The Maximum Edge-Weighted Clique Problem in Complete Multipartite Graphs[1]

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Abstract. The problem we consider is: Given a complete multipartite graph G with integral weights on the edges, and given an integer m, find a clique C in G such that the weight-sum of the edges of C is greater than or equal to m. We prove that where G has k parts, each with at most two nodes, the edge-weights are 0–1, and $m = \binom{k}{2}$, this problem is equivalent to 2-conjunctive normal form satisfiability, and hence is polynomially solvable. However, if either each part has at most three nodes or m is arbitrary, the problem is NP-complete. We also show that a related problem which is equivalent to a 0–1 weighted version of 2-CNF satisfiability is NP-complete.

The maximum edge-weighted clique problem in complete multipartite graphs arises in transit scheduling, where it is called the schedule synchronization problem.

Let $V_1, V_2, \ldots, V_k$ be pairwise disjoint non-empty sets. The complete multipartite graph $G(V_1, V_2, \ldots, V_k)$ is defined to be the graph with node-set $\bigcup_{i=1}^{k} V_i$ and an edge between nodes u and v (denoted $[u, v]$) if and only if $u \in V_h, v \in V_i, i \neq h$.

A clique C in graph G is a subgraph of G such that if nodes $u, v \in C$, then edge $[u, v] \in C$.

We consider problems of the following form: Given a complete multipartite graph G with integral weights $w[u, v]$ on the edges, and given an integer m, find a clique C in G such that the weight-sum, $\sum_{[u,v]\in C} w[u, v]$, of the edges of C equals m or is greater than or equal to m.

Consider a complete multipartite graph $G(V_1, V_2, \ldots, V_k)$ with 0–1 edge weights. We will show that the problem of finding a clique of weight $\binom{k}{2}$ is polynomially solvable if $|V_i| \leq 2\ \forall i$, but NP-complete if $|V_i| \leq 3\ \forall i$. We will also show that given an integer m, the problem of finding a clique of weight $\geq m$ is NP-complete even if $|V_i| \leq 2\ \forall i$. One other related problem is shown to be NP-complete.

A Boolean expression B is said to be in conjunctive normal form (CNF) if it consists of a conjunction ("and") of disjunctions ("or") of literals (variables x_j or their negatives $\bar{x}_j$); that is, $B = \wedge_{i=1}^{k} (\vee_{\ell_j \in D_i} \ell_j)$ where D_i is a set of literals. A Boolean expression B is satisfiable if there is an assignment of true (T) and false (F) values to the variables such that $B = T$.

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The satisfiability problem is: Given a Boolean expression, it is satisfiable? A 2-CNF is a CNF with two literals per disjunction, and a 3-CNF is defined similarly. The 2-CNF satisfiability problem is polynomially solvable (e.g., see [2]), but the 3-CNF satisfiability problem is NP-complete [1].

Problem 1. *Given a complete multipartite graph* $G = G(V_1, V_2, \ldots V_k)$, *with* $|V_i| = 2\ \forall i$, *and with* 0–1 *weights* $w[u,v]$ *on the edges, find a clique of weight* $\binom{k}{2}$ *or determine that there isn't one.*

Theorem 1. *Problem* 1 *is equivalent to: given a* 2*-CNF, find a satisfying assignment or determine that there isn't one.*

Theorem 1 is proved using the following Construction 1 and its reverse, Construction 2.

Construction 1. *Given an instance of Problem* 1, *construct a* 2*-CNF as follows. For each* i, *create a variable* x_i, *and let the two nodes in* V_i *represent* $x_i = F$ *and* $x_i = T$. *For each edge* $[u,v]$ *of* G *with* $w[u,v] = 0$, *let* $D[u,v]$ *be the two-literal disjunction which would be false if the assignments corresponding to* u *and* v *were chosen. (For example,* $D[x_h = F, x_i = T] = x_h \vee \bar{x}_i$). *Let* $B = \wedge_{w[u,v]=0} D[u,v]$.

Construction 2. *Given a* 2*-CNF* $B = \wedge_{j=1}^{d} (\ell_{j_1} \vee \ell_{j_2})$ *where* $\ell_{j_1}, \ell_{j_2} \in \{x_i, \bar{x}_i : 1 \le i \le k\}$, *construct an instance of Problem* 1 *as follows. For each variable* x_i, *let* V_i *be a set of two nodes, one representing* $x_i = F$ *and the other* $x_i = T$. *Give all edges of* $G(V_1, V_2, \ldots, V_k)$ *weight* 1 *except, for each disjunction* $\ell_{j_1} \vee \ell_{j_2}$ *of* B, *give the edge which corresponds to* $[l_{j_1} = F, l_{j_2} = F]$ *weight* 0.

Claim 1. *In Construction* 1 *(or* 2*),* G *has a clique of weight* $\binom{k}{2}$ *if and only if* B *is satisfiable.*

Proof: Suppose C is a clique in G of weight $\binom{k}{2}$. Then $|C \cap V_i| = 1\ \forall i$, and all edges of C have weight 1. Assign x_i the value specified by $C \cap V_i$. Because all edges of C have weight 1, this assignment satisfies B.

Reversing this argument proves the "if" part of Claim 1. ∎

Proof of Theorem 1: Theorem 1 now follows from Constructions 1 and 2 and Claim 1. ∎

Construction 1 enables us to use any algorithm for finding a satisfying assignment for a 2-CNF to solve Problem 1. Knowing Construction 1, it is quite easy to transform known polynomial algorithms for 2-CNF satisfiability (e.g. see [2]) into polynomial algorithms for Problem 1. Any algorithm for Problem 1 is easily modified to solve the same problem where $|V_i| \le 2\ \forall i$.

Problem 2. *Given a complete multipartite graph* $G = G(V_1, V_2, \ldots, V_k)$ *with* $|V| = 2\ \forall i$, *and with* 0–1 *weights* $w[u,v]$ *on the edges, and given an integer* m, *is there a clique which hits each* V_i, *and has weight* $\ge m$?

Theorem 2. *Problem 2 is NP-complete.*

Proof: Problem 2 is clearly in NP. The following problem is NP-complete: Given a 2-CNF B, and given an integer q, find an assignment which satisfies at least q of the clauses [4]. This problem can be reduced to Problem 2 by using Construction 2 because:

Claim 2. *In Construction 2, G has a clique which hits each set V_i and which has weight $\binom{k}{2} - p$ if and only if B has an assignment in which all but p of the disjunctions are satisfied.* ∎

Where d is the number of disjunctions in B, choose $m = \binom{k}{2} - (d - q)$. ∎

Problem 3. *Given a complete multipartite graph $G = G(V_1, V_2, \ldots, V_k)$, with $|V_i| = 3 \ \forall i$, and with 0–1 weights $w[u,v]$ on the edges, is there a clique of weight $\binom{k}{2}$?*

Theorem 3. *Problem 3 is NP-complete.*

Proof of Theorem 3: Problem 3 is clearly in NP. Construction 3 below transforms any 3-CNF satisfiability problem into an instance of Problem 3. Since 3-CNF satisfiability is NP complete, so is Problem 3. ∎

Construction 3. *Given a CNF, $B = \wedge_{i=1}^{k}(\vee_{\ell_j \in D_i} \ell_j)$, let G be the complete multipartite graph $G = G(V_1, V_2, \ldots, V_k)$ where $V_i = \{(\ell_j, i) : \text{literal } \ell_j \in D_i\}$. Assign weights to the edges of G as follows:*

$$\text{For } i \neq h, \quad w[(\ell_j, i), (\ell_p, h)] = \begin{cases} 1 & \text{if } \ell_j \neq \bar{\ell}_p \\ 0 & \text{otherwise} \end{cases}$$

Claim 3. *There is a clique of weight $\binom{k}{2}$ in G if and only if B is satisfiable.* ∎

The referee pointed out that Construction 3 is essentially the same as the construction used in [5], p.522, to prove that the following problem is NP-complete: Given a graph H and an integer m, is there a clique in H with $\geq m$ nodes? The construction in [5] is, given a CNF B, construct G as in Construction 3, except delete all the edges of weight 0. Let m be the number of disjunctions in B. Then there is a clique with $\geq m$ nodes in the graph constructed if and only if B is satisfiable.

We remark that Construction 3 transforms any 2-CNF satisfiability problem into an instance of Problem 1, but not the same instance of Problem 1 that Construction 2 would produce. In Construction 3, if we consider the subgraph of G formed by the edges with weight 0, each connected component is a complete bipartite graph.

Problem 4. *Given a complete multipartite graph $G = G(V_1, V_2, \ldots, V_k)$ with $|V_i| = 2 \ \forall i$ with 0–1 weights on the edges and 0–1 weights on the nodes, and* *given an integer m, is there a clique in G with edge-weight $\binom{k}{2}$ and node-weight $\geq m$?*

Problem 4 is an extension of Problem 1 and hence of 2-CNF satisfiability. In the 2-CNF form what it says is: Given a 2-CNF B and for each variable x_i two 0–1 weights, one for setting $x_i = F$ and one for setting $x_i = T$, and given an integer m, is there a satisfying assignment such that the sum of the weights of the chosen values is $\geq m$?

Theorem 4. *Problem 4 is NP-complete.*

Proof: Problem 4 is clearly in NP. We will prove that it is NP-complete by reducing the following NP-complete problem to it: Given a graph H and an integer m, is there a clique in H with $\geq m$ nodes?

Construction 4. *Given a graph H with k nodes, and given an integer m, construct an instance of Problem 4 as follows. For each node i of H, let $V_i = \{u_i, v_i\}$, and let $G = G(V_1, V_2, \ldots, V_k)$. Give all edges of G weight 1 except edges of the form $[u_h, u_i]$ where $[h, i]$ is not an edge of H. Give the nodes u_i weight 1 and the nodes v_i weight 0.*

Claim 4. *H has a clique with $\geq m$ nodes if and only if the instance of Problem 4 constructed above has a yes answer.*

Proof of Claim 4: Let C be a clique in H with p nodes. Then $C' = \{u_i : i$ is a node of $C\} \cup \{v_i : i$ is not a node of $C\}$ is the node-set of a clique in G with edge weight $\binom{k}{2}$ and node-weight p.

Conversely, let C' be a clique in G with edge weight $\binom{k}{2}$ and node-weight p. Since all edges of C' have weight $1, C = \{i : u_i$ is a node of $C'\}$ is the node-set of a clique in H, and $|C| = p$. ∎

The maximum edge-weighted clique problem in a complete multipartite graph arises in transit scheduling; see [6], where it is called the schedule synchronization problem. The exact form of the problem given in [6] is: Given a complete multipartite graph $G(V_1, V_2, \ldots, V_k)$ with weights on the edges, find a minimum weight clique which hits each set V_i.

By modifying the proof of Theorem 2, it follows that this problem is NP- hard even if $|V_i| = 2$ $\forall i$ and the edge weights are 0 or 1.

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