The disjoint 1-factors of (d, d + 1)-graphs

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Abstract. A graph G is called $(d, d+1)$-graph if the degree of every vertex of G is either d or $d+1$. In this paper, the following results are proved: A $(d, d+1)$-graph G of order $2n$ with no 1-factor and no odd component, satisfies $|V(G)| \geq 3d+4$; A $(d, d+1)$-graph G of order $2n$ with $d(G) \geq n$, contains at least $[(n+2)/3] + (d-n)$ edge disjoint 1- factors. These results generalized the theorems due to W. D. Wallis, A.J.W. Hilton and C. Q. Zhang.

A 1-factor in a graph G is a set of disjoint edges of which together cover all vertices of G. In this paper we only consider simple graphs. A graph G is said to be $(d, d+1)$-graph, if the degree of each vertex of G is either d or $d+1$. We derive a lower bound for the number of vertices in a $(d, d+1)$- graph which has no 1-factor and no odd component. And we also obtain the lower bound of the edge-disjoint 1-factors of $(d, d+1)$-graph G of order $2n$ with $d \geq n$. We use the following well-known theorems:

Lemma 1.[2]. *A graph G has no 1-factor if and only if there is some set K of k vertices such that deletion of K (all edges touching it) from G leaves a graph with at least $k+1$ odd components.*

Lemma 2.[3]. *If G has an even number of vertices without 1-factor, then there is some set K of k vertices such that $G-K$ has at least $k+2$ odd components.*

Theorem 1. *A $(d, d+1)$-graph G of order $2n$ with no 1-factor and no odd component, satisfies: $|V(G)| \geq 3d+4$.*

Proof: By contradiction. Suppose that $|V(G)| \leq 3d+3$, and G has no 1-factor, and no odd component. Since G is a $(d, d+1)$-graph of order $2n$, then by Lemma 2, there is set K of k vertices such that $G-K$ has at least $k+2$ odd components.

Suppose $G-K$ has a component with p vertices, where $1 \leq p \leq d$. The number of edges within the component is at most $p(p-1)/2$. But in G, each vertex has degree at least d, so the number of edges joining the component to K must be at least

$$pd - p(p-1)$$

For the sum of degrees of these p vertices in $G-K$ is at most $p(p-1)$, but in G each vertex has degree at least d, so the sum of their degree is at least pd.

For fixed d and for integer p satisfying $1 \leq p \leq d$, this function has minimum value d. So any odd component with d or less vertices is joined to K at least d edges.

Now let $G - K$ contain a_+ odd components with more than d vertices and a_- odd components with d or less vertices. It is obvious that

$$a_+ + a_- \geq k + 2 \tag{1}$$

Each of the a_- smaller components has at least one vertex and each of the a_+ larger components has at least $d + 1$ vertices, so the number of $|V(G)|$ satisfies:

$$|V(G)| \geq k + a_- + (d+1)a_+ \tag{2}$$

Since $|V(G)| \leq 3d + 3$, by (2), it follows that $a_+ \leq 2$, and by (1), $a_- \geq k$.

Note that the number of edges leading from K to the odd components is at least $a_+ + da_-$ Thus we have that

$$a_+ + da_- \leq k(d+1) \tag{3}$$

Rearrange the inequality (3), we have that

$$a_- \leq k + [(a_- - a_+)/(d+1)] \tag{4}$$

Then, $a_- - a_+ \geq 0$, since $a_- \geq k$. By (1), we have

$$2a_- \geq a_- + a_+ \geq k + 2,$$

Then, $a_- \geq (k+2)/2 \geq 3/2$, it follows that $a_- \geq 2$.

Case 1: If $a_- - a_+ \geq d + 1$, then $a_- \geq d + 1 + a_+$. By (3), $k \geq [a_+ + d(d+1+a_+)]/(d+1)$, then, if $a_+ \geq 1, k \geq d+1$; if $a_+ = 0, k \geq d$.

Subcase 1.1: If $a_+ \geq 1$, then by (2), $|V(G)| \geq k + (a_- + a_+) + da_+ \geq d + 1 + d + 1 + 2a_+ + d \geq 3d + 4$. This is a contradiction.

Subcase 1.2: If $a_+ = 0$, then by (1), $a_- \geq k+2$, and by (3), we have, $d(k+2) \leq da_- \leq k(d+1)$. It follows that, $k \geq 2d$. Hence, by (2), $|V(G)| \geq k + a_- \geq 2d + (2d+2) \geq 4d + 2$. Since G contains no 1-factor and no odd component, we can exclude the trivial case $d \leq 1$, therefore $4d + 2 \geq 3d + 4$, a contradiction.

Case 2: If $a_- - a_+ \leq d$, then by (4), $a_- \leq k$, and by (1), $a_+ \geq 2$. Therefore $a_+ = 2, a_- = k$.

Suppose C is one of the small odd components; consider a vertex x in C. Since vertex x and all its neigbors are in $(K + C)$, then $|K| + |C| \geq d + 1$, so

$$|V(G)| \geq d + 1 + (a_- - 1) + 2(d+1) \geq 3d + 4, \tag{5}$$

since $a_- - 1 \geq 1$. This is a contradiction. This completes the proof. ∎

We point out that all possible orders can be realized (except for the trivial case $d = 1$, when the graph must be a union of even paths, and have a one-factor). Here is an easy construction. Take a set K of $d+1$ vertices. Then $G - K$ has a set S of $d+2$ components of size 1 and one large component H of size $d+1$ (d even) or $d+2$ (d odd). One vertex x of H is distinguished. When d is even, H is K_{d+1}; when d is odd, form H from K_{d+2} by deleting one edge through x. Then add one edge from one vertex of K to x, and $d^2 + 2d$ edges from the vertices of K to S, in such a way as to form a connected graph. This can be done in many ways.

Corollary [3]. *Let G be a d-regular graph of even order without 1-factor and odd component. Then $|V(G)| \geq 3d+4$.*

By theorem 1, we have the following result:

Theorem 2. *If G is a $(d, d+1)$-graph of order $2n$ and $d \geq n$, then G contains at least $[(n+2)/3] + (d-n)$ disjoint 1-factors.*

Proof: If $d \geq n+1$, then we can use the Dirac theorem to find $d-n$ disjoint 1-factors in G. Hence, it is sufficent to prove the theorem by considering an $(n, n+1)$-graph G of order $2n$.

We assume that $n \geq 5$. Let $F_1, \ldots, F_t$ be a maximum set of disjoint 1-factors of G. We prove the theorem by contradiction. Suppose $t < (n+2)/3$. Then $H = G - \cup_{j=1}^{t} F_j$ is an $(h, h+1)$-graph, where $h = n - t$ and H is of order at most $3h+1$.

If H is connected, by the previous theorem, H has a 1-factor and this contradicts the choice of $F_1, \ldots, F_t$. Hence, H must be disconnected and contains some odd components. Since each component of H is of order at least $h+1$, H has exactly two components C_1 and C_2, each of which is of odd order. Without loss of generality, let $|V(C_1)| \leq |V(C_2)|$. Then

$$h+1 \leq |V(C_1)| \leq |V(H)|/2,$$

$$|V(H)|/2 \leq |V(C_2)| \leq 2h,$$

Since C_2 is an odd component,

$$|V(H)|/2 \leq |V(C_2)| \leq 2h-1$$

We claim that there is $F_j \in \{F_1, \ldots F_t\}$ such that $e_{F_j}(C_1, C_2) \geq 3$. If not, then $e_{F_j}(C_1, C_2) = 1$, for each $F_i \in F_1, \ldots, F_t$, because C_1 is an odd component and $e_{F_j}(C_1, C_2)$ is odd. Since

$$\sum_{i=1}^{t} e_{F_i}(C_1, C_2) = t.$$

$$t \leq h < |V(C_1)|.$$

There must be a vertex v of C_1 such that the neighbor of v in each F_i is contained in C_1. Thus, all vertices adjacent to v in G are contained in C_1 and hence, $|V(C_1)| \geq n+1$. This contradicts the assumption that $|V(C_1)| \leq |V(H)|/2 \leq n$.

Without loss of generality, let F_1 be such that $e_{F1}(C_1, C_2) \geq 3$, and (x_1, x_2), (y_1, y_2), (z_1, z_2) be edges of F_1 such that $x_i, y_i, z_i \in V(C_i)$ for $i = 1, 2$. Since $|V(C_i) - \{x_i\}| \leq 2h - 2$ and the minimum degree of $H(C_i - x_i)$ is $\geq h - 1$, by the Dirac theorem, let $P_i = v_1^i \dots v_{|C_i|-1}^i v_1^i$ be a Hamilton cycle in $H(C_i - x_i)$, for $i = 1, 2$. Thus, if $F_0 = \{(x_1, x_2)\} \cup \{(v_{2j-1}^1, v_j^1) : j = 1, \dots, \frac{|V(C_1)|-1}{2}\} \cup \{(v_{2j-1}^2, v_j^2) : j = 1, \dots, (|V(C_2)| - 1)/2\}$, then F_0 is a 1-factor of $H \cup F_1$. Since $|V(C_i)| \leq 2h - 1$ and the minimum degree of $H(C_i) - F_0$ is at least $h - 1$, $H(C_i) - F_0$ is still connected. Therefore, $[H \cup F_1] - F_0$ is also connected because (y_1, y_2) and (z_1, z_2) are edges joining the two connected parts $[F_1 \cup H(C_1)] - F_0$ and $[F_1 \cup H(C_2)] - F_0$. By theorem 1, the connected $(h, h+1)$-graph $H \cup F_1 - F_0$ has a 1-factor F_{t+1}, which contradicts the choice of $F_1, \dots, F_t$. This completes the proof. ∎

Corollary [4]. *A d-regular graph G of even order $2n$ with $d \geq n$, contains at least $[(n+2)/3] + (d - n)$ disjoint 1-factors.*

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