The Integrity of the Cube is Small

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Abstract. It is shown that the integrity of the n-dimensional cube is $O(2^n \log n/\sqrt{n})$.

Barefoot, Entringer and Swart [1] introduced the graphical parameter integrity. Let $m(H)$ denote the maximum number of vertices in a component of a graph H. Then the integrity of a graph G, denoted $I(G)$, is defined by $I(G) = \min\{\, |S| + m(G - S) : S \subset V(G)\,\}$.

In [2] the values of the integrity of some product graphs were calculated. One such graph for which the integrity was not calculated was the cube. The n-dimensional cube Q_n has 2^n vertices, and may be defined as the cartesian product of n copies of Q_1, where Q_1 is the complete graph on 2 vertices.

In [2] it was conjectured that $I(Q_n)$ was $2^{n-1} + 1$. We show here that this value is considerably too high.

The Upper Bound

One approach to cutting up the cube is as follows. You find a suitable set $\mathcal{D}_n$ of integers, and then select any vertex x and for each $d \in \mathcal{D}_n$, remove all the vertices of distance d from x. Indeed, this approach can be used to show that $I(Q_n)$ is $O(2^n/n^{1/4})$, and thus that the conjectured value of $I(Q_n)$ is incorrect. But the approach described below is better.

Consider an n-dimensional cube Q_n where n is a power of 2. For any vertex x, we define the *cut* C_x to be the set of vertices at distance $n/2$ from x. We investigate removing a series of "orthogonal" cuts.

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Choose a collection $x_1, x_2, \ldots, x_r$ of vertices which are pairwise distance $n/2$ apart as follows: x_1 is given by the string O^n, x_2 by $O^{n/2}1^{n/2}$, x_3 by $O^{n/4}1^{n/4}O^{n/4}1^{n/4}$ etc. Note that r is at most $1+\log n$ (where log is base 2). Then let $G_r = Q_n - (C_{x_1} \cup C_{x_2} \cup \ldots \cup C_{x_r})$.

Lemma. *If $n \geq 4$ then G_r has 2^r isomorphic parts, each the union of components. Thus $m(G_r) \leq 2^{n-r}$.*

Proof: Label a vertex v in G_r with an r-tuple $(y_1, \ldots, y_r)$ where $y_i = +1$ if v is less than $n/2$ away from x_i in Q_n, and -1 otherwise. This labeling partitions G_r into 2^r *parts*. It is easy to see that there is no edge connecting two parts.

To show that these parts are isomorphic, observe that there exists an automorphism φ_i of Q_n which maps x_i to its complement $\overline{x}_i$ but keeps the other x_j fixed. (For example, consider $n = 4$: the function which maps string x to the reversal of its complement fixes x_2 and x_3 but maps x_1 to 1111.) Thus the composition of the relevant φ_i is the desired isomorphism. ∎

Hence we have established that: If n is a power of 2 then

$$I(Q_n) \leq \min_{0 \leq r \leq 1+\log n} \left\{ 2^{n-r} + r \binom{n}{n/2} \right\}. \tag{1}$$

(It can be shown that the overestimates in this bound are asymptotically insignificant.)

A standard result is that

$$\binom{n}{n/2} \sim \frac{2^n}{\sqrt{n\pi/2}}.$$

Therefore calculus shows that the value of r which minimizes the expression in Equation 1 is approximately $\log(\ln 2\sqrt{n\pi/2}) + O(\log\log n)$. We choose r to be $\lceil (\log n)/2 \rceil$. Thus there is a constant c such that, when n is a power of 2, $I(Q_n)$ is at most $c2^n \log n/\sqrt{n}$.

Now let n be any positive integer. We consider Q_n as the cartesian product $Q_m \times Q_l$, where m is the largest power of 2 not exceeding n. As a consequence of a general observation in [2], it holds that $I(Q_n) \leq I(Q_m) \times 2^l$. Thus

$$I(Q_n) \leq c2^m \log m/\sqrt{m} \times 2^l < c2^n \log n/\sqrt{n/2},$$

as $m > n/2$.

Hence we may conclude:

Theorem. *The integrity $I(Q_n)$ of the n-dimensional cube Q_n is $O\left(2^n \log n/\sqrt{n}\right)$*

We suspect that this might be the right (order of) value. An $\Omega(2^n/\sqrt{n})$ lower bound on the integrity is an immediate consequence of the following result:

Proposition. [3] *If $m(Q_n - S) \leq 2^{n-1}$ then $|S| \geq \binom{n}{\lfloor n/2 \rfloor}$.*

References

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