Changing and Unchanging the Domination Number of a Graph

Julie Carrington
Department of Computer Science
University of Central Florida
Orlando, FL 32816

Frank Harary
Department of Computer Science
New Mexico State University
Las Cruces, NM 88003

Teresa W. Haynes
Department of Computer Science
East Tennessee State University
Johnson City, TN 37601

Abstract. A dominating set in a graph G is a set D of nodes such that every node of G is either in D or is adjacent to some node in D. The domination number $\alpha(G)$ is the minimum size of a dominating set. The purpose of this paper is to explore the changing or unchanging of $\alpha(G)$ when either a node is deleted, or an edge is added or deleted.

1. Introduction

Let $G = (V, E)$ be a graph as in [5], undirected with no loops or multiple edges. Notation and terminology not introduced here is found in the book [5]. A set S of nodes is a *dominating set* if every node not in S is adjacent to at least one node in S. The *domination number* $\alpha(G)$ is the minimum size of a dominating set for G. We study the problems involving the changing or the unchanging of the domination number of a graph G under three different situations: (a) deleting a node, (b) deleting an edge and (c) adding an edge. Let $\overline{E} = E(\overline{G})$, the edge set of the complement of G. Formally, we have six subproblems: to characterize those graphs $G = (V, E)$ for which

(1) $\alpha(G - v) \neq \alpha(G)$ for all $v \in V$
(2) $\alpha(G - v) = \alpha(G)$ for all $v \in V$
(3) $\alpha(G - e) \neq \alpha(G)$ for all $e \in E$
(4) $\alpha(G - e) = \alpha(G)$ for all $e \in E$
(5) $\alpha(G + e) \neq \alpha(G)$ for all $e \in \overline{E}$
(6) $\alpha(G + e) = \alpha(G)$ for all $e \in \overline{E}$.

Some of these individual subproblems have been approached in the literature. Our main objective is to tie known information with new results in order to present the current status of all six of these problems.

JCMCC 9 (1991), pp. 57-63

2. Removal of a Node

It will be useful to partition the nodes of G into three sets according to how their removal affects the domination number. Thus we define

$$V^0 = \{v \in V | \alpha(G - v) = \alpha(G)\},$$
$$V^+ = \{v \in V | \alpha(G - v) > \alpha(G)\} \quad \text{and}$$
$$V^- = \{v \in V | \alpha(G - v) < \alpha(G)\}.$$

Notation: Throughout, we write D for a generic minimum dominating set (mds) of G.

We study two classes of graphs in this section: the graphs satisfying $V^0 = \emptyset$ (Class 2.1) and those in which $V^0 = V$ (Class 2.2). Let I be the set of all isolated nodes of G. It is obvious that we always have $|V^+| \leq \alpha(G)$ and $I \subset V^-$.

We will use the following notation: $N(v)$ is the *open neighborhood* of v, i.e., the set of all nodes adjacent to v, $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v.

Class 2.1: The graphs for which the domination number is changed when any node is removed.

Since $V^0 = \emptyset$, the nodes of G in this case are partitioned into V^+ and V^-. We first state some facts about these sets. Bauer, Harary, Nieminen and Suffel [1] characterized nodes whose removal increases α, i.e., the nodes in V^+.

Theorem A [1]. *In a graph G, node $v \in V^+$ if and only if*

(1) *v is not isolated and is in every mds for G, and*
(2) *there is no dominating set for $G - N[v]$ having α nodes which also dominates $N(v)$.*

Theorem 1. *If $v \in V^+$ then in every mds D, v dominates at least two nonadjacent nodes u, w of G not dominated by $D - v$.*

Proof: We already know from Theorem A that each $v \in V^+$ is in every mds and is not isolated. Let D be any mds. Let $S \subset N(V)$ be the set of those nodes of D dominated only by $v \in D$. If S is empty then $D - v$ together with any neighbor of v form an mds not containing v, contradicting the fact that v is in every mds. Suppose next S is not empty, and that it induces a complete subgraph, contrary to the existence of two nonadjacent nodes u, w dominated only by v. Then again $D - v$ together with any node of S will serve as an mds, a contradiction. Hence v must dominate at least two nonadjacent nodes of S. ∎

Let d_v denote the degree of v. Theorem 1 shows that $d_v \geq 2$ for $v \in V^+$. We also note that the converse of Theorem 1 is not true. For a counterexample, consider graph G of Figure 1. Obviously, $\alpha(G) = 2$ and v is in every mds. Thus

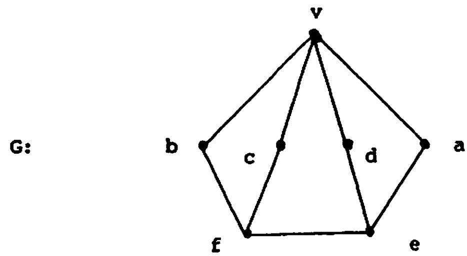

Figure 1. A counterexample to the converse of Theorem 1.

the conditions of the converse of Theorem 1 are met, but the set $\{e, f\}$ dominates $G - v$, implying that v is not in V^+.

Theorem 2. *If $x \in V^+$ and $y \in V^-$, then x and y are not adjacent.*

Proof: Suppose x is adjacent to y. Let D_y be an mds of $G - y$ of size $\alpha(G) - 1$. If D_y contains x, then D_y dominates G, a contradiction. On the other hand, if D_y does not contain x, then $D_y \cup \{y\}$ dominates G and does not contain x, violating Theorem A. ∎

Corollary 2a. *Every graph satisfies $|V^0| \geq 2|V^+|$.*

Proof: By Theorem 1, each $v \in V^+$ has, for each mds D, at least two neighbors in $V - D$ which are not dominated by $D - v$. Since v is not adjacent to a node in V^- by Theorem 2 and no nodes of V^+ are in $V - D$, v has at least two neighbors in V^0 which are not dominated by $D - v$. ∎

Corollary 2b. *If a graph G is in Class 2.1, then $V = V^-$.*

Proof: By hypothesis $\alpha(G - v) \neq \alpha(G)$ for all $v \in V$. Thus V^+ and V^- partition V. If $x \in V^+$ then by Corollary 2a, V^0 is nonempty, a contradition. Hence $V = V^-$. ∎

Thus in any graph G such that $\alpha(G - V) \neq \alpha(G)$ for all $v \in V$, it must be the case that $\alpha(G - v) < \alpha(G)$. These are precisely the graphs which Brigham, Chinn and Dutton [2] call "vertex critical" or just "α-critical". For consistency with research involving other invariants, motivated by the general approach of [6], we call these graphs *α-node-minimal*. We mention one property of such graphs which establishes when G is not one of them.

Theorem B [2]. *If G has a nonisolated node v such that $N(v)$ is complete, then G is not α-node-minimal.*

According to [2], attempts to characterize these graphs have been unsuccessful, and the problem remains unsolved. Furthermore, they show it is not possible to characterize these graphs in terms of forbidden subgraphs. On the other hand, they did characterize those α-node-minimal graphs having $p = \alpha + \Delta$ nodes, the minimum number possible, in the following result.

Theorem C [2]. *A graph G having $p = \alpha + \Delta$ nodes is α-node-minimal if and only if G has the form shown in Figure 2 where B is the neighborhood of a node v of maximum degree Δ and*

(1) *the $\alpha - 1$ nodes of C are independent,*
(2) *every B node is adjacent to exactly one C node,*
(3) *C is the only set of $\alpha - 1$ nodes which dominates $B \cup C$,*
(4) *for each $b \in B$, there is a set of $\alpha - 1$ nodes of $B \cup C$ which dominates $B - b$ and includes a node of B.*

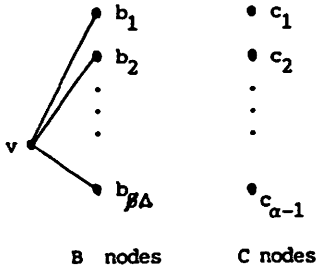

Figure 2. Part of a construction for Theorem C.

Class 2.2: The graphs for which the domination number is unchanged when any node is removed.

In this family of graphs $\alpha(G - v) = \alpha(G)$ for all $v \in V$, so $V = V^0$. We characterize graphs having this property after making a preliminary observation.

Lemma 3. *If $v \in V^-$ and v is not isolated, then there exists an mds D such that $v \notin D$.*

Proof: Since $v \in V^-$ there is an mds D' such that $D' \cap N(v) = \emptyset$. Then $v \in D'$ since v is necessary in D' to dominate itself. We form another mds from D' by replacing v by any one of its neighbors. ∎

Theorem 3. *A graph G is in Class 2.2 if and only if G has no isolated nodes, and for each node v, either*

(1) *there is an mds D' such that $v \notin D'$ and for all mds D such that $v \in D$, v is necessary in D to dominate at least one node of $G - v$, or*
(2) *v is in every mds and there is a subset of $\alpha(G)$ nodes in $G - N[v]$ which dominates $G - v$.*

Proof: First consider a graph G such that $\alpha(G - v) = \alpha(G)$ for all $v \in V$. Then obviously there are no isolated nodes. Assume there exist mds D, D' such that $v \in D$ but $v \notin D'$. Suppose $v \in D$ dominates only itself. Then $D - \{v\}$ dominates $G - v$, implying $v \in V^-$, a contradiction. Thus (1) holds.

Next consider a node v in every mds. Since $v \in V^+$, Theorem A shows there must be a set of $\alpha(G)$ nodes of $G - N[v]$ which dominates $G - v$, proving (2).

Conversely, suppose G has no isolated nodes and either (1) or (2) holds. If (1) holds for node v, then $\alpha(G - v) \leq \alpha(G)$ since some D' does not contain v and hence dominates $G - v$. If $\alpha(G - v) < \alpha(G)$ then there is a D in which v is needed to dominate only itself, a contradiction. Thus $\alpha(G - v) = \alpha(G)$ in this case. If (2) holds for a node v, then $v \notin V^-$ by Lemma 3, so $\alpha(G - v) \geq \alpha(G)$. But $\alpha(G-v) > \alpha(G)$ is excluded by Theorem A, so again $\alpha(G-v) = \alpha(G)$. ∎

3. Removal of an Edge

Just as in the preceding section on the removal of a node, we now consider the two classes of graphs where

$$\alpha(G - e) > \alpha(G) \quad \text{for any edge } e \text{, and}$$
$$\alpha(G - e) = \alpha(G) \quad \text{for all } e.$$

Class 3.1: The graphs in which the domination number is changed when any edge is removed.

Here we treat the family of graphs where the removal of any edge from G results in a change in the domination number, i.e., $\alpha(G - e) \neq \alpha(G)$ for all $e \in E$. Clearly, the removal of an edge cannot decrease the domination number, so such graphs have the property that $\alpha(G - e) > \alpha(G)$ for all $e \in E$. These graphs are *α-edge-minimal*. They are easy to characterize as shown independently by Bauer, Harary, Nieminen and Suffel [1] and by Walikar and Acharya [12]. A *galaxy* is a graph in which every component is a star.

Theorem D [1], [12]. *A graph G satisfies $\alpha(G - e) > \alpha(G)$ for each edge e if and only if G is a galaxy.*

Class 3.2: The graphs where the domination number is unchanged when any edge is removed.

The characterization of these graphs is much more difficult than of those in Class 3.1. Here the removal of an arbitrary edge from G does not change the domination number, that is, $\alpha(G-e) = \alpha(G)$ for all $e \in E$. Dutton and Brigham [3] call connected graphs with this property "α-insensitive" graphs. In general, α-insensitive graphs seem to be difficult to characterize, and the problem remains open. The problem has been extended to consider the removal of $k > 1$ edges in [7], [8]. Applications of graphs having the unchanging property when $k \geq 1$ edges are removed have been explored in [4], [9], [10].

4. Addition of an Edge

Again we consider separately the graphs satisfying $\alpha(G + e) < \alpha(G)$ for each edge e in the complement $\overline{G}$, and $\alpha(G + e) = \alpha(G)$ for all $e \in E(\overline{G})$.

As before, one of these characterizations seems intractable and the other is routine.

Class 4.1: The graphs for which the domination number is changed when any edge is added.

Just as the removal of an edge cannot decrease the domination number α, the addition of an edge cannot increase α. Thus, in this problem, we look at those graphs G which for each $u, v \in V$, where uv is not an edge, $\alpha(G + uv) < \alpha(G)$.

This hard problem was studied by Sumner and Blitch [11]. They were able to characterize graphs in Class 4.1 only in the special cases where $\alpha(G) = 1$ or 2 and where $\alpha(G) = 3$ when $p \leq 8$.

Class 4.2: The graphs for which the domination number is unchanged when any edge is added.

In this problem $\alpha(G + uv) = \alpha(G)$ for all $u, v \in V$ where uv is not an edge. A characterization of graphs having this property is staightforward.

Theorem 5. *A graph G is in Class 4.2 if and only if V^- is empty.*

Proof: First consider G as unchanging with respect to domination when any edge of $\overline{G}$ is added, and assume G contains a node $x \in V^-$. Thus $\alpha(G - x) < \alpha(G)$. Let D_x be an $(\alpha - 1)$-dominating set of $G - x$. Then adding an edge e joining x and any node of D_x yields $\alpha(G + e) = \alpha(G) - 1$, a contradiction.

To prove the converse, suppose G has no nodes in V^- and $\alpha(G + uv) = \alpha(G) - 1$ for some pair of nonadjacent nodes u and v. Then any $(\alpha - 1)$-dominating set D of $G + uv$ must include exactly one of u or v, say u, and furthermore D must dominate $G - v$. Thus $v \in V^-$ which is a contradiction. ∎

The result of Theorem 4 relates our criterion for Class 4.2 to that for Class 2.1.

5. Remarks

It is interesting to note that straightforward characterizations were possible for subproblems (2), (3) and (6) listed in Section 1. On the other hand, their counterparts (1), (4) and (5), respectively, do not seem to lend themselves to useful characterizations.

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