ON AFFINE AND PROJECTIVE FAILED DESIGNS

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Abstract. An affine (respectively projective) failed design D, denoted by AFD(q) (respectively PFD(q)) is a configuration of $v = q^2$ points, $b = q^2 + q + 1$ blocks and block size $k = q$ (respectively $v = q^2 + q + 1$ points, $b = q^2 + q + 2$ blocks and block size $k = q + 1$) such that every pair of points occurs in at least one block of D and D is minimal, that is, D has no block whose deletion gives an affine plane (respectively a projective plane) of order q. These configurations were studied by Mendelsohn and Assaf and they conjectured that an AFD(q) exists if an affine plane of order q exists and a PFD(q) never exists. In this paper, it is shown that an AFD(5) does not exist and, therefore, the first conjecture is false in general, AFD(q^2) exists if q is a prime power and the second conjecture is true, that is, PFD(q) never exists.

1. Introduction.

In [3] Mendelsohn and Assaf defined an imbrical design ID($v, k, 1; b$) to be a configuration D of v points, b blocks, block size k such that D is a covering design, that is, every point-pair of D occurs in at least one block but D is *minimal* w.r.t. this property, that is, deletion of a block of D does not result in a covering design for any block of D or equivalently every block of D contains one point-pair which does not occur in any other block. In that paper the authors determined all the possible values of b such that an ID($v, k, 1; b$) exists with $k = 3$ and 4 (with a finite number of values of b still in doubt). In the same paper, the authors defined a failed design FD(v, k) to be an ID($v, k, 1; b$) say D with $b = \frac{v(v-1)}{k(k-1)} + 1$, that is, D has one more block than a ($v, k, 1$) BIBD. Note that the definition given in [3] is (numerically) incorrect.

In particular, an affine failed design AFD(q) is just an FD(q^2, q) while a projective failed design PFD(q) is just an FD($q^2 + q + 1, q + 1$). These are minimal covering designs with the same number of points as an affine (respectively projective) plane but with one extra block and with the property that no block can be deleted to produce an affine (projective) plane. In [3], the authors made the following two conjectures:

AFD Conjecture. *AFD(q) exists if an affine plane of order q exists.*

PFD Conjecture. *PFD(q) does not exist for any value of q.*

In this paper, *the AFD conjecture is shown to be false by showing that AFD(5) does not exist.* This is shown in the last section 5 using structural results on AFD(q) obtained in section 4. On the other hand, *section 2 shows that AFD(q^2) exist for all prime powers q* using a construction obtained through Baer subplane of a projective plane of order q^2. *Also in section 3 we show that the PFD conjecture is true.*

2. A construction of affine failed designs.

We recall the definition: An *affine failed design* AFD(q) is a configuration D of $v = q^2$ points, $b = q^2 + q + 1$ blocks, block size $k = q$ such that every point-pair of D is covered at least once and D is minimal w.r.t. this property, that is, D has no block whose deletion will result in an affine plane of order q; equivalently every block of D contains a point-pair occurring precisely once.

In [3], AFD(3) and AFD(4) were constructed and these will be shown to be unique in section 5 of our paper. Here we prove

Theorem 2.1. *(a) An AFD(q^2) exists for every prime power q.*
(b) If there exists a pairwise balanced design PBD $((q^4 - q), \{q^2, q^2 - q\})$ D (with block sizes q^2 and $q^2 - q$) such that D has $q^2 + q + 1$ blocks of size $q^2 - q$ forming a spread (parallel class partitioning the point-set) then there exists an AFD (q^2).
(c) If there exists a projective plane Π of order q^2 containing a Baer subplane Π^* (of order q) then a PBD $(q^4 - q, \{q^2, q^2 - q\})$ as stipulated in (b) exists.

Proof: (a) follows from (b) and (c) after noting that a projective plane as stipulated in (c) exists. Consider (b) and suppose a PBD as in (b) exists. Then construct a new configuration $\overline{D}$ as follows: let S be a set of q points not in D. The point-set of $\overline{D}$ is a union of the point-set of D and S. All the blocks of size q^2 of D are blocks of $\overline{D}$ and for every block B of D, $B \cup S$ is a block of $\overline{D}$. It is easily checked that $\overline{D}$ is an AFD(q^2).

Consider (c). Delete the points of Π^* from Π to get a PBD (with truncated blocks) as required. This PBD has a spread of lines of size $q^2 - q$ since these lines form lines of Π^* and, hence, do not meet in $\Pi \setminus \Pi^*$ (also note that every point of $\Pi \setminus \Pi^*$ is on a unique line of Π^*). ∎

3. Projective failed designs do not exist.

We recall from [3] that a projective failed design PFD(q) is a configuration D of $v = q^2 + q + 1$ points, $b = q^2 + q + 2$ blocks, block size $k = q + 1$ such that every point-pair of D is covered by at least one block of D but D has no block whose deletion produces a projective plane of order q; equivalently every block of D contains a point-pair covered by that block alone. In [3], it was shown that PFD(2) and PFD(3) do not exist. This section is devoted to the proof of

Theorem 3.1. *For every q, PFD(q) does not exist.* Hence, the PFD conjecture is true.

Theorem 3.1 will be proved after proving several 'assertions'. Our notations and terminology are simple: x, y, z, etc., denote points, and X, Y, Z, etc., denote blocks. For a point x, $r(x)$ denotes the number of blocks containing x and for a point-pair (x, y), $r(x, y)$ denotes the number of blocks containing x and y. *From this point on, let D denote a PFD(q). We show that this leads to a contradiction.*

Assertion 1. *For every point* x, $r(x) \geq q+1$ *with equality if and only if* $r(x,y) = 1$ *for all* $y \neq x$.

Proof: Trivial after noting that $v-1 = q^2+q$, $k-1 = q$ and $r(x,y) \geq 1$ for all $y \neq x$. ∎

A point x will be called *regular* if $r(x) = q+1$ and irregular otherwise. A point-pair (x,y) is called *regular* if $r(x,y) = 1$ and irregular otherwise. Let R (*respectively* I) denote the *set of regular* (*respectively irregular*) points of D. Clearly an irregular pair has both points irregular because for a regular point x, $r(x,y) = 1$ for all $y \neq x$. Finally, call a *block regular if it is contained in* R.

Assertion 2. $|I| \leq q+1$ *with equality if and only if* $r(x) = q+2$ *for all* $x \in I$.

Proof: A two-way counting produces $\Sigma\ r(x) = bk = q^2 + (q+2)(q+1)$.

So $\Sigma(r(x) - (q+1)) = q+1$ since $v = q^2+q+1$. By Assertion 1, every summand is non-negative and is positive if and only if $x \in I$. So $|I| \leq q+1$ and with equality if and only if $r(x) - (q+1) = 1$ for all $x \in I$. ∎

Assertion 3. *D has a regular block.*

Proof: Suppose not. Let $x \in R$. Then the $q+1$ blocks on x contain at least one point of I. Since for all points y other than x, $r(x,y) = 1$, we have $|I| \geq q+1$ which by Assertion 2 implies that $|I| = q+1$ and every point z of I has $r(z) = q+2$. Since equality holds everywhere, our argument also shows that all the $q+1$ blocks on $x \in R$ contain exactly one point of I. Therefore, every block containing some point of R contains just one point of I. Since the number of point-pairs of $R \times I$ covered by such a block is q while $R \times I$ has cardinality $q^2(q+1)$, it follows that there are precisely $q(q+1)$ such blocks (note that every pair in $R \times I$ is regular). The remaining two blocks say I_1 and I_2 must be completely contained in I. So I_1 and I_2, as point-sets, are both equal to I. Deletion of I_2 then clearly leaves us with a projective plane of order q, which is a contradiction. ∎

Assertion 4. *The set of points of* I *forms a block (say* I^**).*

Proof: Let C be a regular block given by Assertion 3. For a point $x \in R$, $x \notin C$, every block on x meets C in at most one point (because C contains regular pairs) and $r(x) = q+1 = |C|$. So x is on no block disjoint from C. How many blocks are disjoint from C? These are $(b-1) - q(q+1) = 1$ in number. Our argument shows that the unique blocks I^* disjoint from C contains no point of R. So $I^* \subseteq I$ and $|I^*| = q+1 \geq |I|$ by Assertion 2. Therefore, $I^* = I$ as desired. ∎

Assertion 5. *Let* $x \in I$. *Then there is a block* X *such that* $X \cap I = \{x\}$.

Proof: By Assertion 4, $|I| = q+1$ and by Assertion 2, $r(x) = q+2$. So $|R| = q^2$ and one block containing x is I^*. If every other block containing x contains one more point of I then $q^2 = |R| \leq (r(x)-1)\cdot(q-1) = q^2-1$, a contradiction. So there is a block X on x for which $X \cap I = \{x\}$. ∎

Proof of Theorem 3.1: It is sufficient to show that for all $x, y, \in I, r(x,y) \geq 2$ holds, since in that case we can delete the block I^* to obtain a PBD D^* with block size $q+1$ which is clearly a projective plane of order q. For $x \in I$, let X be a block with $X \cap I = \{x\}$ given by Assertion 5. Every point-pair of X is regular. So no block meets X in more than one point. But for $y \neq x, y \in I$, Assertion 4 and Assertion 2 imply that $r(y) = q+2 = 1+|X|$. Our argument also shows that X meets $(q+1) + q \cdot q = q^2 + q + 1$ blocks, that is, no block is disjoint from X. Since $r(y,z) \geq 1$ for all $z \in X$, we have $r(y,z) = 2$ for precisely one $z \in X$. Clearly, then z is irregular. But $X \cap I = \{x\}$. So $z = x$ and $r(y,x) = 2$ as desired. ∎

4. Affine failed designs revisited.

We again recall the definition of an affine failed design AFD(q) given in section 2 to be a covering design D with q^2 points, q^2+q+1 blocks, block size q such that no block of D can be deleted to obtain an affine plane of order q. Since AFD(2) does not exist, *throughout this section D will stand for an AFD(q) with $q \geq 3$.* This section is devoted to the structural investigations of D. Our terminology is borrowed from section 3 (paragraphs 2 and 5) and is, therefore, only partially repeated. Our main theorems give some necessary conditions on D which will be used in studying AFD(q) with $q = 3, 4$ and 5 in section 5.

Lemma 4.1. *For every $x, r(x) \geq q+1$ with equality if and only if $r(x,y) = 1$ for all $y \neq x$.*

Proof: Trivial after noting that $r(x,y) \geq 1$ and $v - 1 = q^2 - 1, k - 1 = q - 1$. ∎

A point-pair (x,y) is called regular if $r(x,y) = 1$ and a point x is called regular if $r(x) = q+1$. As in section 3, R and I denote the sets of regular and irregular points respectively. Finally a block X is regular if $X \subseteq R$.

Lemma 4.2. $|I| \leq q$ *with equality if and only if $r(x) = q+2$ for all $x \in I$.*

Proof: A two-way counting produces $\Sigma(r(x) - (q+1)) = bk - v(q+1) = q$ where, by Lemma 4.1 every summand on the L.H.S. is non-negative and is positive if and only if $x \in I$. Hence, the proof. ∎

Proposition 4.3. *Let X be a block such that $X \cap I = \{x\}$. Let $X_i, i = 2, 3, \ldots, s$ be the set of (all the) $s-1$ blocks disjoint from X. Write $\overline{X} = \{X_1 = X, X_2, \ldots, X_s\}$. Then the following assertions hold.*

- (a) *If y is a regular point, then y is on a unique $X_i, i = 1, \ldots, s$.*
- (b) *For $i \neq j$, $X_i \cap X_j$ is contained in I.*
- (c) $s = q$.
- (d) *If some X_i is a regular block then the set of points of I forms a block say I^* and then for all $z \in I, r(z) = q+2$.*
- (e) *If X_i is irregular for every i then $|X_i \cap I| = 1$ for every i, $|I| = q$ and the set $\overline{X}$ forms a spread, that is, the blocks of $\overline{X}$ partition the point-set of D.*

Proof: Since X contains no irregular point-pair, no block on y meets X in two or more points. Now $r(y) = q + 1 = 1 + |X|$ proves (a) after noting that $r(y, z) = 1$ for all $z \in X$; (b) is clearly a consequence of (a). If $s \leq q - 1$ then the blocks of $\overline{X}$ can cover at most $(q-1)q - 1 = q^2 - q - 1$ points of R because $|X_1 \cap R| = q - 1$. By Lemma 4.2, $|R| \geq q^2 - q$ and by (a) and (b) every point of R is on a unique member of $\overline{X}$. This contradiction shows that $s \geq q$. On the other hand, count the number of blocks meeting X. Since no block meets X in two or more points and since $r(x) \geq q + 2$ this number is at least $(q - 1)q + (q + 1) = q^2 + 1$. So the number $s - 1$ of blocks disjoint from X is at most $(b - 1) - (q^2 + 1) = q - 1$, that is, $s \leq q$. Therefore, $s = q$ and (c) is proved. Also the equality $s = q$ implies $r(x) = q + 2$ (else $s < q$). Now consider (d). Let $X_i = y$ be regular. Then the number of blocks meeting Y is $q \cdot q = q^2$ and, hence, Y is disjoint from $(b - 1) - q^2 = q$ blocks. For a point $y \notin Y$ and $y \in R$, y is on a unique block not meeting $Y = X_i$ which must be some X_j by (a) and (b). Thus, $X_j, j \neq i$ are $q - 1$ blocks disjoint from Y and the unique remaining block $I^\star$ disjoint from Y can not contain any regular point. Therefore, $I^\star \subseteq I$. But $|I^\star| = q$ and $|I| \leq q$ by Lemma 4.2. This proves (d).

Finally, consider (e). By assumption $|X_i \cap R| \leq q - 1$ for all i and since X_i's partition R, Lemma 4.2 gives $q^2 - q \leq |R| = \Sigma |X_i \cap R| =\leq q(q-1)$. Therefore, equality must hold everywhere and $(X_i \cap R) = q - 1$ for all i and $|R| = q^2 - q$. So $|X_i \cap R| = q - 1$ for all i and $|I| = q$. Hence, by Lemma 4.2, $r(z) = q + 2$ for all $z \in I$. Let $z \neq x$ and $z \in I$. Since no block meets X in two points, z is on at most two blocks not meeting X. Suppose that is the case and let z *be on* $X_i \cap X_j, i \neq j$. *We show that this leads to a contradiction.* Since $q \geq 3$, X_j has distinct points $y, w \in R$. For any X_k, $k \neq i, j$, $X_k \cap X_i$ is contained in I. But $X_i \cap I = \{z\}$. If $X_k \cap X_i$ is non-empty then $X_k \cap X_i = \{z\}$. But then z will be on three blocks disjoint from X, a contradiction. This contradiction shows that X_k is disjoint from X_i, $k \neq i, j$, $k = 1, 2, \ldots, q$. In the present proposition application of (a) through (e) with X replaced by X_1 shows that $X_i = Y$ is disjoint from exactly $q - 1$ blocks and $q - 2$ of these are given by X_k, $k \neq i, j$. Let Z be the remaining unique block disjoint from X_i. By (a) (with X replaced by Y) every regular point is either on some X_k, $k \neq j$ or is on Z. Since $y, w \in X_j$, y, w are not on any X_k, $k \neq j$. So $y, w \in Z$. But then $r(y, w) = 1$ implies $Z = X_j$, which is a contradiction.

This contradiction shows that each $z \in I$ is on at most one member of $\overline{X}$. But every member of $\overline{X}$ meets I in a single point and $|\overline{X}| = |I| = q$. Therefore, every point of I is on a unique X_i and conversely every X_i meets I in a unique point. This completes the proof of (e). ∎

Theorem 4.4. *D has no block $I^\star$ such that $I^\star = I$ (as a point-set).*

Proof: Suppose D has a block $I^\star = I$. Then by Lemma 4.2, $r(x) = q + 2$ for all $x \in I$. If every block on x contains at least two points of I (note that $I^\star$ contains no point of R) then $|R| \leq (q+2-1)\cdot(q-2) = q^2-q-2$ which is a contradiction of Lemma 4.2. So there is some block X for which $X \cap I = \{x\}$. Form the set $\overline{X}$ of q blocks exactly as in Proposition 4.3. ∎

Case 1. $\overline{X}$ has no regular block.

By Proposition 4.3(e), every y in I with $y \notin X$ is on exactly one block disjoint from X. Since $r(y) = q+2$ and since no block meets X in two or more points we have $r(y,z) = 2$ for some $z \in X$. Clearly such a z must be irregular. But $X \cap I$ contains the only point x. So $z = x$ and $r(x,y) = 2$.

Case 2. $\overline{X}$ has a regular block say $X_i = Y$.

Then the number of blocks disjoint from Y is $(q^2+q) - q^2 = q$ and of these $q-1$ are given by $X_k, k \neq i, k = 1,2,\ldots,q$ by Proposition 4.3(b). The remaining block disjoint from Y is, of course, I^*. Let $y \in I, y \neq x$ and suppose y is on two blocks X_j and X_k disjoint from X. Since X_j, X_k meet I, they are not regular. So X_j, X_k are also disjoint from $Y = X_i$. Thus, y is on three blocks (including I^*) disjoint from Y. But no block meets Y in two or more points and, therefore, $r(y) \geq |Y| + 3 = q+3$, a contradiction since $r(y) = q+2$ holds by Proposition 4.3(d). So y is on at most one block disjoint from X. Again no block meets X in more than one point and $r(y) = q+2 = |X|+2$. Therefore, there is some $z \in X$ for which $r(y,z) \geq 2$. But then $z \in X \cap I = \{x\}$. So $z = x$ and $r(x,y) \geq 2$.

In both the cases we have proved that $r(x,y) \geq 2$ holds for every point-pair (x,y) in I. Clearly then deletion of the block I^* from D results in a *smaller covering* D^* with q^2 points and q^2+q blocks (and block size q). It is easily seen that D^* is an affine plane of order q which contradicts the assumption that D is an AFD.

Theorem 4.5. *D contains no block pair (X,Y) such that X is disjoint from Y, $|X \cap I| = 1$ and Y is regular.*

Proof: If D contained such a block pair then Proposition 4.3(d) implies the existence of a block I^* which is prohibited by Theorem 4.4. ∎

Theorem 4.6. *Let $S = \{X \mid X$ is a block of D meeting I in a single point$\}$. Assume that S is non-empty and define a relation $||$ on S by: $X||Y$ if $X = Y$ or X and Y are disjoint. Then the following assertions hold.*

(a) *$||$ is an equivalence relation on S.*
(b) $|I| = q$.
(c) *Every $||$ class partitions the point-set of D.*
(d) *q divides $|S|$.*
(e) *If $|S| = s$ then every regular point is on s/q blocks of S.*
(f) $|S| \neq q^2$.
(g) $|S| \neq q^2 - q$.

Proof: (a) Let $X \in S$ and suppose $Y, Z \in S$ such that $Z||X||Y$ and $Z \neq Y$. From the set $\overline{X}$ as in Proposition 4.3 and by Theorem 4.5 no member of $\overline{X}$ is regular.

Since Y, Z are in S and are $\|$ to X, Y, Z belong to $\overline{X}$ and by Proposition 4.3(e) members of $\overline{X}$ are pairwise disjoint. So Z and Y are disjoint, that is, $Z \| Y$. (b) is clear from Proposition 4.3(e). For (c) observe that the $\|$ class of X is actually $\overline{X}$ (from the discussion in (a)) and use Proposition 4.3(e). (d) is then obvious since every $\|$ class has q blocks. For (e), notice that any regular point X is on a single block of every $\|$ class of S. Consider (f) and suppose $|S| = q^2$. Then every regular point x is on q members of S and these blocks pairwise intersect in x alone. So x is on no other block meeting I. But $r(x) = q + 1$. So x is on a unique regular block. Since this is true for every $x \in R$ and since $|R| = q^2 - q$, D has $(q^2 - q)/q = q - 1$ regular blocks. It is also clear that no blocks except those in S contain points of both R and I. Therefore, D has $b - (|S| + q - 1) = 2$ blocks say I_1 and I_2 which are equal to I as point-sets. Deletion of I_2 from D then clearly results in an affine plane of order q, a contradiction.

Consider (g). If $|S| = q^2 - q$ then every point $x \in R$ is on $q - 1$ blocks meeting I. These blocks certainly do not intersect in I and cover exactly $q - 1$ points of I. But $|I| = q$ and, therefore, there is one more block say X containing x such that $X \cap I$ is non-empty and $X \notin S$. But then X must intersect I in two or more points and, hence, for some $Y \in S$ and $x \in Y$, $x \cap Y \cap I$ is non-empty which is a contradiction because x is a regular point. ∎

Theorem 4.7.

(a) $|I| \geq 2$. *If* $|I| = 2$ *then* $q = 4$.

(b) $|I| = 3$ *implies* $q = 3$ *or* $q = 9$.

(c) $|I| = 4$ *implies* q *is an even number between 6 and 16.*

Proof: In all the three cases, if there is a block with $|X \cap I| = 1$ then by Theorem 4.6, $|I| = q$. In (a) this is impossible since $q \geq 3$, by assumption. Therefore, in (a) every irregular block properly contains I and all such blocks do not meet in R and, hence, the set of these blocks partitions R. This shows that $q - 2$ divides $|R| = q^2 - 4$, that is, $q = 4$. Consider (b) and note again that if we have a block X meeting I in a single point then $q = 3$. Suppose no such block exists. Let $x \in R$. If x is on some block meeting I in two points then x must be on some (unique) block meeting I in the remaining single (third) point. This contradiction shows that every irregular block contains I properly and the set of all such blocks partitions R.

Hence, for a point $x \in R$, x is on one irregular block and q regular blocks. So the number of regular blocks is $(q^2 - 3) \cdot q/q = q^2 - 3$. Since $b = q^2 + q + 1$, the number of irregular blocks is $q + 4$ and these partition R. Therefore, $(q - 3)(q + 4) = q^2 - 3$, the solution of which is $q = 9$. This proves (b).

The proof of (c) is rather tedious and is divided into two major cases.

Case 1. $q = 4$.

Let a_{ij} denote the number of blocks meeting I in i points and R in j points,

$i + j = 4$. By Theorem 4.4, $a_{40} = 0$. Hence, counting of blocks, point-pairs in $I \times R$ and point-pairs in $R \times R$ obtains the following equations

$$a_{13} + a_{31} + a_{22} + a_{04} = b = 21 \tag{1}$$

$$3a_{13} + 3a_{31} + 4a_{22} = 4.12 = 48 \ldots \tag{2}$$

$$a_{22} + 3a_{13} + 6a_{04} = \binom{12}{2} = 66 \ldots . \tag{3}$$

Let S snd T respectively denote the sets of blocks meeting I in one (two) points. Then by Theorem 4.6(d), 4 divides $|S| = a_{13}$ and, hence, Equation (3) implies that 6 divides a_{22}. Therefore, Equation (2) yields $a_{22} = 0, 6$ or 12. First let $a_{22} = 0$. Then the unique solution of (1), (2) and (3) gives $a_{13} = 12 = q(q-1)$ which contradicts Theorem 4.6(g). Let $a_{22} = 6$. Then the unique solution of (1), (2) and (3) gives $a_{13} = 6$ which contradicts Theorem 4.6(d). Let $a_{22} = 12$ then (1), (2) and (3) imply $a_{13} = a_{31} = 0$ and $a_{04} = 9$. So D has 9 regular blocks. For a point $x \in R$, the blocks on x meeting I intersect I in two points. So x is on two such blocks and, hence, on $5 - 2 = 3$ regular blocks. It follows that any two regular blocks intersect in a single point and if R^* denotes the incidence structure with point-set R and block-set the set of regular blocks then R^* is dual of a $(9, 12, 4, 3, 1)$ BIBD, that is, R^* is dual of an affine plane of order 3. So there exist three 'parallel' points say x, y, z in R^*. Clearly then the blocks containing xy, xz, and yz are not regular and, hence, are members of T. But these blocks intersect in R and, therefore, cannot intersect in I. So $|I| \geq 3 \cdot 2 = 6$, a contradiction.

Case 2. $q \geq 5$.

By Theorem 4.6(b), D has no block meeting I in a single point. Let $x \in R$ such that x is on some block X meeting I in three points. Then the unique other block Y on x meeting I must have $|Y \cap I| = 1$, a contradiction. This contradiction shows that every irregular block meets I in 2 or 4 points and let x and y respectively denote the number of such blocks. The calculation of Lemma 4.1 shows that $\Sigma(r(p) - (q+1)) = bk - v(q+1) = q$ and a summand on the L.H.S. is positive if and only if $p \in I$. Therefore, restricting p to I obtains $\Sigma r(p) = q + 4(q+1) = 5q + 4$. Clearly the L.H.S. equals $2x + 4y$. So we have:

$$2x + 4y = 5q + 4 \ldots . \tag{4}$$

Similarly count double-flags to get $\Sigma(r(p,s) - 1) = bk(k-1) - v(v-1) = q(q-1)$. Again a summand on the L.H.S. is positive only if $p, s \in I$. So restricting p, s to I, $\Sigma r(p,s) = q(q-1) + 4 \cdot 3$. But the L.H.S. equals $2x + 12y$. Hence:

$$2x + 12y = q^2 - q + 12 \ldots \tag{5}$$

(4) and (5) can be solved to get

$$y = 1 + \frac{q(q-6)}{8} \ldots . \tag{6}$$

So q *is even*. Also every block containing 4 points of I contains $q-4$ points of R and all such blocks are clearly disjoint in R. Therefore, $y\cdot(q-4) \leq |R| = q^2-4$. Substitution of the value of y from (6) then leads to $q \leq 16$. Hence, $6 \leq q \leq 16$ and q is even as desired.

5. A disproof of the AFD conjecture and concluding remarks.

We make a modest beginning by proving

Theorem 5.1. *and AFD (3) and AFD (4) are unique.*

Proof: First note that both the configurations were constructed in [3] and AFD (4) is also constructed in our section 2. However, there is an error in the solution of AFD (3) given in [3]. Actually one block (out of 13) is missing and, hence, we give the following complete solution: The point set is the set of numbers 0 through 8 and the blocks are 147, 345, 057, 013, 048, 237, 156, 678, 258, 246, 026, 128, 368. In [3], the first block is missing. Also note that $I = \{2,6,8\}$.

The proof of Theorem 5.1 is easy: Lemma 4.2 implies $|I| \leq q$ and Theorem 4.7 implies that $|I| = 3$ for $q = 3$ and $|I| = 2$ for $q = 4$. For $q = 3$, it is easily seen that we have 4 regular blocks, 6 blocks meeting I in two points each and 3 blocks meeting I in a single point each such that these three blocks partition the point-set of D in view of Theorem 4.6(b). The uniqueness of AFD(3) is then easily established.

Let $q = 4$. Then $|I| = 2$. If D had a block meeting I in a single point, then by Theorem 4.6, $q-1$ divides $|R| = 14$, that is, 3 divides 14, a contradiction. So every irregular block contains both points of I and, hence, there are 7 such blocks. It is, therefore, sufficient to show that the PBD$(14, \{4,2\})$ $R^\star$ induced on R with 14 blocks of size 4 and 7 of size 2 forming a partition of R is unique. This is done as follows: Every block of size 4 of $R^\star$ is disjoint from a unique block of size 4 and three blocks of size 2 such that these 5 blocks are pairwise disjoint. Also we obtain 7 parallel classes, any block of size 4 contained in a unique parallel class while any block of size 2 occurs in three parallel classes and any block-pair (disjoint) occurs in a unique parallel class. Addition of 7 new points, one to blocks of each parallel class produces a projective plane Π of order 4 and it is easily seen that $R^\star$ is actually $\Pi \backslash \Pi^\star$ where $\Pi^\star$ is a Baer subplane of Π.

We now offer a disproof of the AFD conjecture (section 1) by showing that

Theorem 5.2. *AFD (5) does not exist.*

From this point on assume that D is an AFD(5), that is, has $v = 25$ points, $b = 31$ blocks each of size 5 such that no block of D can be deleted to produce an affine plane of order 5 and D is a covering design. *We show that this leads to a contradiction.*

By Lemma 4.2, $|I| \leq 5$ and Theorem 4.7 rules out $|I| = 2,3,4$. Therefore, $|I| = 5$. Let a_{ij} denote the number of blocks meeting I in i points and R in j

points $(i + j = 5)$. Then by Theorem 4.4, $a_{50} = 0$. Therefore, counting of blocks, point-pairs of $R \times I$, $R \times R$ and $I \times I$ covered by the blocks obtains:

$$a_{41} + a_{14} + a_{32} + a_{23} + a_{05} = 31 \ldots \tag{7}$$

$$4(a_{41} + a_{14}) = 6(a_{32} + a_{23}) = 20.5 = 100 \ldots \tag{8}$$

$$6a_{14} + a_{32} + 3a_{23} + 10a_{05} = \frac{20.19}{2} = 190 \ldots \tag{9}$$

$$6a_{41} + 3a_{32} + a_{23} = 20 \ldots \tag{10}$$

where the last equation is obtained as follows: we have the sum of various point pairs covered by blocks $= \frac{bk(k-1)}{2} = 310$. Now subtract (8) and (9) from this to get the point-pairs in $I \times I$ which obtains (10).

We have $a_{41} \leq 3$ in view of (10). If $a_{41} = 3$ then (10) gives $a_{32} = 0, a_{23} = 2$ which by (8) implies $a_{14} = 19$. *But Theorem 4.6(d) already tells us that* a_{14} *is a multiple of* 5. Hence, $a_{41} \neq 3$.

Let $a_{41} = 2$. Then (8) implies that $a_{14} \equiv 2 \pmod 3$ which using Theorem 4.6(d) and the fact that $a_{14} \leq 25$ given by (8) obtains $a_{14} = 5$ or 20. If $a_{14} = 5$ then (8) and (10) have no non-negative solution for a_{23} and a_{32}. Also $a_{14} = 20 = 5.4$ is ruled out by Theorem 4.6(g). Hence, $a_{41} \neq 2$.

Let $a_{41} = 1$. Then (8) implies that 3 divides a_{14} which by Theorem 4.6 (d) and (8) gives $a_{14} = 15$. Then the unique solution of (7) through (10) is: $a_{32} = 4$, $a_{23} = 2$ and $a_{05} = 9$, that is, D has 9 regular blocks. Therefore, some point p of D is on at least $\lceil 9 \cdot 5/20 \rceil = 3$ regular blocks. If S denotes the set of blocks meeting I in a single point (as in Theorem 4.6) then $|S| = a_{14} = 15$ and Theorem 4.6(e) tells us that p is on 3 blocks of S. Since $r(p) = 6$, p is on 3 regular blocks and 3 blocks of S. But $|I| = 5$ and, therefore, there are two points $t \in I$ for which $r(p,t) = 0$, a contradiction. Hence, $a_{41} \neq 1$.

Finally, let $a_{41} = 0$. Then (8) implies that $a_{14} \leq 25$ and $a_{14} = 1 \pmod 3$, which by Theorem 4.6(d) implies $a_{14} = 10$ or 25. The second possibility is ruled out by Theorem 4.6(f). So $a_{14} = 10$ and the unique solution of (7) through (10) gives: $a_{32} = a_{23} = 5$ and $a_{05} = 11$. Let T be the set of all the blocks meeting I in two or three points. Then $|T| = a_{23} + a_{32} = 10$. Let $p \in R$ such that $p \in X \cap Y$ and $X, Y \in T$, $X \neq Y$. We have, by Theorem 4.6(e) exactly two blocks on p which meet I in a single point. Since the blocks on p do not meet in I, $2 + |X \cap I| + |Y \cap I| \leq |I| = 5$. But $|X \cap I|$ and $|Y \cap I|$ is each at least 2. This contradiction shows that the blocks of T do not meet in R. Hence, $20 = |R| \geq 3a_{23} + 2a_{32} = 3.5 + 2.5 = 25$, a contradiction.

This completes the proof of Theorem 5.2. ∎

We end the paper with some questions.

(1) Theorem 5.2 shows that the AFD conjecture is not true in general, while Theorem 2.1 shows that AFD(q) exists for infinitely many values of q. It is perhaps true that AFD(q) does not exist if q is a prime, $q \geq 5$.

(2) Define an (s,r)-FN (failed net) D as follows: D is a configuration of $v = s^2$ points, $b = sr + 1$ blocks, block size $k = s$ such that given any point p, $r(p,x) \geq 1$ holds for at least $r(s-1)$ points x and such that the deletion of any block of D destroys the above property, that is, every block B contains a point-pair (p,x) such that with the deletion of B either p occurs with less than $r(s-1)$ points or x occurs with less than $r(s-1)$ points. When does an (s,r)-FN exist? Observe that an AFD is an $(s,s+1)$-FN.

Acknowledgement.

The author thanks the Department of Mathematics, Central Michigan University, Mt. Pleasant, Michigan for its hospitality while doing this work.

Notes added in the proof:

After writing this paper in October, 1989 and presenting it in the Fourth Carbondale conference on Combinatorics, the following two papers were brought to the author's attention.

(1) Z. Füredi in the paper *A projective plane is an outstanding 2-cover* Discrete Math., 74/3 (1989), 321–324, contains a proof of the PFD conjecture.

(2) A. Blokhuis, H.A. Wilbrink, C.A. Baker, and A.E. Brouwer in the paper *Characterization theorems for failed projective and affine planes* to appear in the IMA proceedings on Combinatorics, Springer-Verlag, give some results on the AFD conjecture. In particular, Blokhuis *et al* prove that if an AFD is not of the Baer type, then it must be a certain type of Bhaskara-Rao type design. However, that paper does not contain a proof of the statement that an AFD(5) does not exist (which is proved in the present paper).

References

1. C. J. Colbourn and A. Rosa, *Quadratic leaves of maximal partial triple systems*, Graphs and Combinatorics **2** (1986), 317–337.
2. D. R. Hughes and F. C. Piper, "Projective Planes, Springer Graduate Texts in Mathematics", Springer-Verlag.
3. E. Mendelsohn and A. Assaf, *Spectrum of imbrical designs, in: Combinatorial Design Theory*, C. J. Colbourn and R. A. Mathon, Eds., Annals of Discrete Math. **34** (1987), 363–370, North-Holland Math. Studies.
4. W. H. Mills, *On the covering of pairs by quadruples I*, Journ. Combin. Theory (A) **13** (1972), 55–78.
5. W. H. Mills, *On the covering of pairs by quadruples II*, Journ. Combin. Theory (A) **15** (1973), 138–166.