Cycle Covers in Graphs Without Subdivisions of K_4

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Abstract. In [B], Bondy conjectured that if G is a 2-edge-connected simple graph with n vertices, then G admits a cycle cover with at most $(2n-1)/3$ cycles. In this note we show that if G is a 2-edge-connected simple graph with n vertices and without subdivisions of K_4, then G has a cycle cover with at most $(2n-2)/3$ cycles and we characterize all the extremal graphs. We also show that if G is 2-edge-connected and has no subdivision of K_4, then G is mod $(2k+1)$-orientable for any integer $k \geq 1$.

Introduction.

Graphs in this note are finite and loopless. For all undefined terms, see Bondy and Murty [BM]. Let G be a graph and $e \in E(G)$. The *contraction* G/e is the graph obtained from G by identifying the two ends of e and deleteing the resulting loops. A *subdivision* of a graph H is a graph obtained from H by subdividing some edges of H, and will be denoted by TH. As in [BM], a *block* in a 2-edge-connected graph G is a maximal 2-connected subgraph. For a real number x, $\lfloor x \rfloor$ denotes the largest integer not bigger than x.

Theorem A. *(Dirac [D]) If G is a nontrivial simple graph without TK_4, then G has a vertex of degree at most 2.* ∎

Let C be a collection of cycles in a graph G. If

$$E(G) \subseteq \bigcup_{c \in C} E(C),$$

then C is called a *cycle cover* of G. It is well known that G has a cycle cover if and only if G has no cut-edges. For a 2-edge-connected graph G, let $cc(G)$ denote the minimum number of cycles in G that are needed to cover $E(G)$. In [B], Bondy conjectured that if G is a 2-edge-connected simple graph with n vertices, then

$$cc(G) \leq \frac{2n-1}{3}.$$

In this note we shall prove that if G is a 2-edge-connected simple graph with n vertices and without TK_4, then

$$cc(G) \leq \frac{2n-2}{3}, \tag{1}$$

and we shall characterize all the extremal graphs and thereby show that the bound in (1) is sharp.

Let $k \geq 1$ be an integer. A graph G is *mod* $(2k+1)$*-orientable* if it has an orientation such that the out-degree of each vertex is congruent (modulo $2k+1$) to the in-degree. (See [J] for further discussion on this subject). Following Jaeger [J], we denote by M_{2k+1} the class of mod $(2k+1)$-orientable graphs. It is observed in [SY] and in [J] that $G \in M_3$ if and only if G has nowhere-zero 3-flows, (see [J] or [Y] for flows). In this note, we shall show that if G is 2-edge-connected and if G does not contain a TK_4, then $G \in M_{2k+1}$, for any $k \geq 1$.

Main Results

Let G be a simple graph. An *arc* of G is an (x,y)-path P of G with $x, y \in V(G)$, where x may equal y, such that all the internal vertices of P have degree 2 in G. A *maximal* arc is one that cannot be extended in G. The length of an arc P is $|E(P)|$. We regard K_2 as an arc of length 1.

Let $\mathcal{A}(G)$ denote the collection of all maximal arcs A with $|E(A)| \geq 2$. For any $A \in \mathcal{A}(G)$, A is a *cycle arc* is $G[E(A)]$ is a cycle in G; A is a *cyclic arc* if $G[E(A)]$ is not a cycle but there is an arc A' in G such that $G[E(A) \bigcup E(A')]$ is a cycle in G; and A is an *acyclic arc* if A is neither a cycle arc nor a cyclic arc.

For each $A \in \mathcal{A}(G)$, define $b_G(A)$ as follows: if A is a cycle arc, then $b_G(A) = |E(A)| - 3$; if A is a cyclic arc, then $b_G(A) = |E(A)| - 2$; and if A is acyclic, then $b_G(A) = |E(A)| - 1$. Note that by Theorem A, if a simple graph G satisfies $\kappa'(G) \geq 2$, and has no TK_4, then $\mathcal{A}(G) \neq \emptyset$. Define

$$b(G) = \sum_{A \in \mathcal{A}(G)} b_G(A).$$

Let $t \geq 3$ and $s_t \geq \cdots \geq s_2 \geq s_1 \geq 1$ be integers. Let the t arcs of length 2 of $K_{2,t}$ be labeled by $A_1, A_2, \ldots, A_t$. Define $K_{2,t}(s_1, \ldots, s_t)$ to be the graph obtained from $K_{2,t}$ by replacing A_i by a path of length s_i, $(1 \leq i \leq t)$. For convenience, we regard a cycle of length $s_1 + s_2$ as a $K_{2,2}(s_1, s_2)$.

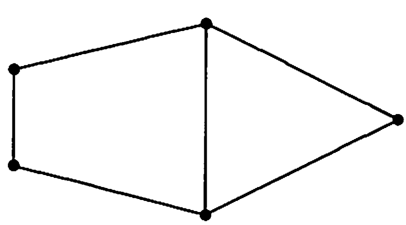

Figure 1: $K_{2,3}(1,2,3)$.

Let $\mathcal{K}$ denote the collection of graphs such that $G \in \mathcal{K}$ if and only if each block of G is a $K_{2,3}(1, s_2, s_3)$, for some $s_3 \geq s_2 > 1$. Let $\mathcal{K}'$ denote the subcollection of $\mathcal{K}$ such that $G \in \mathcal{K}'$ if and only if each block of G is a $K_{2,3}(1,2,2)$. Note that by definition, every graph in $\mathcal{K}$ is simple.

Theorem 1. *Let G be a 2-edge-connected simple graph with n vertices. If G has no TK_4,*

$$cc(G) \leq \frac{2\,(n-1-b(G))}{3}, \qquad (2)$$

where equality holds if and only if $G \in \mathcal{K}$. Moreover, if $b(G) = 0$, then equality holds in (2) if and only if $G \in \mathcal{K}'$.

Theorem 2. *Let G be a 2-edge-connected graph. If G has no TK_4, then for any integer $k \geq 1$, $G \in M_{2k+1}$.*

The Proofs

Lemma 1. *[LL] Let G be a 2-connected graph without TK_4. Then either G is a cycle or G is the union of two subgraphs G_1 and G_2 such that the intersection of G_1 and G_2 is an arc in G of length at least 1 and such that $\kappa'(G_1) \geq 2$ and $\kappa'(G_2) \geq 2$.* ∎

Let H be a subgraph of G. The set of all vertices in $V(H)$ that are incident with at least one edge in $E(G) - E(H)$, denoted by $A_G(H)$, is called the *vertices of attachment* of H in G. If $H = K_{2,t}(S_1, s_2, \ldots, s_t)$ is a subgraph of G such that either $G = H$ or $A_G(H)$ consists of two vertices of degree t in H, then H is called a *$K_{2,t}$-block* of G.

Lemma 2. *Let G be a 2-connected graph without TK_4. Then for some $t \geq 2$, G has a $K_{2,t}$-block.*

Proof: We argue by induction on $|V(G)|$. Assume that G is not a cycle (in which case $G = K_{2,2}(s_1, s_2)$). By Lemma 1, G is the union of G_1 and G_2 such that the intersection of G_1 and G_2 is an arc of length at least 1. By induction, either G_1 or G_2 contains such a subgraph H, or both G_1 and G_2 are cycles. If both G_1 and G_2 are cycles, then since the intersection of G_1 and G_2 is an arc in G, G must be a $K_{2,3}(s_1, s_2, s_3)$, and so Lemma 2 follows In any case. ∎

Lemma 3. *Let G be a 2-edge-connected graph and let G_1 and G_2 be two subgraphs of G such that*

$$G = G_1 \bigcup G_2 \text{ and } V(G_1) \bigcap V(G_2) = \{v\}.$$

Then $cc(G) = cc(G_1) + cc(G_2)$.

Proof: By definition, we have $cc(G) \leq cc(Gl) + cc(G2)$. Conversely, since G_1 and G_2 are separated by a single vertex v, any cycle cover of G induces cycle covers of G_1 and of G_2, and so $cc(G) \geq cc(Gl) + cc(G2)$. ∎

Lemma 4. *Let G be a 2-edge-connected graph and let $A \in \mathcal{A}(G)$ and $e \in E(A)$. Then*

$$cc(G) = cc(G/e).$$

Proof: Since A is an arc of length at least 2, any cycle containing an edge in A contains all edges in A. ∎

Proof of Theorem 1: We argue by induction on $n = |V(G)|$, and so we may assume that G is not a cycle.

Suppose that $\kappa(G) = 1$ and so there are two nontrivial subgraphs H_1, H_2 of G such that $|V(H_1) \bigcap V(H_2)| = 1$. Note that by definition, $b(G) \leq b(H_1) + b(H_2)$ and so by induction and by Lemma 3,

$$\begin{aligned} cc(G) &= cc(H_1) + cc(H_2) \\ &\leq \sum_{i=1}^{2} \frac{2|V(H_i)| - 2 - 2b(H_i)}{3} \\ &\leq \frac{2n - 2 - 2b(G)}{3}. \end{aligned} \tag{3}$$

If $cc(G) = (2n - 2 - 2b(G))/3$, equalities hold in (3) everywhere and so by induction, both H_1 and H_2 are in $\mathcal{K}$. It follows that $G \in \mathcal{K}$. Thus we may assume that

$$\kappa(G) \geq 2. \tag{4}$$

If G is a cycle, then Theorem 1 holds trivially. Thus by (4) we may also assume that

$$G \text{ has no cycle arcs.} \tag{5}$$

If $b(G) > 0$, then by (5), G has no cycle arcs and so G has either a cyclic arc A with $|E(A)| > 2$ or an acyclic arc A with $|E(A)| > 1$. Choose an edge $e \in E(A)$. Then G/e is simple, and by the definition of $b(G)$,

$$b(G) - 1 = b(G/e). \tag{6}$$

By induction, by Lemma 4 and by (6),

$$cc(G) \leq \frac{2(n-1) - 2 - 2b(G/e)}{3} = \frac{2n - 2 - 2b(G)}{3}. \tag{7}$$

Again, if $cc(G) = (2n - 2 - 2b(G))/3$, then equalities hold everywhere in (7) and so by induction, each block of G/e is in $\mathcal{K}$. Let $L' = K_{2,3}(1, s_2, s_3)$ be the block in G/e that contains the vertex to which e is contracted, and let L be the preimage of L' under the contraction, (*i.e.* $L/e = L'$). If $L = K_{2,3}(2, s_2, s_3)$, then since $b(L) = s_2 + s_3 - 4$ and $|V(L)| = 1 + s_2 + s_3$,

$$\frac{2(|V(L)| - 1 - b(L))}{3} = \frac{8}{3} > 2 = cc(L).$$

Thus by Lemma 3, $cc(G) < (2n-2-2b(G))/3$, a contradiction. Hence L must be in $\mathcal{K}$, and so Theorem 1 is proved by induction in this case.

Hence we may assume that $b(G) = 0$. By a similar argument, we can assume that

$$\text{every arc in } \mathcal{A}(G) \text{ has length 2 and lies in a } K_3 \text{ of } G. \tag{8}$$

In fact, let A be an arc in $\mathcal{A}(G)$. By $b(G) = 0$, A is cyclic and of length 2. If A is not lying in a 3-cycle K_3 in G, then for any edge $e \in E(A)$, G/e is simple, and so by repeating the previous paragraph, we can conclude that

$$cc(G) < \frac{2(n-1)}{3}.$$

By Lemma 2, G has a maximal $K_{2,t}$-block $H = K_{2,t}(s_1, s_2, \ldots, s_t)$. Choose H so that t is maximized. By (7) and since G is simple, we may assume that

$$1 = s_1 < s_2 = \cdots = s_t = 2. \tag{9}$$

Suppose first that $G = H$. By (5), $t > 2$. Since $G = K_{2,t}(1, 2, \ldots, 2)$, $cc(G) = \lfloor (t+1)/2 \rfloor$. Note that $n = t + 1$ and $b(G) = 0$. Thus for $t \geq 3$,

$$cc(G) \leq \frac{t+1}{2} \leq \frac{2t}{3},$$

and equalities hold if and only if $t = 3$, which implies that $G \in \mathcal{K}'$. Hence we may assume that

$$G \neq H. \tag{10}$$

Suppose that $t \geq 3$. Let A_i, $(1 \leq i \leq t)$ denote the arc of length s_i in H and let $H' = G[\bigcup_{i=t-1}^{t} E(A_i)]$. By (9), H' is a cycle of order 4 in G. Let $G' = G - (V(H') - A_G(H))$. By (10), by (4) and by $t \geq 3$, $\kappa'(G') \geq 2$. Thus by $b(G) = 0$, by $|V(H')| = 4$ and by induction,

$$\begin{aligned} cc(G) &\leq cc(G') + 1 \\ &< \frac{2|V(G')| - 2}{3} + \frac{2|V(H')| - 2}{3} \\ &= \frac{2n-2}{3}. \end{aligned}$$

Hence $t = 2$ and so by the maximality of t and by (8), we may assume that in G,

$$\text{every maximal } K_{2,t}\text{-block is a } K_{2,2}(1, 2). \tag{12}$$

Since $G \neq H$, G/H is also simple and nontrivial. It follows by Theorem A that $|\mathcal{A}(G)| \geq 2$. Let A_1 and A_2 be two distinct arcs in $\mathcal{A}(G)$. By (8) and (12), each

A_i lies in a 3-cycle H_i and has exactly one vertex v_i of degree 2, and so $H_i - v_i$ contains exactly one edge e_i in $G - v_i$, $(1 \le i \le 2)$. By (12), $e_1 \ne e_2$. Since H_1 and H_2 are $K_{2,2}$-blocks of G and by (4), $G - \{v_1, v_2\}$ is also 2-connected, and so by Menger's Theorem ([BM], page 46), there is a cycle C' in $G - \{v_1, v_2\}$ that contains both e_1 and e_2. Let

$$C = G\left[E(C') \bigcup E(H_1) \bigcup E(H_2) - \{e_1, e_2\}\right]$$

Then C is a cycle in G containing v_1 and v_2.

Let $\mathcal{C}'$ be a cycle cover of $G - \{v_1, v_2\}$ such that

$$cc(G - \{v_1, v_2\}) = |\mathcal{C}'|.$$

Define $\mathcal{C} = \mathcal{C}' \bigcup \{C\}$. Then by the definition of C and $\mathcal{C}'$, $\mathcal{C}$ is a cycle cover of G and

$$cc(G) \le |\mathcal{C}| = |\mathcal{C}'| + 1.$$

Since $|V(G - \{v_1, v_2\})| = n - 2$, by induction and by $b(G) = 0$,

$$\begin{aligned} cc(G) &\le cc(G - \{v_1, v_2\}) + 1 \\ &\le \frac{2(n-2) - 2}{3} + 1 \\ &< \frac{2n - 2}{3}. \end{aligned} \tag{13}$$

Hence Theorem 1 is proved by induction. ∎

Proof of Theroem 2: We shall prove Theorem 2 by induction on the number of edges of G.

If G is a cycle, then any orientation that makes G a directed cycle will do. Hence we may assume that G is not a cycle.

If G has a cut-vertex v, then G has two subgraphs H_1 and H_2 with $G = H_1 \bigcup H_2$ and $V(H_1) \bigcap V(H_2) = \{v\}$. Since $\kappa'(G) \ge 2$ and since v is a cut-vertex, both $\kappa'(H_1) \ge 2$ and $\kappa'(H_2) \ge 2$. Hence by induction, $H_1, H_2 \in M_{2k+1}$ and so $G \in M_{2k+1}$.

Thus we may assume that $\kappa(G) \ge 2$. By Lemma 1 and since G is not a cycle, G is the union of two 2-edge-connected subgraphs G_1 and G_2 such that the intersection of G_1 and G_2 is an arc A of length at least 1 in G. Since $\kappa'(G_2) \ge 2$, G_1 has fewer edges than G and so by induction, $G_1 \in M_{2k+1}$. Similarly, $G_2 \in M_{2k+1}$.

Observation 1: If D is a mod $(2k + 1)$-orientation of a graph L, then D^-, then orientation obtained from D by reversing all directions in D, is also a mod $(2k + 1)$-orientation.

Observation 2: If D is a mod $(2k+1)$-orientation of a graph L, if A is an arc of length at least 1 in L, then under D, all edges in A have the same direction.

These two observations above are immediate from the definitions of arcs and of mod $(2k+1)$-orientations. Since both G_1 and G_2 are in M_{2k+1} and by the above two observations, we may assume that there are mod $(2k+1)$- orientations D_1 and D_2 such that both D_1 and D_2 agree on A, the arc in G commonly shared by G_1 and G_2. (If they do not agree, then by Observations 1 and 2, D_1 and D_2^- must agree). Thus we can combine D_1 and D_2 to obtain a mod $(2k+1)$-orientation of G and so $G \in M_{2k+1}$. ∎

References

[B] J. A. Bondy, *Small cycle covers of graphs*. Research Report CORR 88–40, University of Waterloo, (1988).

[BM] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier, New York (1976).

[D] G. A. Dirac, *A property of 4-chromatic graphs and some remarks on critical graphs*. J. London Math. Soc. 27 (1952) 85–92.

[J] F. Jaeger, *Nowhere-zero flow problems*, Selected Topics in Graph Theory, 3. eds. by L. W. Beineke and R. J. Wilson, Academic Press, London, (1988).

[LL] H.-J. Lai and H. Y. Lai, *Graphs without K_4-minors*, submitted.

[SY] R. Steinberg and D. H. Younger, *Grotzsch's theorem for the pro- jective plane*. Ars Combin. 28 (1989) 15–31.

[Y] D. H. Younger, *Integer flows*. J. Graph Theory 7 (1983) 349–357.