5,8)$ and then adding 1 new point to every group of it. Thus a $(121,33;1;K_4)$-IPBD and then a $(121,33;3;\{4\})$-IPBD is obtained.

From Lemma 2.12 and Lemma 2.5 we know that $(117, 32)$ and $(133, 36) \in E_4$.

For other values of v, $4u - 11 \le v \le 4u - 4$, let $d = 5, 8, 9$ or 12 in Lemma 3.4. Now the conclusion comes from Lemma 1.4 and Proposition 3.5. ∎

4. The embeddings of $S_3(2,4,u)$ for $u \equiv 12$ and $13 \pmod{16}$.

In this section we shall discuss the case $u \equiv 12$ or $13 \pmod{16}$ and $u \ge 44$. From the embeddings of $S_3(2,4,u)$ for $u = 8$ or 9 and Lemma 2.9 we easily obtain the following proposition.

Proposition 4.1. *Let $u \equiv 12$ or $13 \pmod{16}$ and $u \ge 44$. If $(v,u) \in A_4$ and $3u + 1 \le v \le 4u - 24$, then $(v,u) \in E_4$.*

Proof: When $u \equiv 12 \pmod{16}$, let $s = (u-8)/4$, $t_1 = 0$ and $t_2 = 8$ in Lemma 2.9. When $u \equiv 13 \pmod{16}$ and $3u + 1 \le v \le 4u - 27$, let $s = (u-9)/4$, $t_1 = 0$ and $t_2 = 9$ in Lemma 2.9. When $u \equiv 13 \pmod{16}$ and $v = 4u - 24$, let $d = 8$ in Lemma 3.4. ∎

Now we shall consider the embeddings of $S_3(2,4,u)$ for $u = 17, 21$, and 24, so that we shall be able to use Lemma 3.4 for the case $4u - 23 \le v \le 4u - 3$.

Proposition 4.2. $(v, 17) \in E_4$ *if and only if* $(v, 17) \in A_4$.

Proof: For $v \in \{53, 56, 61, 64\}$, we use Lemma 2.9 by letting $s = 4$, $t_1 = 0$ and $t_2 = 1$. Here $v - 3u \equiv 1$ or $2 \pmod{3}$, so the condition $t_2 > 1$ can be changed to $t_2 = 1$.

For $v = 57$, the conclusion comes from Lemma 2.12. For $v = 60$, see Lemma 2.13.

For $v \in \{88, 89, 92, \ldots, 157\}$, make use of Lemma 2.10 by taking $n = 4$, $n' = 1$, $a = 0$ and $v = 4m + u + m'$ as following:

$$88 \le v \le 97,\quad v = 4 \times 16 + 17 + b \text{ where } 7 \le b \le 16$$
$$98 \le v \le 117,\quad v = 4 \times 20 + 17 + b \text{ where } 1 \le b \le 20$$
$$118 \le v \le 142,\quad v = 4 \times 25 + 17 + b \text{ where } 1 \le b \le 25$$
$$143 \le v \le 157,\quad v = 4 \times 28 + 17 + b \text{ where } 14 \le b \le 28.$$

The required $IA_3(20,4)$, $IA_3(24,4)$, $IA_3(29,4)$ and $IA_3(32,4)$ come from Lemma 2.2 by the fact $24 = 4 \times 5 + 4$, $20 = 4 \times 5$, $29 = 4 \times 7 + 1$, and $32 = 4 \times 7 + 4$. The required IPBDs come from the embeddings of $S_3(2,4,4)$. For $v > 157$, the conclusion follows from the embeddings of $S_3(2,4,52)$ and Lemma 3.11. This completes the proof. ∎

Proposition 4.3. $(v, 21) \in E_4$ *if and only if* $(v, 21) \in A_4$.

Proof: For $v \in \{73, 76, 77\}$, use Lemma 2.9 by letting $s = 5$, $t_1 = 0$ and $t_2 = 1$. In this case we can let $t_2 = 1$ for $v - 3u \equiv 1$ or $2 \pmod{3}$.

For $v \in \{108, 109, 112, \ldots, 192\}$, we apply Lemma 2.10 as follows, here the existence of $IA_3(m+n, n)$ comes from Lemma 2.2.

v	m	n	a	$IA_3(m+n,n)$
108 − 121	20	5	0	$25 = 4 \times 5 + 5$
122 − 141	24	5	0	$29 = 4 \times 7 + 1$
142 − 161	28	5	0	$33 = 4 \times 7 + 5$
162 − 181	32	5	0	$37 = 4 \times 8 + 5$
182 − 192	35	5	0	$40 = 5 \times 8$

The required IPBDs come from Proposition 3.3.

Using Lemma 3.4 with $d = 5$, we have $(80, 21) \in E_4$. $(v, 21) \in E_4$ for $v = 65, 68, 69, 72$ by Proposition 3.5.

For $v > 192$, the conclusion follows from the embeddings of $S_3(2, 4, 64)$ and Lemma 3.11. ∎

Proposition 4.4. $(v, 24) \in E_4$ *if and only if* $(v, 24) \in A_4$.

Proof: From Lemma 2.5 we know that $\{(85, 24), (88, 24)\} \subset E_4$. $(89, 24) \in E_4$ comes from Lemma 2.12.

For $v = 121$, add a new point to every group of an $IA_2(30, 6)$ and fill in groups with $(31, 6; 3; \{4\})$-IPBD which is displayed below:

points: $Z_{25} \cup \{\infty_i\colon\ 1 \le i \le 6\}$
blocks: $\{0,1,4,7\}$ $\{0,5,12,16\}$ $\{0,8,13,18\}$ $\{\infty_1,0,4,10\}$
$\{\infty_2,0,1,9\}$ $\{\infty_3,0,1,3\}$ $\{\infty_4,0,10,12\}$ $\{\infty_5,0,11,19\}$
$\{\infty_6,0,9,11\}$ mod 25

For $v = 124$, use Lemma 2.11 by taking $k = 5$, $m = 25$, $m' = 24$ and $n = 0$.

For $v = 125$ or 128, use Construction 2.7 by starting from an $IGD[K_4, 1]$ of type $(25, 5)^4\ (m, 4)^1$ where $m = 21$ or 24, which is obtained by deleting some points from one group of an $IA_3(25, 5)$. Let $b = 4$ and $a = 0$. The required $(29; 4, 5; 0; K_4)$-◇-IPBD can be obtained by adding a point to a $(28, 4, 1)$-RBIBD. The required $(25; 4, 4; 0; K_4)$-◇-IPBD and $(28; 4, 4; 0; K_4)$-◇-IPBD are just $(25, 4, 1)$-BIBD and $(28, 4, 1)$-BIBD.

For $v \in \{129, 132, 133, \ldots, 220\}$, we use Lemma 2.10 as following, the existence of $IA_3(m+n, n)$ comes from Lemma 2.2.

v	m	n	a	$IA_3(m+n,n)$
129 − 144	24	5	0	$29 = 4 \times 7 + 1$
145 − 164	28	5	0	$33 = 4 \times 7 + 5$
165 − 184	32	5	0	$37 = 4 \times 8 + 5$
185 − 199	35	5	0	$40 = 5 \times 8$
200 − 224	40	5	0	$45 = 5 \times 9$.

The required IPBDs come from Proposition 3.3.

Using Lemma 3.4 with $d = 5$, we have $(92,24) \in E_4$. $(v,24) \in E_4$ for $v = 76,77,80,81,84$ by Proposition 3.5.

For $v > 220$, the conclusion comes from the embeddings of $S_3(2,4,73)$ and Lemma 3.11. ∎

Lemma 4.5. $\{(157,45),(160,45),(161,45),(164,45),(221,60),(224,61)\} \subseteq E_4$.

Proof: Make use of Construction 2.8 by starting from an $IA_3(11,2)$ whose groups are denoted by G_i, $1 \le i \le 5$. Let $W = (\cup_{i=1}^{4} G_i) \cup U$ where $U = \{g\}$ for some $g \in G_5$. The weight of the points and the IGDDs filled in blocks are the same as those in the proof of Lemma 2.9. Thus we can obtain a GD[$\{4\}$, 3] of type $33^4\ n^1$ where $n = 24, 27, 28$ or 31, which contains two sub-GDDs. One sub-GDD is of type $11^4\ 1^1$ and the other is of $6^4\ m^1$ where $m = 0$ or 3. Now use Construction 2.6 by letting Y_1 be the point set of the sub-GDD of type $6^4\ m^1$ and Y_2 be the point set of other sub-GDD. Let $|S_1| = |S_2| = 1$ and $S_3 = \phi$. The required $\diamondsuit_3$-IPBD can be easily obtained by giving weight 3 to every point of a $(11,2;3;\{4\})$-IPBD and then adding one new point. The required $\diamondsuit$-IPBD can be formed in a similar way. The required IPBD comes from Proposition 3.15. This shows that$(v,45) \in E_4$ for $v \in \{157,160,161,164\}$.

For $(221,60)$ and $(224,61)$, we use a method which is similar to what we have done for the case $(92,25)$ in Lemma 3.12. Delete 2 points of a $T(6,12)$ to obtain a GD[$\{5,6\}$, 1] of type $12^5\ 10^1$. Let W consist of all points of the groups with size 12. The weight of the points and the GDDs filled in the blocks are the same as those in the proof of Lemma 3.12. Let $|S_1| = 1$ and $S_2 = \phi$ or $|S_1| = 4$ and $|S_2| = |S_3| = 1$. This shows that $\{(221,60),(224,61)\} \subseteq E_4$ and completes the proof. ∎

Lemma 4.6. $\{(153,44),(156,44),(157,44),(160,44), (217,60),(220,60),(221,61),(225,61)\} \subseteq E_4$.

Proof: From Lemma 2.5 we know that $\{(157,44),(160,44), (217,60), (220,60)\} \subseteq E_4$. A $(153,44;3;\{4\})$-IPBD can be obtained by adding one point to an $IA_2(38,11)$ and filling in groups of $(39,11;3;\{4\})$-IPBD which is displayed below:

points: $Z_{28} \cup \{\infty_i\colon\ 1 \le i \le 11\}$

blocks: $\{0,7,14,21\}$ $\{0,1,6,13\}$ $\{\infty_1,0,2,10\}$ $\{\infty_2,0,3,14\}$
$\{\infty_3,0,4,9\}$ $\{\infty_4,0,1,8\}$ $\{\infty_5,0,2,6\}$ $\{\infty_6,0,3,12\}$
$\{\infty_7,0,10,25\}$ $\{\infty_8,0,11,23\}$ $\{\infty_9,0,4,15\}$ $\{\infty_{10},0,6,8\}$
$\{\infty_{11},0,9,10\}$ mod 28

A $(156,44;3;\{4\})$-IPBD can be obtained by letting $m = 28$, $n = 8$, $m' = 0$, $n' = 8$ and $a = 4$ in Lemma 2.10. The required IPBD comes from Proposition 3.15 and the $IA_3(36,8)$ see [21].

Let $k = 5, m = m' = 41$, and $n = 20$ in Lemma 2.11. This shows $(225, 61) \in E_4$.

Lemma 2.12 shows $(221, 61) \in E_4$. This completes the proof. ∎

Theorem 4.7. *Let* $u \equiv 12$ *or* $13 \pmod{16}$ *and* $u > 44$. *Then* $(v, u) \in E_4$ *for every* $(v, u) \in A_4$.

Proof: From Lemma 1.4 and Proposition 4.1 we know that we need only discuss the case $4u - 23 \leq v \leq 4u - 3$. Some pairs are showed in Lemma 4.5 and Lemma 4.6. For the other pairs, let $d = 4, 5, 8, 9, 12, 17, 21$, and 24 in Lemma 3.4. The proof is complete. ∎

5. The proof of Theorem 1.3.

In this section we shall complete the proof of Theorem 1.3. To do this, we need the proof of following cases: $u = 13, 3u + 1 < v < 9u + 4$; $u = 16, 3u + 1 < v < 9u + 4$; $u = 20, 4u - 11 \leq v < 4u - 3$ and $5u < v < 9u + 4$; $u = 25$, $5u < v < 9u + 4$; and $u = 29, 3u + 1 < v < 4u - 3$.

Proposition 5.1. $(v, 13) \in E_4$ *if and only if* $(v, 13) \in A_4$.

Proof: Apply Lemma 2.10 by taking $n = 3, n' = 0, a = 1$ and $v = 4m + u + m'$ as following:

$$68 \leq v \leq 77, \quad v = 4 \times 13 + 13 + b \text{ where } 3 \leq b \leq 13$$
$$77 \leq v \leq 93, \quad v = 4 \times 16 + 13 + b \text{ where } 0 \leq b \leq 16$$
$$93 \leq v \leq 113, \quad v = 4 \times 20 + 13 + b \text{ where } 0 \leq v \leq 20.$$

The required $IA_3(16, 3)$ and $IA_3(19, 3)$ see [13], the $IA_3(23, 3)$ can be obtained from Lemma 2.2 and the fact $23 = 4 \times 5 + 3$. The required IPBDs come from the embeddings of $S_3(2, 4, 4)$.

A $(117, \{9, 13\})$-PBD exists, since $TD(9, 13)$ exists. Hence $(117, 13) \in E_4$. Deleting one point from $TD(9, 13)$, we have a $(116, \{8, 9, 12, 13\})$-PBD. Hence $(116, 13) \in E_4$. There exists an IGD$[\{8, 9\}, 1]$ with type $(13, 1)^8 (12, 1)^1$, which is obtained by deleting one point from $TD(9, 13)$. By adding four new points into the design and using the following input designs, a $(17, 5; 3; \{4\})$-IPBD and a $(16, 5; 3; \{4\})$-IPBD, we have $(120, 13) \in E_4$. For $49 \leq v \leq 65$, using Lemma 3.4 with $d = 0, 1$ and 4, we have $(v, 13) \in E_4$. For $v \in \{41, 44, 45\}$, see Lemma 2.13. For $v = 48$, the conclusion comes from Lemma 2.12. This completes the proof. ∎

Proposition 5.2. $(v, 16) \in E_4$ *if and only if* $(v, 16) \in A_4$.

Proof: Use Lemma 3.1 by letting $n = 4$, $n' = a = 0$ and $v = 4m + u + m'$ as following:

$$81 \le v \le 96, \quad v = 4 \times 16 + 16 + b \text{ where } 1 \le b \le 16,$$
$$97 \le v \le 116, \quad v = 4 \times 20 + 16 + b \text{ where } 1 \le b \le 20,$$
$$117 \le v \le 141, \quad v = 4 \times 25 + 16 + b \text{ where } 1 \le b \le 25,$$
$$141 \le v \le 148, \quad v = 4 \times 28 + 16 + b \text{ where } 13 \le b \le 20.$$

The required $IA_3(20,4)$, $IA_3(24,4)$, $IA_3(29,4)$ and $IA_3(32,4)$ come from Lemma 2.2 by the fact $24 = 4 \times 5 + 4, 29 = 4 \times 7 + 1, 20 = 4 \times 5$ and $32 = 4 \times 7 + 4$. The required IPBDs come from the embeddings of $S_3(2,4,4)$.

For $v = 53$ and 57, see Lemma 2.13. For $v = 52$, see Lemma 2.5. For $v = 56$, see Lemma 2.12. For $v = 60$, use Lemma 2.11 by letting $k = 5$, $m = m' = 11$ and $n = 5$. For $61 \le v \le 80$, use Lemma 3.4 with $d = 0, 1$ and 4. The proof is complete. ∎

Proposition 5.3. $(v, 29) \in E_4$ *if and only if* $(v, 29) \in A_4$.

Proof: Apply Construction 2.8 by starting from a GD[$\{4,5\}$, 1] of type $7^4 n^1$, where $0 \le n \le 7$, which can be obtained by deleting some points from a $T(5,7)$. Let $t(x) = 3$ for every x in this GDD. Let W be the set consisting of all points in the groups of size 7. Fill in blocks of GD[$\{4\}$, 1] with group type 3^5 or 3^4. Thus we can obtain a IGD[$\{4\}$, 3] of type $(21,7)^4\ (3n,0)^1$. Now we use Construction 2.7 for this IGDD by letting $a = 1$, b and n as follows.

v	b	n
92	8	0
97	4	3
100	4	4
101	8	3
104	8	4

The required $(29; 8,8; 1; \{4\})$-◇-IPBD can be obtained by adding a new point to an $IA_2(4,7)$. The required $(25; 8,4; 1; \{4\})$-◇-IPBD is a $(25, 8; 3; \{4\})$-IPBD. This shows $(v, 29) \in E_4$ for $v \in \{92, 97, 100, 101, 104\}$.

For $v = 105$, use Lemma 2.10 by taking $m = 19$, $n = 5$, $m' = 0$, $n' = 5$ and $a = 4$. The $IA_3(24,5)$ comes from Lemma 2.2 and the fact $24 = 4 \times 5 + 4$.

For $v = 89, 93$ and 96, see Lemma 2.13. For $v = 108$, 109 and 112, let $d = 5$ and 9 in Lemma 3.4. This completes the proof. ∎

Proposition 5.4. *Let $u = 20$ or 25. Then $(v,u) \in E_4$ if and only if $(v,u) \in A_4$.*

Proof: Delete 4 points from a group of a $T(5,5)$ to obtain a $B[\{4,5\},1,21]$ which has two disjoint blocks of size 5. It also has a block of size 4 and a block of size 5 which are disjoint. So we can see it as a $(21;5,5;0;K_4)$-◇-IPBD or $(21;4,5;0;K_4)$-◇-IPBD. Now use Construction 2.7 with $(b,a) = (5,0)$ and $(t_i,u_i) = (16,5)$ for $1 \le i \le 4$ or with $(b,a) = (4,0)$ and $(t_i,u_i) = (17,5)$ for $1 \le i \le 4$. This shows that $\{(69,20),(72,20)\} \subset E_4$. $\{(73,20),(76,20)\} \subset E_4$ is proved in Lemma 2.5.

For $u = 20$ and $v \in \{101,104,105,\ldots,184\}$, the proof is similar to that of $u = 21$. We apply Lemma 2.10 by taking the values of m, n and a as same as those in the case $u = 21$.

For $u = 25$ and $v \in \{128,129,132,\ldots,145\}$, use Lemma 2.10 by letting $m = 24$, $n = 6$ and $a = 1$. The required $IA_3(30,6)$ see [18] and the IPBDs see Lemma 2.1.

For $v \in \{148,149,152,\ldots,225\}$, use Lemma 2.10 by letting the values of m, n and a as same as those in the case $u = 24$.

For $v = 228$, let $m = 48$, $n = 6$, $m' = 11$, $n' = 0$ and $a = 1$ in Lemma 2.10. The $IA_3(54,6)$ see [18], and the $(55,7;3;\{4\})$-IPBD see Lemma 2.1. The proof is completed. ∎

Proof of Theorem 1.3: The conclusion follows from Theorem 3.16, Theorem 4.7, Proposition 3.3, 3.14, 3.15, 4.2, 4.3, 4.4 and the propositions of this section directly. ∎

Acknowledgement.

The author wishes to thank Professor L. Zhu for his encouragement and guidance, and to thank Professor H. Shen for helpful discussions. The author also wishes to thank the referee for his/her valuable suggestions.

Note added.

Recently, G. Kong and L. Zhu have proved that any $S_6(2,4,u)$ can be embedded in some $S_6(2,4,v)$ iff $u \ge 4$ and $v \ge 3u+1$ in their paper "Embeddings of $S_\lambda(2,4,u)$". The existence problem of embeddings for $S_\lambda(2,4,v)$ is then finished.

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Appendix

A (61, 8; 3; {4})-IPBD

points:	$Z_{53} \cup \{\infty_i: 1 \le i \le 8\}$			
blocks:	{0, 1, 11, 23}	{0, 2, 15, 29}	{0, 4, 22, 25}	{0, 3, 19, 20}
	{0, 5, 15, 11}	{0, 6, 18, 13}	{0, 7, 21, 16}	{0, 8, 23, 25}
	{0, 9, 17, 28}	{∞_1, 0, 1, 10}	{∞_2, 0, 2, 14}	{∞_3, 0, 4, 22}
	{∞_4, 0, 6, 26}	{∞_5, 0, 7, 26}	{∞_6, 0, 8, 29}	{∞_7, 0, 20, 23}
	{∞_8, 0, 29, 13}			mod 53

A (32, 9; 3; {4})-IPBD*

points:	$Z_{23} \cup \{\infty_i: 1 \le i \le 9\}$			
blocks:	{0, 3, 6, 7}{∞_1, 0, 10, 11}{∞_2, 0, 2, 6}		{∞_3, 0, 2, 7}	{∞_4, 0, 1, 9}
	{∞_5, 0, 2, 10}	{∞_6, 0, 3, 12}	{∞_7, 0, 4, 11}	{∞_8, 0, 6, 14}
	{∞_9, 0, 5, 10}			mod 23

A (56, 9; 3; {4})-IPBD

points:	$Z_{47} \cup \{\infty_i: 1 \le i \le 9\}$			
blocks:	{0, 1, 12, 22}	{0, 2, 15, 23}	{0, 3, 9, 27}	{0, 14, 19, 31}
	{0, 4, 7, 13}	{0, 7, 11, 25}	{0, 1, 20, 32}	{∞_1, 0, 2, 10}
	{∞_2, 0, 5, 22}	{∞_3, 0, 1, 21}	{∞_4, 0, 3, 9}	{∞_5, 0, 4, 14}
	{∞_6, 0, 13, 29}	{∞_7, 0, 5, 7}	{∞_8, 0, 11, 30}	{∞_9, 0, 8, 23}
				mod 47

A (65, 12; 3; {4})-IPBD

points:	$Z_{53} \cup \{\infty_i: 1 \le i \le 12\}$			
blocks:	{0, 1, 11, 23}	{0, 2, 15, 29}	{0, 3, 19, 20}	{0, 4, 22, 25}
	{0, 5, 9, 17}	{0, 6, 19, 21}	{0, 7, 16, 33}	{∞_1, 0, 8, 10}
	{∞_2, 0, 8, 11}	{∞_3, 0, 7, 11}	{∞_4, 0, 14, 24}	{∞_5, 0, 18, 25}
	{∞_6, 0, 9, 23}	{∞_7, 0, 20, 25}	{∞_8, 0, 21, 27}	{∞_9, 0, 22, 34}
	{∞_{10}, 0, 15, 16}	{∞_{11}, 0, 13, 18}	{∞_{12}, 0, 24, 30}	mod 53

A (41, 13; 3; {4})-IPBD

points:	$Z_{28} \cup \{\infty_i\colon\ 1 \le i \le 13\}$			
blocks:	$\{0,7,14,21\}$	$\{\infty_1,0,1,3\}$	$\{\infty_2,0,4,9\}$	$\{\infty_3,0,6,13\}$
	$\{\infty_4,0,8,18\}$	$\{\infty_5,0,11,23\}$	$\{\infty_6,0,1,4\}$	$\{\infty_7,0,2,8\}$
	$\{\infty_8,0,9,14\}$	$\{\infty_9,0,11,13\}$	$\{\infty_{10},0,11,12\}$	$\{\infty_{11},0,3,12\}$
	$\{\infty_{12},0,8,21\}$	$\{\infty_{13},0,4,10\}$		mod 28

A (44, 13; 3; {4})-IPBD

points:	$Z_{31} \cup \{\infty_i\colon\ 1 \le i \le 13\}$			
blocks:	$\{0,1,4,25\}$	$\{\infty_1,0,2,11\}$	$\{\infty_2,0,5,13\}$	$\{\infty_3,0,12,15\}$
	$\{\infty_4,0,1,14\}$	$\{\infty_5,0,2,6\}$	$\{\infty_6,0,5,12\}$	$\{\infty_7,0,8,17\}$
	$\{\infty_8,0,10,25\}$	$\{\infty_9,0,4,11\}$	$\{\infty_{10},0,11,26\}$	$\{\infty_{11},0,2,10\}$
	$\{\infty_{12},0,3,12\}$	$\{\infty_{13},0,1,14\}$		mod 31

A (45, 13; 3; {4})-IPBD

points:	$Z_{32} \cup \{\infty_i\colon\ 1 \le i \le 13\}$			
blocks:	$\{0,8,16,24\}$	$\{0,1,7,22\}$	$\{\infty_1,0,2,5\}$	$\{\infty_2,0,4,13\}$
	$\{\infty_3,0,12,26\}$	$\{\infty_4,0,1,8\}$	$\{\infty_5,0,2,11\}$	$\{\infty_6,0,3,13\}$
	$\{\infty_7,0,4,16\}$	$\{\infty_8,0,14,29\}$	$\{\infty_9,0,1,5\}$	$\{\infty_{10},0,9,19\}$
	$\{\infty_{11}0,15,20\}$	$\{\infty_{12},0,6,8\}$	$\{\infty_{13},11,25\}$	mod 32

A (53, 16; 3; {4})-IPBD

points:	$Z_{37} \cup \{\infty_i\colon\ 1 \le i \le 16\}$			
blocks:	$\{0,2,5,18\}$	$\{\infty_1,0,1,7\}$	$\{\infty_2,0,4,12\}$	$\{\infty_3,0,9,10\}$
	$\{\infty_4,0,11,14\}$	$\{\infty_5,0,15,17\}$	$\{\infty_6,0,4,9\}$	$\{\infty_7,0,6,13\}$
	$\{\infty_8,0,8,18\}$	$\{\infty_9,0,11,23\}$	$\{\infty_{10},0,15,31\}$	$\{\infty_{11},0,17,20\}$
	$\{\infty_{12},0,9,11\}$	$\{\infty_{13},0,4,14\}$	$\{\infty_{14},0,5,13\}$	$\{\infty_{15},0,12,19\}$
	$\{\infty_{16},0,15,16\}$			mod 37

A (57, 16; 3; {4})-IPBD

points:	$Z_{41} \cup \{\infty_i: 1 \le i \le 16\}$			
blocks:	$\{0,1,9,20\}$	$\{0,2,5,18\}$	$\{\infty_1,0,4,10\}$	$\{\infty_2,0,7,24\}$
	$\{\infty_3,0,12,26\}$	$\{\infty_4,0,1,9\}$	$\{\infty_5,0,2,20\}$	$\{\infty_6,0,3,7\}$
	$\{\infty_7,0,4,15\}$	$\{\infty_8,0,5,19\}$	$\{\infty_9,0,6,22\}$	$\{\infty_{10},0,12,13\}$
	$\{\infty_{11},0,2,18\}$	$\{\infty_{12},0,5,11\}$	$\{\infty_{13},0,7,15\}$	$\{\infty_{14},0,9,21\}$
	$\{\infty_{15},0,10,24\}$	$\{\infty_{16},0,3,13\}$		mod 41

A (60, 17; 3; {4})-IPBD

points:	$Z_{43} \cup \{\infty_i: 1 \le i \le 17\}$			
blocks:	$\{0,1,10,21\}$	$\{0,2,6,19\}$	$\{\infty_1,0,3,8\}$	$\{\infty_2,0,7,12\}$
	$\{\infty_3,0,14,16\}$	$\{\infty_4,0,15,18\}$	$\{\infty_5,0,1,7\}$	$\{\infty_6,0,4,12\}$
	$\{\infty_7,0,9,19\}$	$\{\infty_8,0,11,25\}$	$\{\infty_9,0,13,20\}$	$\{\infty_{10},0,15,21\}$
	$\{\infty_{11},0,2,22\}$	$\{\infty_{12},0,4,13\}$	$\{\infty_{13},0,5,17\}$	$\{\infty_{14},0,16,17\}$
	$\{\infty_{15},0,10,28\}$	$\{\infty_{16},0,3,14\}$	$\{\infty_{17},0,8,24\}$	mod 43

A (93, 28; 3; {4})-IPBD

points:	$Z_{65} \cup \{\infty_i: 1 \le i \le 28\}$			
blocks:	$\{0,1,5,30\}$	$\{0,2,8,19\}$	$\{\infty_1,0,9,21\}$	$\{\infty_2,0,3,10\}$
	$\{\infty_3,0,13,27\}$	$\{\infty_4,0,15,31\}$	$\{\infty_5,0,18,41\}$	$\{\infty_6,0,20,22\}$
	$\{\infty_7,0,26,32\}$	$\{\infty_8,0,1,28\}$	$\{\infty_9,0,4,18\}$	$\{\infty_{10},0,7,17\}$
	$\{\infty_{11},0,8,19\}$	$\{\infty_{12},0,9,21\}$	$\{\infty_{13},0,15,35\}$	$\{\infty_{14},0,22,24\}$
	$\{\infty_{15},0,25,51\}$	$\{\infty_{16},0,29,32\}$	$\{\infty_{17},0,25,31\}$	$\{\infty_{18},0,4,28\}$
	$\{\infty_{19},0,5,21\}$	$\{\infty_{20},0,7,17\}$	$\{\infty_{21},0,8,31\}$	$\{\infty_{22},0,9,20\}$
	$\{\infty_{23},0,12,27\}$	$\{\infty_{24},0,13,43\}$	$\{\infty_{25},0,18,19\}$	$\{\infty_{26},0,13,29\}$
	$\{\infty_{27},0,3,26\}$	$\{\infty_{28},0,32,60\}$		mod 65

A (96,28; 3; {4})-IPBD**

points: $Z_{68} \cup \{\infty_i: 1 \le i \le 28\}$

blocks:

{0,7,29,32}	{0,19,21,28}	{0,17,34,51}	(repeated 3 times)
$\{\infty_1,0,12,25\}$	$\{\infty_2,0,20,33\}$	$\{\infty_3,0,28,30\}$	$\{\infty_4,0,23,28\}$
$\{\infty_5,0,20,42\}$	$\{\infty_6,0,19,32\}$	$\{\infty_7,0,18,27\}$	$\{\infty_8,0,14,26\}$
$\{\infty_9,0,30,61\}$	$\{\infty_{10},0,15,25\}$	$\{\infty_{11},0,12,23\}$	$\{\infty_{12},0,6,9\}$
$\{\infty_{13},0,27,33\}$	$\{\infty_{14},0,24,32\}$	$\{\infty_{15},0,29,60\}$	$\{\infty_{16},0,31,64\}$
$\{\infty_{17},0,5,29\}$	$\{\infty_{18},0,6,30\}$	$\{\infty_{19},0,11,27\}$	$\{\infty_{20},0,10,26\}$
$\{\infty_{21},0,10,21\}$	$\{\infty_{22},0,8,23\}$	$\{\infty_{23},0,5,21\}$	$\{\infty_{24},0,4,22\}$
$\{\infty_{25},0,4,18\}$	$\{\infty_{26},0,1,3\}$	$\{\infty_{27},0,1,15\}$	$\{\infty_{28},0,1,20\}$
			mod 68

A (89,29; 3; {4})-IPBD

points: $Z_{60} \cup \{\infty_i: 1 \le i \le 29\}$

blocks:

{0,15,30,45}	$\{\infty_1,0,1,29\}$	$\{\infty_2,0,2,18\}$	$\{\infty_3,0,3,20\}$
$\{\infty_4,0,5,27\}$	$\{\infty_5,0,6,13\}$	$\{\infty_6,0,8,19\}$	$\{\infty_7,0,9,21\}$
$\{\infty_8,0,23,48\}$	$\{\infty_9,0,24,26\}$	$\{\infty_{10},0,1,26\}$	$\{\infty_{11},0,3,19\}$
$\{\infty_{12},0,4,15\}$	$\{\infty_{13},0,5,29\}$	$\{\infty_{14},0,6,28\}$	$\{\infty_{15},0,7,17\}$
$\{\infty_{16},0,8,21\}$	$\{\infty_{17},0,9,23\}$	$\{\infty_{18},0,18,45\}$	$\{\infty_{19},0,20,30\}$
$\{\infty_{20},0,1,20\}$	$\{\infty_{21},0,2,16\}$	$\{\infty_{22},0,3,24\}$	$\{\infty_{23},0,4,31\}$
$\{\infty_{24},0,5,13\}$	$\{\infty_{25},0,6,23\}$	$\{\infty_{26},0,25,51\}$	$\{\infty_{27},0,7,11\}$
$\{\infty_{28},0,12,22\}$	$\{\infty_{29},0,18,46\}$		mod 60

A (93,29; 3; {4})-IPBD

points: $Z_{64} \cup \{\infty_i: 1 \le i \le 29\}$

blocks:

{0,16,32,48}	{0,1,15,34}	$\{\infty_1,0,2,29\}$	$\{\infty_2,0,3,26\}$
$\{\infty_3,0,4,25\}$	$\{\infty_4,0,5,13\}$	$\{\infty_5,0,6,18\}$	$\{\infty_6,0,7,17\}$
$\{\infty_7,0,9,20\}$	$\{\infty_8,0,22,28\}$	$\{\infty_9,0,1,24\}$	$\{\infty_{10},0,2,26\}$
$\{\infty_{11},0,3,15\}$	$\{\infty_{12},0,4,14\}$	$\{\infty_{13},0,5,30\}$	$\{\infty_{14},0,7,27\}$
$\{\infty_{15},0,8,19\}$	$\{\infty_{16},0,9,22\}$	$\{\infty_{17},0,16,33\}$	$\{\infty_{18},0,18,46\}$
$\{\infty_{19},0,21,29\}$	$\{\infty_{20},0,1,32\}$	$\{\infty_{21},0,2,17\}$	$\{\infty_{22},0,3,30\}$
$\{\infty_{23},0,4,20\}$	$\{\infty_{24},0,5,29\}$	$\{\infty_{25},0,6,28\}$	$\{\infty_{26},0,7,21\}$
$\{\infty_{27},0,9,19\}$	$\{\infty_{28},0,11,23\}$	$\{\infty_{29},0,13,38\}$	mod 64

A (96,29; 3; {4})-IPBD

points:	$Z_{67} \cup \{\infty_i: 1 \le i \le 29\}$			
blocks:	{0,1,8,30}	{0,2,15,33}	$\{\infty_1,0,3,23\}$	$\{\infty_2,0,4,28\}$
	$\{\infty_3,0,5,21\}$	$\{\infty_4,0,6,17\}$	$\{\infty_5,0,9,19\}$	$\{\infty_6,0,14,40\}$
	$\{\infty_7,0,12,44\}$	$\{\infty_8,0,1,25\}$	$\{\infty_9,0,2,31\}$	$\{\infty_{10},0,3,16\}$
	$\{\infty_{11},0,4,32\}$	$\{\infty_{12},0,6,27\}$	$\{\infty_{13},0,7,18\}$	$\{\infty_{14},0,8,20\}$
	$\{\infty_{15},0,9,26\}$	$\{\infty_{16},0,14,33\}$	$\{\infty_{17},0,22,38\}$	$\{\infty_{18},0,1,25\}$
	$\{\infty_{19},0,2,33\}$	$\{\infty_{20},0,3,18\}$	$\{\infty_{21},0,4,32\}$	$\{\infty_{22},0,6,27\}$
	$\{\infty_{23},0,7,30\}$	$\{\infty_{24},0,8,19\}$	$\{\infty_{25},0,9,22\}$	$\{\infty_{26},0,12,26\}$
	$\{\infty_{27},0,17,47\}$	$\{\infty_{28},0,5,10\}$	$\{\infty_{29},0,10,25\}$	mod 67