Hyperovals in the Translation Planes of Order 16

William E. Cherowitzo

Department of Mathematics
University of Colorado at Denver
1200 Larimer Street
Denver, CO 80217-3364 (USA)

Abstract. The translation planes of order 16 have been classified by Dempwolff and Reifart [4]. Using this classification, and in particular the spreads given in that paper, we have conducted a complete computer search for the hyperovals (18-arcs) in each of these planes. With few exceptions, the hyperovals obtained are hyperbolic (having two points on the special line at infinity) and are of a type we call translation hyperovals. The only exceptions occur in the plane over the semifield with kernel $GF(2)$. In this plane there also appear a class of elliptic (having no points on the special line at infinity) hyperovals and two classes of hyperbolic hyperovals which are not translation hyperovals. The automorphism groups of the hyperovals are also determined.

1. Introduction

In a projective plane of order n, an *oval* is a set of $n+1$ points, no three of which are collinear. In a coordinatized Desarguesian plane, a *conic* is the set of points whose coordinates satisfy a non-degenerate quadratic equation. While every conic is easily seen to be an oval, the converse, proved in 1955 by B. Segre [9] for Desarguesian planes of odd order, is a rather surprising result. Segre's result cannot be extended to non-Desarguesian planes nor to Desarguesian planes of even order. Work on the classification problem for ovals in these planes has suffered from the lack of examples. Besides Segre's result the only complete determination of ovals has been made in small order planes. Conics are essentially (see below) the only ovals which arise in the unique planes of orders up to and including 8. The ovals have all been determined in the four planes of order 9 (for a survey, see Cherowitzo, Kiel, and Killgrove [2]). The only other plane in which this determination has been made is the Desarguesian plane of order 16 (Hall [6]). The current work extends this determination to the seven remaining translation planes of order 16.

In the even order case, every oval can be uniquely extended to a set of $n+2$ points, no three of which are collinear. These sets are called *hyperovals* and the additional point which is the intersection of all the tangents to the oval is called the *knot*. As an oval can be recovered from a hyperoval by the removal of any one of its points, it is clear that to determine all ovals we need only determine all hyperovals in planes of even order. It is in this sense that the word "essentially" was used above, i.e., in the Desarguesian planes of-even order up to 8, the conics determine the set of hyperovals. Hyperovals are sets of type $(0,2)$, that is, every line of the plane intersects a hyperoval in either 0 or 2 points. Lines can thus be classified as being either exterior or secant (respectively) with reference to a given hyperoval.

Besides the Desarguesian planes, the translation planes are the most widely studied class of projective planes. A translation plane $\mathcal{P}$ of order $q = p^n$ is a $2n$-dimensional vector space W over $GF(p)$ having a collection π (spread) $V_\infty, V_0, V_1, \ldots, V_{q-1}$ of n-dimensional pairwise disjoint subspaces which are called components. The lines of $\mathcal{P}$ different from ℓ_∞ are the cosets $w + U$ where $w \in W$ and U is a component. The points off of ℓ_∞ are the elements of W, while those on it can be identified with the components. Let V be an n-dimensional $GF(p)$-vector space and let $W = V \oplus V$. If, by a suitable choice of basis, we have $V_\infty = \{(0, v) \mid v \in V\}$, $V_0 = \{(v, 0) \mid v \in V\}$, and $V_1 = \{(v, v) \mid v \in V\}$, then there are elements $1 = t_1, t_2, \ldots, t_{q-1} \in GL(n, p)$ with $V_i = \{(v, vt_i) \mid v \in V\}$ for $i = 1, 2, \ldots, q-1$ and $t_i t_j^{-1}$ is a fixed point free transformation on V for $1 \leq i < j \leq q-1$. The set $M = \{1, t_2, t_3, \ldots, t_{q-1}\} \subseteq GL(n, p)$ is called a *coordinate set*.

Using the well known fact that $A_8 \approx GL(4, 2)$, Dempwolff and Reifart [4] were able to classify all the coordinate sets in A_8. They showed that there are exactly eight classes, giving rise to eight translation planes: the Desarguesian plane (DES), the semifield plane with kern $GF(2)$ (SEMI2), the semifield plane with kern $GF(4)$ (SEMI4), the Hall plane (HALL), the Lorimer-Rahilly plane (LMRH), the Johnson-Walker plane (JOWK), the derived semifield plane (DSFP), and the Dempwolff plane (DEMP). The translation planes other than the Desarguesian one are called proper translation planes and have the property that every collineation of the plane leaves the line ℓ_∞ invariant.

Two hyperovals are said to be *projectively equivalent* if there exists a collineation of the plane which maps one to the other. Our determination of hyperovals is up to projective equivalence, but as the results indicate, this may be too fine a classification; reflecting more the structure of the collineation group of the plane than the structure of the hyperovals.

In the non-Desarguesian translation planes the unique line at infinity may be either a secant line or an exterior line of a given hyperoval in the plane. This gives a crude classification of hyperovals as either *hyperbolic*, when the line at infinity is a secant line, or *elliptic*, when it is an exterior line. In the proper translation planes these types are projectively inequivalent.

2. Translation Hyperovals

In anticipation of the results, we will define and examine some of the properties of a special type of hyperoval found in each of the planes that were examined. For definitions and results not included here, a general reference is Dembowski [3].

If Ω is an oval in a projective plane, a line g is called a *symmetry line of* Ω if for each pair of distinct points $A, B \in \Omega \backslash g$ there exists an elation τ with axis g which interchanges A and B and leaves Ω invariant.

Proposition 1. *If an oval Ω in a projective plane of order n ($n \geq 4$) has a symmetry line g, then n is even and g is a tangent to Ω.*

Proof: Consider the tangent line h to Ω at a point $P \in \Omega \backslash g$, and let $Q = h \cap g$. Suppose that there is a secant line $\neq g$ through Q, with oval points C and D. Then the elation $\tau : C \leftrightarrow D$ has center Q and is not the identity. Since τ leaves Ω invariant, P must be a fixed point of τ, but this implies that P lies on g which is a contradiction. Thus no secant line (other than possibly g) can pass through Q. Since $n \geq 4$, there must exist more than two tangents to Ω which pass through Q. Therefore, n is even and Q is the knot of Ω. Since only tangents pass through the knot of an oval, g must be a tangent line. ∎

Thus it is natural to define a *translation oval* as an oval in a projective plane of even order having a tangent as a line of symmetry.

Proposition 2. *In a proper translation plane the line of symmetry of a translation oval is ℓ_∞.*

Proof: Let Ω be a translation oval with line of symmetry g in a translation plane. Let τ be an elation with axis g fixing Ω. Being a collineation, τ must leave ℓ_∞ invariant in a strict translation plane. Thus the center of τ is $g \cap \ell$. Let A, B, and C be distinct points of $\Omega \backslash g$. Let σ be the elation so that $\sigma : A \leftrightarrow B$ and let ρ be the elation such that $\rho : A \leftrightarrow C$. The centers of σ and ρ are distinct since A does not lie on g and they must both appear on g and on ℓ_∞; thus $g = \ell_\infty$. ∎

A hyperoval which contains a translation oval is called a *translation hyperoval*. Note that in the proper translation planes, translation hyperovals must be of hyperbolic type (see below). Also note that for a translation hyperoval the removal of one of only two specific points will result in a translation oval.

In the affine plane obtained by the removal of ℓ_∞ from a translation plane, the elations with axis ℓ_∞ become translations. If coordinates are introduced so that one of the points of the restriction of a translation hyperoval to this affine plane is the origin, then it is easy to see that the coordinates of the points of this restriction are in one-to-one correspondence with the translations which leave the restriction invariant. Thus, if Ω^* is the restriction of the translation hyperoval Ω and $(a, b) \in \Omega^*$ then the translation $\tau : (x, y) \rightarrow (x + a, y + b)$ leaves Ω^* invariant provided $(0, 0) \in \Omega^*$.

The conics in Desarguesian planes are translation ovals. Payne [7] has determined all the translation ovals in finite Desarguesian planes of even order. In a Desarguesian plane of order 2^h, an oval is a translation oval if and only if the plane can be coordinatized so that the oval contains the point (∞) and its affine points are given by $y = x^{2^n}$, where $(n, h) = 1$.

Denniston [5] has shown that translation ovals exist in certain Andre planes. These results together with those of the current work lead one to the natural conjecture that translation hyperovals exist in all translation planes of even order.

3. Search Procedure

To reduce the search time, use was made of the collineation groups of each of the planes examined. These groups were determined from the information in Dempwolff and Reifart [4] . In searching for hyperbolic hyperovals, we first determined the orbits on the line at infinity of unordered pairs of points on that line. Then choosing a representative from each orbit we searched for hyperovals passing through this pair and also passing through the origin. This condition is not restrictive since by the use of an appropriate translation, every hyperoval is projectively equivalent to one which has the same infinite points and passes through any specified affine point. The hyperovals containing such a triple of points have the same hyperbolic type. In the elliptic case, we can again assume that the hyperovals pass through the origin and since every line through the origin must contain another affine point of the hyperoval, we can limit the search in a manner analogous to the hyperbolic case. In particular, we choose the line $x = 0$ and consider the orbits of its affine points under the group of collineations which fix this line and the origin. We again search for hyperovals passing through an orbit representative and the origin, and any such hyperovals are of the same elliptic type.

The search procedure used for both types of hyperovals was a straight forward backtracking algorithm. For each plane considered, the coordinate set given in Dempwolff and Reifart [4] was converted by the standard isomorphism (see Tsuzuku [10]) to a set of matrices in $PSL(4,2)$ which were then used to determine the spreads in coordinate form. From the spreads, the incidence matrix of each plane was constructed and stored in a compact form. All incidence operations required by the program were performed by table look-up. A matrix, each of whose rows correspond to a component of the spread, is used as a bookkeeping device. The general algorithm is to systematically pick a point in a row, add it to a list of previously selected points, determine all the points on each of the secant lines which are the joins of the last point with each of the previous points on the list and remove these points from the matrix. A check is made to see if all the points of any row have been eliminated without one being selected. If this occurs, the last point added to the list is removed and another point in its row is selected; otherwise a selection is made from the next row. If a point can be selected from each row in this manner, this set together with the origin (which is not represented in the matrix) forms a hyperoval. This algorithm finds the hyperovals of elliptic type. The modification for hyperovals of hyperbolic type is simply to skip in checking and selecting those two rows which contain the two infinite points and to reject any point which would form a collinear triple passing through one of these infinite points.

The programs were written in Turbo Pascal and run on a microcomputer. No attempt was made to optimize the code nor was any attention paid to precise timing. Searching each class took approximately two weeks on a 5 MHz machine.

4. Results

The following is a brief description of the hyperovals found in each of the planes. More detailed information is contained in the tables at the end of the next section. There are no tables for the Desarguesian plane and the information about it is included here only for the sake of completeness.

DES: There are two classes of hyperovals in the Desarguesian plane of order 16. The first class consists of the hyperovals arising from conics, and these are translation hyperovals. The other class was first discovered using a computer by Lunelli and Sce [8] in 1958. Hall [6] showed that these were the only two classes of hyperovals in this plane. Hyperovals in the second class are not translation hyperovals and they have the interesting property of admitting an automorphism group which is transitive on the hyperoval.

SEMI2: The semifield plane with kern $GF(2)$ is the most interesting from the point of view of hyperovals. There are 17 equivalence classes of hyperovals. Only three of these classes are not translation hyperovals. This is the only plane in which hyperovals of elliptic type exist. There is one class of these and each hyperoval of this class is fixed by a single generalized homology of order 3. The 18 hyperovals of class HI.I have an algebraic description that is given in Cherowitzo [1].

SEMI4: The semifield plane with kern $GF(4)$ is remarkable only in its paucity of hyperoval types. There are three classes of "strictly" translation hyperovals, that is to say, hyperovals whose automorphism group consists only of the group of translations which fix the hyperoval. This plane exhibits the phenomenon of "forbidden pairs", i.e., pairs of infinite points through which no hyperoval passes. There are 64 such pairs in this plane.

HALL: The Hall plane has 10 forbidden pairs and 15 equivalence classes of translation hyperovals. There are three types of abstract automorphism groups that appear for these hyperovals.

LMRH: There are no forbidden pairs and six classes of translation hyperovals in the Lorimer-Rahilly plane. Three types of automorphism groups are exhibited.

JOWK: As in the LMRH plane there. are no forbidden pairs and six classes of translation hyperovals in the Johnson-Walker plane. The distribution and types of hyperovals in this plane are identical to those of the LMRH plane, which is to be expected due to the close connection between these planes.

DSFP: The derived semifield plane has 19 forbidden pairs and 22 classes of translation hyperovals. Only one of these classes consists of non-"strictly" translation hyperovals.

DEMP: The Dempwolff plane has no forbidden pairs and 15 classes of translation hyperovals. Only three classes consist of non-"strictly" translation hyperovals.

Table A below records the total number of hyperovals (not just those through the origin) in each of the planes.

5. Tables

The arithmetic tables (Tables 1–8) are provided to permit the reconstruction of the planes that were investigated. Table 1 gives the addition table to be used for all the planes (the elementary abelian group of order 16). Tables 2–8 provide the multiplicative loops of the quasifields which define the various planes. In each case the plane can be reconstructed in the standard manner by the use of Hall coordinates. It should be noted that these are left quasifields so that the affine points of the non-vertical lines of the planes are given by the equations of the form $y = xm + b$. To conserve space these tables are not bordered.

Table A
Total Number of Hyperovals

DES	119,857,920
SEMI2	102,912
SEMI4	82,944
HALL	89,280
LMRH	70,848
JOWK	70,848
DSFP	73,440
DEMP	68,832

Tables 9–15 give the results of the searches in each of the planes. Under the heading "UType" we provide a crude classification of the hyperovals as either hyperbolic (H) or elliptic (E). A further refinement of the hyperbolic hyperovals is given by considering the orbits of pairs of infinite points. The orbit size is given in parentheses and a representative of the orbit is listed below. For each type "Class" refers to a projective equivalence class of hyperovals, and "#" is the number of hyperovals in this class. An entry of "$\emptyset$" indicates that there are no hyperovals of this type. The affine coordinates of a representative of the class are provided under "Representative". The infinite points of these class representatives are the orbit representatives of the respective type. Finally, under "Group" we list the abstract automorphism group for that class of hyperovals. In this listing the symbol T_Ω refers to the subgroup of translations, $\tau(a, b) : (x, y) \rightarrow (x+a, y+b)$, where (a, b) are the coordinates of any affine point of the particular hyperoval. The hyperovals whose automorphism groups contain this subgroup are translation hyperovals.

Acknowledgment

The author would like to thank Professor Marialuisa de Resmini of the Università di Roma "La Sapienza" for her continual encouragement (or, as she would have it, "pestering"), without whom this paper may never have been written.

The author would also like to thank Professor David Glynn and Professor Gunter Steinke of the University of Canterbury for their independent verification and correction of the results reported here.

Table 1: Addition Table for All Quasigroups

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	5	9	15	2	11	14	10	3	8	6	13	12	7	4
2	5	0	6	10	1	3	12	15	11	4	9	7	14	13	8
3	9	6	0	7	11	2	4	13	1	12	5	10	8	15	14
4	15	10	7	0	8	12	3	5	14	2	13	6	11	9	1
5	2	1	11	8	0	9	13	4	6	15	3	14	7	12	10
6	11	3	2	12	9	0	10	14	5	7	1	4	15	8	13
7	14	12	4	3	13	10	0	11	15	6	8	2	5	1	9
8	10	15	13	5	4	14	11	0	12	1	7	9	3	6	2
9	3	11	1	14	6	5	15	12	0	13	2	8	10	4	7
10	8	4	12	2	15	7	6	1	13	0	14	3	9	11	5
11	6	9	5	13	3	1	8	7	2	14	0	15	4	10	12
12	13	7	10	6	14	4	2	9	8	3	15	0	1	5	11
13	12	14	8	11	7	15	5	3	10	9	4	1	0	2	6
14	7	13	15	9	12	8	1	6	4	11	10	5	2	0	3
15	4	8	14	1	10	13	9	2	7	5	12	11	6	3	0

Table 2: Multiplication Table for SEMI2

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	2	5	12	3	1	14	10	9	7	11	13	15	8	4	6
0	3	10	5	1	12	15	2	13	11	8	14	4	7	6	9
0	4	14	7	5	9	1	13	6	3	12	15	2	10	11	8
0	5	1	10	7	2	8	6	12	15	14	4	11	3	9	13
0	6	15	14	9	13	3	4	10	8	7	2	1	11	12	5
0	7	11	13	2	8	4	14	15	5	9	3	10	6	1	12
0	8	7	6	13	11	10	15	4	14	5	1	9	12	2	3
0	9	4	11	15	14	13	12	3	2	1	10	6	5	8	7
0	10	12	2	11	3	7	9	5	4	15	6	8	1	13	14
0	11	8	15	14	7	2	3	1	12	6	9	13	4	5	10
0	12	3	1	6	10	9	11	7	13	2	8	5	14	15	4
0	13	6	9	12	15	5	8	11	10	4	7	14	2	3	1
0	14	9	8	10	4	12	1	2	6	13	5	3	15	7	11
0	15	13	4	8	6	11	5	14	1	3	12	7	9	10	2

Table 3: Multiplication Table for SEMI4

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	2	14	7	6	13	1	10	15	12	8	5	11	9	4	3
0	3	12	9	11	10	8	2	14	1	15	13	7	4	6	5
0	4	9	8	10	14	12	1	11	5	13	6	3	7	15	2
0	5	13	4	12	7	11	6	2	8	1	3	15	10	9	14
0	6	5	15	1	9	10	4	3	13	2	7	8	14	12	11
0	7	8	12	14	11	9	5	10	2	6	15	4	3	13	1
0	8	10	5	3	1	15	11	9	4	12	2	14	6	7	13
0	9	7	1	13	15	14	12	6	3	5	4	2	11	8	10
0	10	4	11	7	2	13	8	12	14	3	9	5	15	1	6
0	11	1	14	15	6	7	3	13	10	4	8	9	2	5	12
0	12	6	2	8	4	3	15	5	7	14	10	13	1	11	9
0	13	3	6	5	8	2	9	4	15	11	14	1	12	10	7
0	14	15	10	9	3	5	13	1	11	7	12	6	8	2	4
0	15	11	13	2	12	4	14	7	6	9	1	10	5	3	8

Table 4: Multiplication Table for HALL

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	2	13	12	3	4	1	10	5	7	9	6	8	14	15	11
0	3	12	9	11	10	8	2	14	1	15	13	7	4	6	5
0	4	15	11	6	9	12	1	3	13	14	2	10	5	7	8
0	5	14	10	7	8	11	6	4	15	13	1	9	2	3	12
0	6	1	8	5	2	10	4	12	14	7	15	11	9	13	3
0	7	11	2	1	13	9	5	15	12	3	14	6	8	10	4
0	8	3	14	10	12	15	11	7	6	2	5	13	1	4	9
0	9	7	1	13	15	14	12	6	3	5	4	2	11	8	10
0	10	6	15	2	14	13	8	11	5	4	3	1	12	9	7
0	11	5	13	8	1	7	3	9	4	6	12	15	10	2	14
0	12	4	7	9	11	3	15	10	2	1	8	14	6	5	13
0	13	10	4	14	3	2	9	1	11	8	7	5	15	12	6
0	14	9	6	15	7	5	13	2	8	12	10	4	3	11	1
0	15	8	5	12	6	4	14	13	10	11	9	3	7	1	2

Table 7: Multiplication Table for DSFP

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	2	5	12	9	1	10	11	3	7	14	15	6	8	4	13
0	3	7	9	13	15	14	12	6	1	5	4	2	11	8	10
0	4	3	11	15	12	2	5	9	13	6	7	8	14	10	1
0	5	1	10	14	2	7	8	13	15	11	12	4	3	9	6
0	6	13	8	10	4	11	15	2	14	12	1	3	7	5	9
0	7	4	2	6	11	13	14	5	12	9	3	15	10	1	8
0	8	9	14	3	7	12	4	10	6	1	2	5	15	13	11
0	9	12	1	11	10	8	2	14	3	15	13	7	4	6	5
0	10	11	15	7	13	4	3	1	5	8	9	14	6	2	12
0	11	14	13	2	8	1	9	15	4	3	6	10	5	12	7
0	12	8	7	5	6	9	10	11	2	4	14	13	1	15	3
0	13	15	4	8	9	5	6	7	11	2	10	1	12	3	14
0	14	10	6	12	3	15	1	4	8	13	5	11	9	7	2
0	15	6	5	1	14	3	13	12	10	7	8	9	2	11	4

Table 8: Multiplication Table for DEMP

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	2	13	12	10	4	11	15	3	7	14	1	6	8	5	9
0	3	7	9	13	15	14	12	6	1	5	4	2	11	8	10
0	4	10	11	12	3	15	1	9	13	6	5	8	14	7	2
0	5	14	10	2	8	1	9	13	15	11	6	4	3	12	7
0	6	5	8	9	1	10	11	2	14	12	15	3	7	4	13
0	7	6	2	1	14	3	13	5	12	9	8	15	10	11	4
0	8	11	14	7	13	4	3	10	6	1	9	5	15	2	12
0	9	12	1	11	10	8	2	14	3	15	13	7	4	6	5
0	10	9	15	3	7	12	4	1	5	8	2	14	6	13	11
0	11	1	13	14	2	7	8	15	4	3	12	10	5	9	6
0	12	15	7	8	9	5	6	11	2	4	10	13	1	3	14
0	13	8	4	5	6	9	10	7	11	2	14	1	12	15	3
0	14	3	6	15	12	2	5	4	8	13	7	11	9	10	1
0	15	4	5	6	11	13	14	12	10	7	3	9	2	1	8

Table 9
Hyperovals Through the Origin (0, 0) in the Semifield Plane with Kern GF(2)

Type	Class	#	Representative	Group
H1 (16)	I	18	0,0 1,1 2,13 3,5 4,8 5,12 6,7 7,4 8,9	T_Ω
∞,0			9,2 10,3 11,14 12,11 13,6 14,15 15,10	
H2 (24)	I	12	0,0 0,1 1,4 1,15 3,6 3,11 4,7 4,14 7,8	T_Ω
0,1			7,10 9,12 9,13 14,1 14,5 15,3 15,9	
	II	12	0,0 0,2 1,13 1,14 4,1 4,5 6,8 6,15 11,3	T_Ω
			11,6 12,4 12,10 13,9 13,11 15,7 15,12	
	III	12	0,0 0,7 1,6 1,10 2,5 2,13 5,9 5,15 7,2	T_Ω
			7,12 12,1 12,14 13,8 13,11 14,3 14,4	
H3 (24)	I	12	0,0 0,1 3,12 3,13 4,4 4,15 5,7 5,14 7,6	T_Ω
0,3			7,11 8,3 8,9 11,2 11,5 13,8 13,10	
	II	12	0,0 0,4 1,6 1,12 3,3 3,7 4,9 4,14 7,1	T_Ω
			7,15 9,2 9,10 14,11 14,13 15,5 15,8	
	III	12	0,0 0,4 2,11 2,13 3,1 3,15 4,2 4,10 6,6	T_Ω
			6,12 7,5 7,8 10,9 10,14 12,3 12,7	
	IV	12	0,0 0,7 3,1 3,14 4,6 4,10 5,5 5,13 7,8	T_Ω
			7,11 8,9 8,15 11,2 11,12 13,3 13,4	
	V	6	0,0 0,4 5,3 5,7 6,6 6,12 7,9 7,14 9,2	$T_\Omega \rtimes Z_2$
			9,10 10,5 10,8 13,1 13,15 15,11 15,13	
	VI	6	0,0 0,6 1,7 1,10 6,13 6,15 7,4 7,12 8,8	$T_\Omega \rtimes Z_2$
			8,14 10,1 10,11 11,5 11,9 14,2 14,3	
	VII	24	0,0 0,4 5,5 5,8 6,1 6,15 7,9 7,14 9,3	$Z_2 \times Z_4$
			9,7 10,11 10,13 13,2 13,10 15,6 15,12	
	VIII	24	0,0 0,6 1,1 1,11 6,13 6,15 7,5 7,9 8,4	$Z_2 \times Z_4$
			8,12 10,8 10,14 11,2 11,3 14,7 14,10	
H4(72)	I	4	0,0 0,1 5,8 5,10 6,12 6,13 7,6 7,11 9,3	T_Ω
0,4			9,9 10,4 10,15 13,7 13,14 15,2 15,5	
	II	4	0,0 0,5 2,10 2,15 3,12 3,14 6,3 6,11 8,4	T_Ω
			8,8 13,6 13,9 14,7 14,13 15,1 15,2	
	III	4	0,0 0,10 3,3 3,12 5,1 5,8 10,5 10,15 11,9	T_Ω
			11,13 12,11 12,14 14,6 14,7 15,2 15,4	
	IV	4	0,0 0,11 2,2 2,9 4,7 4,8 9,10 9,14 10,12	T_Ω
			10,15 11,4 11,13 13,3 13,5 14,1 14,6	
E	I	1728	0,0 0,1 1,0 1,7 2,2 2,6 7,2 7,9 9,4 9,11	Z_3
			10,1 10,5 11,5 11,9 14,4 14,6 15,7 15,11	

Table 10
Hyperovals Through the Origin (0, 0) in the Hall Plane

Type	Class	#	Representative	Group
H1 (5) ∞,0	∅			
H2 (5) ∞,1	∅			
H3 (60) ∞,2	I	5	0,0 1,0 2,4 3,13 4,1 5,4 6,11 7,12 8,15 9,13 10,15 11,11 12,6 13,6 14,12 15,1	$T_\Omega \rtimes Z_2$
	II	5	0,0 1,0 2,12 3,4 4,1 5,12 6,6 7,15 8,13 9,4 10,13 11,6 12,11 13,11 14,15 15,1	$T_\Omega \rtimes Z_2$
	III	5	0,0 1,1 2,9 3,4 4,1 5,3 6,14 7,15 8,9 9,15 10,3 11,7 12,7 13,14 14,4 15,0	$T_\Omega \rtimes Z_2$
	IV	5	0,0 1,4 2,14 3,1 4,4 5,9 6,7 7,8 8,13 9,15 10,9 11,3 12,3 13,7 14,1 15,0	$T_\Omega \rtimes Z_2$
	V	5	0,0 1,5 2,5 3,3 4,6 5,0 6,11 7,2 8,6 9,11 10,9 11,3 12,1 13,2 14,1 15,9	$T_\Omega \rtimes Z_2$
	VI	5	0,0 1,5 2,5 3,11 4,2 5,0 6,3 7,9 8,2 9,3 10,1 11,11 12,6 13,9 14,6 15,1	$T_\Omega \rtimes Z_2$
H4 (30) 2,4	I	20	0,0 0,1 1,6 1,11 4,0 4,1 6,12 6,13 11,4 11,15 12,12 12,13 13,4 13,15 15,6 15,11	T_Ω
	I	20	0,0 0,1 2,7 2,14 4,0 4,1 9,2 9,5 10,7 10,14 11,12 11,13 13,12 13,13 14,2 14,5	T_Ω
	III	2	0,0 0,2 2,4 2,10 4,13 4,14 9,9 9,11 10,9 10,11 11,13 11,14 13,0 13,2 14,4 14,10	T_Ω
H5 (30) 2,5	I	20	0,0 0,1 2,6 2,11 3,0 3,1 6,6 6,11 8,4 8,15 13,4 13,15 14,12 14,13 15,12 15,13	T_Ω
	II	20	0,0 0,1 5,12 5,13 6,12 6,13 7,7 7,14 9,0 9,1 10,2 10,5 13,2 13,5 15,7 15,14	T_Ω
	III	20	0,0 0,2 1,9 1,11 3,0 3,2 4,3 4,6 7,3 7,6 9,9 9,11 14,1 14,5 15,1 15,5	T_Ω
H6 (6) 2,6	I	10	0,0 0,1 1,8 1,10 3,2 3,5 8,8 8,10 9,4 9,15 10,0 10,1 12,2 12,5 13,4 13,15	$T_\Omega \rtimes (Z_5 \times Z_2)$
	II	10	0,0 0,3 1,5 1,11 3,4 3,7 4,0 4,3 7,4 7,7 9,8 9,13 14,8 14,13 15,5 15,11	$T_\Omega \rtimes (Z_5 \times Z_2)$
	III	10	0,0 0,5 5,7 5,13 6,12 6,14 7,0 7,5 9,1 9,2 10,12 10,14 13,7 13,13 15,1 15,2	$T_\Omega \rtimes (Z_5 \times Z_2)$

Table 11
Hyperovals Through the Origin (0, 0) in the
Lorimer-Rahilly Plane

Type	Class	#	Representative	Group
H1 (3)	I	48	0,0 1,1 2,4 3,5 4,8 5,9 6,12 7,13 8,10	$T_\Omega \rtimes Z_7$
∞,0			9,11 10,14 11,15 12,2 13,3 14,6 15,7	
H2 (42)	I	24	0,0 1,4 2,2 3,6 4,6 5,2 6,4 7,0 8,14	T_Ω
∞,2			9,10 10,12 11,8 12,8 13,12 14,10 15,14	
	II	6	0,0 1,3 2,0 3,3 4,1 5,2 6,1 7,2 8,14	$T_\Omega \rtimes Z_4$
			9,13 10,14 11,13 12,15 13,12 14,15 15,12	
H3 (84)	I	12	0,0 0,2 2,5 2,7 4,0 4,2 6,5 6,7 8,13	T_Ω
2,4			8,15 10,8 10,10 12,13 12,15 14,8 14,10	
	II	12	0,0 0,2 1,5 1,7 6,4 6,6 7,1 7,3 10,0	T_Ω
			10,2 11,5 11,7 12,4 12,6 13,1 13,3	
H4 (7)	I	144	0,0 0,1 3,4 3,5 4,0 4,1 7,4 7,5 8,8	T_Ω
2,3			8,9 11,12 11,13 12,8 12,9 15,12 15,13	

Table 12
Hyperovals Through the Origin (0, 0) in the
Johnson-Walker Plane

Type	Class	#	Representative	Group
H1 (3)	I	48	0,0 1,3 2,2 3,13 4,7 5,6 6,14 7,5 8,10	$T_\Omega \rtimes Z_7$
∞,0			9,8 10,12 11,15 12,1 13,9 14,11 15,4	
H2 (42)	I	24	0,0 1,3 2,13 3,6 4,14 5,8 6,15 7,8 8,6	T_Ω
∞,2			9,2 10,2 11,14 12,3 13,0 14,13 15,15	
	II	6	0,0 1,4 2,3 3,0 4,1 15,7 6,3 7,11 8,8	$T_\Omega \rtimes Z_4$
			9,4 10,5 11,7 12,5 13,8 14,13 15,13	
H3 (84)	I	12	0,0 0,2 1,8 1,15 3,3 3,6 4,13 4,14 7,8	T_Ω
2,3			7,15 9,13 9,14 14,0 14,2 15,3 15,6	
	II	12	0,0 0,3 5,0 5,3 6,5 6,11 7,14 7,15 9,5	T_Ω
			9,11 10,10 10,12 13,14 13,15 15,10 15,12	
H4 (7)	I	144	0,0 0,1 1,12 1,13 3,0 3,1 4,7 4,14 7,7	T_Ω
2,5			7,14 9,12 9,13 14,2 14,5 15,2 15,5	

Table 13
Hyperovals Through the Origin (0, 0) in the Derived Semifield Plane

Type	Class	#	Representative	Group
H1 (3) ∞,0	I	18	0,0 1,1 2,6 3,9 4,12 5,11 6,5 7,8 8,15 9,3 10,4 11,2 12,14 13,7 14,10 15,13	$T_\Omega \rtimes Z_4$
H2 (36) ∞,2	I	6	0,0 1,0 2,13 3,15 4,11 5,13 6,6 7,12 8,4 9,15 10,4 11,6 12,1 13,1 14,12 15,11	T_Ω
	II	6	0,0 1,1 2,7 3,2 4,5 5,14 6,12 7,1 8,12 9,5 10,13 11,13 12,14 13,7 14,0 15,2	T_Ω
	III	6	0,0 1,3 2,2 3,5 4,0 5,6 6,1 7,5 8,6 9,11 10,2 11,9 12,1 13,9 14,11 15,3	T_Ω
	IV	6	0,0 1,3 2,13 3,14 4,14 5,8 6,2 7,0 8,6 9,15 10,2 11,6 12,13 13,8 14,3 15,15	T_Ω
	V	6	0,0 1,6 2,12 3,13 4,1 5,4 6,1 7,12 8,15 9,15 10,13 11,11 12,0 13,6 14,4 15,11	T_Ω
H3 (6) ∞,3	∅			
H4 (24) 2,3	I	9	0,0 0,1 2,6 2,11 3,6 3,11 4,4 4,15 6,0 6,1 7,12 7,13 10,12 10,13 12,4 12,15	T_Ω
	II	9	0,0 0,1 4,12 4,13 5,6 5,11 6,0 6,1 8,4 8,15 9,6 9,11 12,12 12,13 14,4 14,15	T_Ω
	III	9	0,0 0,1 2,4 2,15 7,6 7,11 8,6 8,11 9,4 9,15 11,0 11,1 12,12 12,13 15,12 15,13	T_Ω
	IV	9	0,0 0,2 1,7 1,12 3,1 3,5 8,7 8,12 9,13 9,14 10,0 10,2 12,1 12,5 13,13 13,14	T_Ω
	V	9	0,0 0,4 3,6 3,12 4,1 4,15 5,0 5,4 7,11 7,13 8,1 8,15 11,6 11,12 13,11 13,13	T_Ω
	VI	9	0,0 0,4 3,11 3,13 4,6 4,12 5,6 5,12 7,1 7,15 8,0 8,4 11,1 11,15 13,11 13,13	T_Ω
H5 (36) 2,4	I	6	0,0 0,1 1,7 1,14 4,6 4,11 6,6 6,11 11,8 11,10 12,0 12,1 13,7 13,14 15,8 15,10	T_Ω
	II	6	0,0 0,1 1,6 1,11 3,8 3,10 8,7 8,14 9,7 9,14 10,8 10,10 12,0 12,1 13,6 13,11	T_Ω
	III	6	0,0 0,2 3,8 3,15 5,4 5,10 10,1 10,5 11,1 11,5 12,4 12,10 14,0 14,2 15,8 15,15	T_Ω
	IV	6	0,0 0,5 4,7 4,13 5,7 5,13 6,6 6,9 8,0 8,5 9,10 9,15 12,10 12,15 14,6 14,9	T_Ω
	V	6	0,0 0,10 1,9 1,13 4,1 4,8 6,1 6,8 11,3 11,12 12,0 12,10 13,9 13,13 15,3 15,12	T_Ω

Table 13, continued
Hyperovals Through the Origin (0, 0) in the
Derived Semifield Plane

Type	Class	#	Representative	Group
H6 (18)	I	12	0,0 0,1 1,4 1,15 2,6 2,11 3,0 3,1 5,12	T_Ω
2,5			5,13 6,6 6,11 9,4 9,15 11,12 11,13	
	II	12	0,0 0,1 2,3 2,9 3,0 3,1 4,7 4,14 6,3	T_Ω
			6,9 7,7 7,14 10,4 10,15 12,4 12,15	
	III	12	0,0 0,2 1,9 1,11 2,4 2,10 5,13 5,14 7,0	T_Ω
			7,2 12,4 12,10 13,13 13,14 14,9 14,11	
	IV	12	0,0 0,3 1,4 1,7 6,0 6,3 7,1 7,9 8,14	T_Ω
			8,15 10,1 10,9 11,4 11,7 14,14 14,15	
	V	12	0,0 0,4 3,5 3,8 5,3 5,7 10,3 10,7 11,11	T_Ω
			11,13 12,11 12,13 14,5 14,8 15,0 15,4	
H7 (1) 2,7	∅			
H8 (1) 3,9	∅			

Table 14
Hyperovals Through the Origin (0, 0) in the
Semifield Plane with Kern GF(4)

Type	Class	#	Representative	Group
H1 (16) ∞,0	∅			
H2 (72)	I	24	0,0 0,1 1,4 1,15 4,12 4,13 6,7 6,14 11,3	T_Ω
0,1			11,9 12,2 12,5 13,8 13,10 15,6 15,11	
	II	24	0,0 0,4 4,9 4,14 5,1 5,15 6,11 6,13 8,3	T_Ω
			8,7 9,6 9,12 12,2 12,10 14,5 14,8	
	III	24	0,0 0,11 3,10 3,14 4,2 4,9 5,1 5,6 7,4	T_Ω
			7,13 8,3 8,5 11,7 11,8 13,12 13,15	
H3 (48) .0,3	∅			

Table 15
Hyperovals Through the Origin (0,0)
in the Dempwolff Plane

Type	Class	#	Representative	Group
H1 (1) ∞,0	I	90	0,0 1,1 2,3 3,8 4,14 5,9 6,13 7,6 8,4 9,10 10,15 11,12 12,2 13,5 14,11 15,7	T_Ω
	II	72	0,0 1,2 2,1 3,10 4,13 5,5 6,8 7,9 8,7 9,4 10,12 11,15 12,3 13,6 14,11 15,14	T_Ω
H2 (15) ∞,1	I	6	0,0 1,0 2,8 3,1 4,12 5,8 6,10 7,13 8,9 9,1 10,9 11,10 12,3 13,3 14,13 15,12	$T_\Omega \rtimes Z_4$
H3 (15) 0,1	I	6	0,0 0,1 1,3 1,9 3,12 3,13 8,2 8,5 9,8 9,10 10,6 10,11 12,4 12,15 13,7 13,14	$T_\Omega \rtimes Z_4$
H4 (60) 1,2	I	6	0,0 0,1 2,0 2,1 3,12 3,14 4,7 4,14 6,12 6,13 7,2 7,5 10,7 10,14 12,2 12,5	T_Ω
	II	6	0,0 0,1 4,12 4,13 5,0 5,1 6,4 6,15 8,12 8,13 9,4 9,15 12,6 12,11 14,6 14,11	T_Ω
	III	6	0,0 0,2 1,13 1,14 2,3 2,6 3,13 3,14 5,8 5,15 6,8 6,15 9,0 9,2 11,3 11,6	T_Ω
H5 (15) 1,3	I	24	0,0 0,12 1,4 1,6 3,2 3,7 8,0 8,12 9,3 9,10 10,4 10,6 12,3 12,10 13,2 13,7	T_Ω
	II	24	0,0 0,13 2,4 2,11 3,0 3,13 4,9 4,10 6,4 6,11 7,9 7,10 10,2 10,14 12,2 12,14	T_Ω
H6 (30) 1,8	I	12	0,0 0,1 2,6 2,11 3,4 3,15 4,12 4,13 6,12 6,13 7,6 7,11 10,4 10,15 12,0 12,1	T_Ω
	II	12	0,0 0,5 1,10 1,15 2,0 2,5 4,12 4,14 5,10 5,15 8,3 8,11 10,12 10,14 15,3 15,11	T_Ω
	III	12	0,0 0,5 2,0 2,5 3,12 3,14 4,3 4,11 6,12 6,14 7,10 7,15 10,3 10,11 12,10 12,15	T_Ω
	IV	12	0,0 0,5 1,6 1,9 2,4 2,8 4,0 4,5 5,12 5,14 8,12 8,14 10,4 10,8 15,6 15,9	T_Ω
	V	12	0,0 0,5 3,1 3,2 4,0 4,5 5,3 5,11 7,1 7,2 8,3 8,11 11,6 11,9 13,6 13,9	T_Ω
	VI	12	0,0 0,6 2,4 2,12 3,1 3,11 4,0 4,6 6,13 6,15 7,1 7,11 10,4 10,12 12,13 12,15	T_Ω

References

1. CHEROWITZO, W., *Translation B-ovals*, Congressus Numerantium **47** (1985), 161–172.
2. CHEROWITZO, W., KIEL, D., and KILLGROVE, R., *Ovals and other configurations in the known planes of order nine*, Congressus Numerantium **55** (1986), 167–179.
3. DEMBOWSKI, P., "Finite Geometries", Springer-Verlag, Berlin, 1968.
4. DEMPWOLFF, U., and REIFART, A., *The classification of the translation planes of order 16, I*, Geom. Ded. **15** (1983), 137–153.
5. DENNISTON, R. H. F., *Some non-Desarguesian translation ovals*, Ars Combinatoria **7** (1979), 221–222.
6. HALL, M., *Ovals in the Desarguesian plane of order 16*, Ann. Mat. Pura App. **(4) 102** (1975), 159–176.
7. PAYNE, S. E., *A complete determination of translation ovoids in finite Desarguesian planes*, Atti Accad. Naz. Lincei Rend. **51** (1971), 328–331.
8. LUNELLI, L., and SCE, M., "K-archi completi nei piani proietivi desarguesiani di rango 8 e 16", Centro Calcoli Numerici, Politecnico di Milano, 1958.
9. SEGRE, B., *Ovals in a finite projective plane*, Can. J. Math. **7** (1955), 414–416.
10. TSUZUKU, T., "Finite Groups and Finite Geometries", Cambridge University Press, Cambridge, 1982.