Supereulerian graphs and the Petersen graph

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Abstract. Using a contraction method, we find some best-possible sufficient conditions for 3-edge-connected simple graphs such that either the graphs have spanning eulerian subgraphs or the graphs are contractible to the Petersen graph.

Introduction

We shall use the notation of Bondy and Murty [2], except for contractions. A graph is *eulerian* if it is connected and every vertex has even degree. An eulerian subgraph C of G is called a *a spanning eulerian subgraph* of G if $V(C) = V(G)$ and is called a *dominating eulerian subgraph* of G if $E(G-V(C)) = \emptyset$. A graph G is called *supereulerian* if G has a spanning eulerian subgraph. The family of supereulerian graphs is denoted by $S\mathcal{L}$. For $v \in V(G)$, we define the *neighborhood* $N(v)$ of v in G to be the set of vertices adjacent to v in G. A *bond* is a minimal nonempty edge cut. For an integer $i \geq 1$, define

$$D_i(G) = \{v \in V(G) | d(v) = i\}.$$

For a graph G with a connected subgraph H, the *contraction* G/H is the graph obtained from G by contracting all edges of H, and by deleting any resulting loops. Note that multiple edges can arise in contractions.

The existence of a spanning eulerian subgraph (or a domination eulerian subgraph) of a graph is especially interesting in view of the following theorem.

Theorem A. (Harary and Nash-Williams [10]) *The line graph $L(G)$ of a graph G contains a hamiltonian cycle if and only if G has a dominating eulerian subgraph or G is isomorphic to $K_{1,s}$ for some $s \geq 3$.* ∎

The following are some of the prior results on spanning eulerian subgraphs and dominating eulerian subgraphs.

Theorem B. (Jaeger [9]) *If a graph is 4-edge-connected or if it has 2 edge-disjoint spanning trees, then it is supereulerian.* ∎

Theorem C. (Benhocine, Clark, Köhler, Veldman [1]) *Let G be a 2-edge-connected graph of order $n \geq 3$. If $d(u) + d(v) \geq \frac{1}{3}(2n+3)$ for every edge uv of G, then G has a dominating eulerian subgraph.* ∎

Theorem D. (Cai [4], Catlin [5]) *If a 2-edge-connected graph G of order $n > 20$ satisfies $\delta(G) \geq \frac{n}{5} - 1$, then either G is supereulerian or G is contractible* *to $K_{2,3}$ such that the preimage of each vertex of $K_{2,3}$ is a subgraph of G on exactly $n/5$ vertices that is either complete or one edge short of being complete.* ∎

In [1], Benhocine, Clark, Köhler and Veldman conjectured that for a connected simple graph G on n vertices, if $G - D_1(G)$ is 2-edge-connected, and if for any edge $uv \in E(G)$, $d(u) + d(v) > \frac{2n}{5} - 2$, then G has a dominating eulerian subgraph. Li proved:

Theorem E. (Li [12]) *Let G be a 2-edge-connected simple graph of order n. If $\delta(G) \geq 4$ and if every edge $uv \in E(G)$ satisfies $d(u) + d(v) > \frac{2n}{5} - 2$, then G is supereulerian.* ∎

In this paper, we will discuss some best possible conditions for 3-edge-connected graphs to be supereulerian, using a reduction method which was introduced by Catlin [5]. We first present a concept that was given by Catlin in [5].

A graph G is called *collapsible* if for any even set $S \subseteq V(G)$, there is a subgraph Γ in G such that

(i) $G - E(\Gamma)$ is connected; and
(ii) S is the set of vertices of odd degree in Γ.

The subgraph Γ satisfying (i) and (ii) is called an *S-subgraph* of G. Note that K_1, K_3, and C_2 (the 2-cycle) are collapsible. K_1 is called a *trivial* collapsible graph.

Note that being collapsible is stronger than being supereulerian. For a collapsible graph G, let S be the set of all odd degree vertices of G. Since G has an S-subgraph Γ satisfying (i) and (ii) above, $G - E(\Gamma)$ is a spanning eulerian subgraph of G.

In [5], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The contraction of G obtained from G by contracting each H_i $(1 \leq i \leq c)$ into a single vertex is called the *reduction* of G. A graph is *reduced* if it is the reduction of some graph. Throughout this paper, we let G' be the reduction of G, and let $d(v)$ and $d'(v)$ denote degree of v in G and G', respectively.

For a graph G, define $F(G)$ to be the minimum number of extra edges that must be added to G to create a spanning subgraph of G having two edge-disjoint spanning trees. Thus, G has two edge-disjoint spanning trees if and only if $F(G) = 0$.

We shall make use of the following theorems:

Theorem F. (Catlin [5],[6]) *Let G be a graph, and let G' be the reduction of G. Then each of the following holds:*

(i) *Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if $G' = K_1$.*
(ii) *Let H_1 and H_2 be two collapsible subgraphs of G. If $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is collapsible.*

(iii) *G' has no nontrivial collapsible subgraph. In particular, G' is simple and K_3-free.*
(iv) *$G \in SL$ if and only if $G' \in SL$.*
(v) *$|E(G')| + F(G') = 2|V(G')| - 2$. In particular, if G' has order at least 3, then*

$$|E(G')| \leq 2|V(G')| - 4. \tag{1}$$

(vi) *$K_{3,3} - e$ ($K_{3,3}$ minus an edge) is collapsible.* ∎

Theorem G. (Catlin and Lai [8]) *Let G be a connected graph. If $F(G) \leq 2$, then exactly one of the following holds:*

(i) *$G \in SL$;*
(ii) *G has exactly one cut-edge;*
(iii) *The reduction of G is $K_{2,t}$ for some odd $t \geq 1$.* ∎

Let G be a graph containing an induced 4-cycle $H = uvzwu$. Let G/π be the graph obtained from $G - E(H)$ by identifying u and z to form a single vertex x, by identifying v and w to form a single vertex y, and by adding an edge $e_\pi = xy$ (see Figure 1). Note that G/π may have multiple edges, even if G has none.

Figure 1

Theorem H. (Catlin [6]) *For the graphs G and G/π defined above, the following holds:*

(i) *If G/π is collapsible then G is collapsible.*
(ii) *If $G/\pi \in SL$ then $G \in SL$.*
(iii) $|V(G)| = |V(G/\pi)| + 2$.
(iv) $|E(G)| = |E(G/\pi)| + 3$. ∎

Main Results

We start with the following lemma:

Lemma 1. *Let G be a simple 2-edge-connected graph of order at most 7, with $\delta(G) \geq 2$ and $|D_2(G)| \leq 2$. Then G is collapsible.*

Proof: Suppose G contains a triangle. If G has two vertex-disjoint triangles (say H_1 and H_2) then since $|V(G)| \leq 7$, $(G/H_1)/H_2$ has order at most 3. By the definition of contraction, $\kappa'((G/H_1)/H_2) \geq \kappa'(G) \geq 2$. Hence, $(G/H_1)/H_2$ is collapsible, and so by (i) of Theorem F, G is collapsible.

If G has no two disjoint triangles then let H be a maximal collapsible subgraph of G containing a triangle. If $H = G$ then the lemma holds. Suppose that $H \neq G$. Then $|V(G/H)| \leq 5$, $D_2|(G/H)| \leq 2$ and $\kappa'(G/H) \geq 2$. Since G has no two disjoint triangles and H is a maximal collapsible subgraph of G containing a triangle, it follows from (ii) of Theorem F that the graph G/H has girth at least 4. By inspection, $G/H \in \{K_{2,3}, C_4, C_5\}$. By the definition of G/H, one can easily check that if $G/H \in \{K_{2,3}, C_4, C_5\}$ then $|D_2(G)| \geq 3$, a contradiction. Thus $H = G$, and so G is collapsible.

Suppose that G is triangle-free.

Case 1. $|V(G)| \leq 6$. Let $M = M(G)$ be a maximum matching of G, and let $m = |M(G)|$. Then $V(G) - V(M)$ is an independent set. Since G is a triangle-free graph with $|D_2(G)| \leq 2$ and $|V(G)| \leq 6$, it is easy to see that $m = 3$ and so $|V(G)| = 6$. Let $M(G) = \{x_1y_1, x_2y_2, x_3y_3\}$. Since $|D_2(G)| \leq 2$, at least one pair of $\{x_i, y_i\}$ $(1 \leq i \leq 3)$ (say x_1, y_1) have degree 3. Since G is triangle-free, without loss of generality we may assume x_1y_2 and $x_1y_3 \in E(G)$. Therefore, $N(y_1) = \{x_1, x_2, x_3\}$. Since G has no triangle and $|D_2(G)| \leq 2$, it follows that either $x_2y_3 \in E(G)$ or $y_2x_3 \in E(G)$ (or both). Thus, we have $G = K_{3,3} - e$ or $K_{3,3}$. By (vi) of Theorem F, G is collapsible.

Case 2. $|V(G)| = 7$. Since G is triangle-free and $|D_2(G)| \leq 2$, it is easy to see $3 \leq \Delta(G) \leq 4$.

Case 2(a). $\Delta(G) = 4$. Let v be a vertex with $d(v) = \Delta(G)$.

Let $N(v) = \{x_1, x_2, x_3, x_4\}$. Since $|D_2(G)| \leq 2$, we may assume that $d(x_1) \geq 3$ and $d(x_2) \geq 3$. Since G is a triangle-free graph on 7 vertices, $N(x_1) - \{v\} = N(x_2) - \{v\} := \{y_1, y_2\}$. Since at least two vertices of $\{y_1, y_2, x_3, x_4\}$ have degree at least 3, by inspection, G contains a collapsible subgraph $H = K_{3,3} - e$. Contracting the graph $K_{3,3} - e$ in G, we have a 2-edge-connected graph $G/(K_{3,3} - e)$ of order 2. Obviously, this graph $G/(K_{3,3} - e)$ is collapsible. By (i) of Theorem F, G is also collapsible.

Case 2(b). $\Delta(G) = 3$. Note that G must have even number of odd degree vertices. Since G has order 7, $\delta(G) \geq 2$, and $|D_2(G)| \leq 2$, it follows that $|D_2(G)| = 1$ and G has girth 4.

Let $C = uvzwu$ be a 4-cycle in G. Let G/π be the graph as defined before and let $e_\pi = xy$ be the new edge in G/π. Since $\Delta(G) = 3$ and $|D_2(G)| = 1$, by the definition of G/π, we have that G/π is a connected graph of order 5 with $\delta(G/\pi) \geq 2$ and $|D_2(G/\pi)| \leq 1$.

If $\kappa'(G/\pi) = 1$, then $e_\pi = xy$ is the only cut edge of G/π, because G has no cut edge. Therefore, $G - E(C)$ has two components, say G_1 and G_2, where $u, z \in V(G_1)$ and $v, w \in V(G_2)$. Without loss of generality, we may assume that $|V(G_1)| \leq |V(G_2)|$. Since G is triangle-free, $uz \notin E(G)$, and so G_1 has at least 3 vertices. Since $|V(G)| = 7$, it follows that $|V(G_1)| = 3$ and $|V(G_2)| = 4$. Let $V(G_2) = \{v, w, v_1, w_1\}$. Then $N(v_1) \subseteq V(G_2)$ and $N(w_1) \subseteq V(G_2)$.

Since at least one of $\{v, w\}$ has degree 3 and the other one has degree at least 2, G_2 must have a triangle, a contradiction.

If $\kappa'(G/\pi) \geq 2$, then since $|E(G/\pi)| = (2+3\times 4)/2 = 7 > 6 = 2|V(G/\pi)|-4$, and by (v) of Theorem F, G/π is not reduced. Let H be a maximum collapsible subgraph of G/π. Then H has order at least 2 and so $(G/\pi)/H$ has order at most 4 and $|D_2((G/\pi)/H)| \leq 2$. It is easy to see that $(G/\pi)/H$ is collapsible. Hence, by (i) of Theorem F, G/π is collapsible, and so by (i) of Theorem H, G is collapsible. Lemma 1 is proved. ∎

Remark. The graph $Q_3 - v$ (the cube minus a vertex) shows that $|D_2(G)| \leq 2$ in Lemma 1 cannot be improved. Let G be the graph obtained from $K_{2,3}$ and K_4 by identifying a vertex of degree 2 in the $K_{2,3}$ with a vertex in the K_4. Then G is a 2-edge-connected graph of order 8 with $|D_2(G)| = 2$, but G is not collapsible. This shows that the condition $|V(G)| \leq 7$ in Lemma 1 is necessary.

In the following we shall let P denote the Petersen graph.

Theorem 1. *Let G be a 3-edge-connected simple graph on $n \leq 11$ vertices. Then either G is collapsible or G is the Petersen graph.*

Proof: By way of contradiction, suppose that G is a smallest counterexample to Theorem 1, i.e. G is a 3-edge-connected simple graph with $|V(G)| \leq 11$, but

$$G \in \{\text{collapsible graphs}\} \text{ and } G \neq \text{ Petersen graph } P. \tag{2}$$

Claim G is reduced.

Let G' be the reduction of G. Then G' is a simple graph with $|V(G')| \leq |V(G)|$. If $G' = K_1$, then G is collapsible, contrary to (2). Suppose that $G' \neq K_1$. By the definition of contraction, $\kappa'(G') \geq \kappa'(G) \geq 3$. If $|V(G')| < |V(G)|$ then since G is a smallest counterexample, G' is collapsible, contrary to (iii) of Theorem F. Thus, $|V(G')| = |V(G)|$, and the claim follows.

Since G is a reduced graph, by (iii) of Theorem F, the girth of G is at least 4.

Case 1. G has a 4 cycle, say $C = uvzwu$. Let G/π be the graph defined as before and let the edge $e_\pi = xy$ by the new edge in G/π. By the definition of G/π and 3-edge-connectivity of G, we have that $\delta(G/\pi) \geq 3, \kappa'(G/\pi) \geq 1$ and

$$|V(G/\pi)| = |V(G)| - 2 \leq 9 \tag{3}$$

Case 1(a). $\kappa'(G/\pi) = 1$. Then the new edge $e_\pi = xy$ is the only cut edge of G/π, because G has no cut edge. Therefore, $G - E(C)$ has two components. Let H_1 and H_2 be the two components of $G - E(C)$, where $u, z \in V(H_1)$, and $v, w \in V(H_2)$. Without loss of generality, we may assume $|V(H_1)| \leq |V(H_2)|$. Since $\delta(G) \geq 3$, H_1 has an edge, say $e = x_1x_2$, which is not incident with any vertices of $\{u, v, z, w\}$. Therefore, $N(x_1) \subseteq V(H_1)$ and $N(x_2) \subseteq V(H_1)$. Since G is K_3-free, $N(x_1) \cap N(x_2) = \emptyset$. Therefore,

$$|V(H_1)| \geq |N(x_1)| + |N(x_2)| \geq \delta(G) + \delta(G) \geq 6,$$

and so

$$|V(G)| \geq |V(H_1)| + |V(H_2)| \geq 2|V(H_1)| \geq 12,$$

contrary to $|V(G)| \leq 11$.

Case 1(b). $\kappa'(G/\pi) = 2$. Let E be an edge cut of G/π with $|E| = 2$. Since G is 3-edge-connected, by the definition of G/π, $e_\pi = xy \in E$, for otherwise E is an edge cut of G, contrary to $\kappa'(G) \geq 3$. Let H_1 and H_2 be the two components of $G/\pi - E$, where $|V(H_1)| \leq |V(H_2)|$ and $x \in V(H_1)$, $y \in V(H_2)$. Since G is 3-edge-connected and $\delta(G/\pi) \geq 3$, it follows that $|V(H_2)| \geq |V(H_1)| \geq 2$, $\delta(H_i) \geq 2$ and H_1 and H_2 are 2-edge-connected. Furthermore, if a vertex v has degree 2 in H_i, $1 \leq i \leq 2$, then v is incident with an edge of E, and so $|D_2(H_i)| \leq |E| \leq 2$.

Since $|V(G/\pi)| \leq 9$ and $|V(H_1)| \geq 2$, it follows that $|V(H_2) \leq 7$. If H_2 is simple, then by Lemma 1, H_2 is collapsible. If H_2 is not simple, then H_2 contains a 2-cycle C_2. Let H' be the graph obtained from H_2 by contracting all C_2's until there is no 2-cycle C_2. Then, by the definition of G/π and the fact that if H_2 contains 2-cycle C_2 then $y \in V(C_2)$, the graph H' is simple and 2-edge-connected with $|V(H')| \leq 6$ and $|D_2(H')| \leq 2$. Therefore, by Lemma 1, H' is collapsible, and hence H_2 is collapsible. Similarly, H_1 is also collapsible. Therefore, $((G/\pi)/H_1)/H_2 = C_2$, which is collapsible, and so G/π is collapsible. By (i) of Theorem H, graph G is collapsible, contrary to (2).

Case 1(c). $\kappa'(G/\pi) \geq 3$.

Let G'_π be the reduction of G/π. If $G'_\pi = K_1$, then G/π is collapsible, and hence G is collapsible, contrary to (2). If $G'_\pi \neq K_1$, then $\kappa'(G'_\pi) \geq \kappa'(G/\pi) \geq 3$ and $|V(G'_\pi)| \leq |V(G/\pi)| \leq 9$. By Theorem F, the reduced graph G'_π is simple, and so G'_π satisfies the conditions of Theorem 1. Since G is a smallest counterexample to the theorem and by (3), G'_π is collapsible. Therefore, G/π is collapsible. By (i) of Theorem H, G is collapsible, contrary to (2) again.

Case 2. G has girth at least 5.

Case 2(a). $|V(G)| \leq 10$. Let v be a vertex of G. Then $d(v) \geq \delta(G) \geq 3$. Let $\{x_1, x_2, x_3\} \subseteq N(v)$. Let $S = \cup_{i=1}^{3}(N(x_i) - v)$. By assumption, G has no 3 and 4-cycles, $\delta(G) \geq 3$ and $|V(G)| \leq 10$. It is routine to show that $|S| = 6$ and $G[S] = C_6$ such that G is the Petersen graph P, contrary to (2).

Case 2(b). $|V(G)| = 11$. Since G has order 11 and $\delta(G) \geq 3$, it follows that $\Delta(G) \geq 4$, because G has evenly many vertices of odd degree. Let $v \in V(G)$

with $d(v) \geq 4$. Let $\{x_1, x_2, x_3, x_4\} \subseteq N(v)$. Let $S = \cup_1^4 (N(x_i) - v)$. Since G has no 3 and 4-cycles, and $\delta(G) \geq 3$, $N(x_i) \cap N(x_j) = \{v\}$ if $i \neq j$. Therefore,

$$|V(G)| \geq 4 + 1 + |S| = 5 + \sum_{i=1}^{4} (|N(x_i)| - 1) \geq 5 + 8 = 13,$$

a contradiction.

Since each case leads to a contradiction, the theorem follows. ∎

Remark. Theorem 1 is best possible in some sense. Let G be a graph obtained from P, the Petersen graph, by replacing a vertex v of P by K_3, where each vertex of the K_3 is incident with exactly one edge of $E(P)$ which was incident with the vertex v. Obviously, this graph G is a 3-edge-connected graph of order 12, but G is not collapsible and $G \neq P$.

Lemma 2. *If G' is a 3-edge-connected reduced graph and G' has no spanning eulerian subgraph, then $|V(G')| \geq |D_3(G')| \geq 10$. Furthermore, either $|V(G')| = 10$ and G' is the Petersen graph P, or $|V(G')| \geq 12$.*

Proof: Write $V(G') = \{v_1, v_2, \ldots, v_c\}$, where $c = |V(G')|$. Since G' is 3-edge-connected, G' has no cut-edge and $G' \neq K_{2,t}$ for any integer t. Therefore, by the assumption $G' \notin S\mathcal{L}$, and by Theorem G, these force $F(G') \geq 3$. By (v) of Theorem F,

$$|E(G')| = 2|V(G')| - 2 - F(G') \leq 2|V(G')| - 5,$$

and so,

$$|E(G')| \leq 2c - 5.$$

Hence,

$$\sum_{i=1}^{c} d'(v_i) \leq 4c - 10. \tag{4}$$

Since G' is 3-edge-connected, $\delta(G') \geq 3$, and so the inequality (4) implies

$$3|D_3(G')| + 4(c - |D_3(G')|) \leq \sum_{i=1}^{c} d'(v_i) \leq 4c - 10.$$

Therefore, $|D_3(G')| \geq 10$. By Theorem 1, if $|V(G')| = 10$, then $G' = P$. Otherwise, $|V(G')| \geq 12$. ∎

Theorem 2. *Let G be a 3-edge-connected graph of order n. If every bond $E \subseteq E(G)$ with $|E| = 3$ satisfies the property that each component of $G - E$ has order at least $n/10$, then exactly one of the following holds:*

(i) $G \in SL$;
(ii) $n = 10s$ *for some integer s, and G can be contracted to $G' = P$ such that the preimage of each vertex of G' is a collapsible subgraph of G on exactly* $/10$ *vertices.*

Proof: Let $H_1, H_2, \ldots, H_c$ be the maximal collapsible subgraphs of G. Let G' be the reduction of G obtained from G by contracting the H_i's to distinct vertices $v_1, v_2, \ldots, v_c$, where $c = |V(G')|$. Without loss of generality, we may assume that

$$d'(v_1) \leq d'(v_2) \leq \cdots \leq d'(v_c).$$

If G' is supereulerian, then by (iv) of Theorem F, G is supereulerian. Hence we may assume that G' is not supereulerian. Since G is 3-edge-connected, it follows that G' is 3-edge-connected. By Lemma 2, we have $|V(G')| \geq |D_3(G')| \geq 10$ and so $d'(v_i) = 3$ for $1 \leq i \leq 10$. Therefore, each preimage H_i of v_i $(1 \leq i \leq 10)$ is joined to the remainder of G by a bond consisting of the $d'(v_i) = 3$ edges that are incident with v_i in G. By the hypothesis of Theorem 2,

$$|V(H_i)| \geq \frac{n}{10} \; (1 \leq i \leq 10). \tag{5}$$

It follows that

$$n = |V(G)| = \sum_1^c |V(H_i)| \geq \sum_1^{10} |V(H_i)| \geq n.$$

Therefore $c = 10$. By Lemma 2, $G' = P$, and the preimage H_i of each vertex v_i of G' has exactly $n/10$ vertices

Theorem 2 is proved. ∎

From the proof of Theorem 2, immediately, we can see that the following theorem holds.

Theorem 3. *Let G be a 3-edge-connected nonsupereulerian simple graph of order n. Let G' be the reduction of G. Let $H_1, H_2, \ldots, H_r$ be the maximal collapsible subgraphs of G corresponding to the vertices in $D_3(G')$, where $r = |D_3(G')|$. If $|V(H_i)| \geq n/10$ $(1 \leq i \leq r)$, then the following holds: $r = 10$, $n = 10m$ for some integer m, and G is contractible to P such that the preimage of each vertex of P is a collapsible subgraph H_i $(1 \leq i \leq 10)$ on exactly $n/10$ vertices.* ∎

Corollaries

The first corollary improves Theorem E (Li [12]) for 3-edge-connected graphs:

Corollary 1. *Let G by a 3-edge-connected simple graph of order n. If $\delta(G) \geq 4$ and if every edge $uv \in E(G)$ satisfies*

$$d(u) + d(v) \geq \frac{n}{5} - 2, \tag{6}$$

then exactly one of the following holds:

(i) $G \in S\mathcal{L}$;

(ii) $n = 10s$ *for some integer* $s \geq 5$, *and G can be contracted to P such that the preimage of each vertex of P is either K_s or $K_s - e$ for some edge $e \in E(K_s)$.*

Proof: At first we show that G satisfies the hypothesis of Theorem 2.

Let E be a bond of G with $|E| = 3$, and let H be a component of $G - E$. Since $|E| = 3$ and $\delta(G) \geq 4$, H has a vertex, say u, which is not an end of any edges of E. Since $d(u) \geq \delta(G) \geq 4$, u has a neighbor in H, say v, that is also not an end of any edges of E. Therefore, $N(u) \subseteq V(H)$ and $N(v) \subseteq V(H)$, and so

$$d(u) \leq |V(H)| - 1,$$
$$d(v) \leq |V(H)| - 1.$$

Since G is simple, and by (6),

$$\frac{n}{5} - 2 \leq d(u) + d(v) \leq (|V(H)| - 1) + (|V(H)| - 1) = 2|V(H)| - 2,$$

and so

$$|V(H)| \geq \frac{n}{10}.$$

By Theorem 2, either G is supereulerian, or $n = 10s$ and G is contractible to P such that the preimage H_i of each vertex v_i of P is a subgraph on exactly $n/10$ vertices. Since $d'(v_i) = 3$ for any $v_i \in V(P)$, there are only 3 edges of G, say e_1, e_2 and e_3, which are incident with at most 3 vertices of H_i. Therefore, by (6) and $\delta(G) \geq 4$, $H_i = K_s$ or $K_s - e$ for some $e \in E(K_s)$, and so it is easy to check $s \geq 5$. ∎

Corollary 2. *Let G be a 3-edge-connected simple graph of order $n \geq 41$. If*

$$\delta(G) \geq \frac{n}{10} - 1,$$

the either G is supereulerian or $n = 10s$ for some integer $s \geq 5$, and G can be contracted to P such that the preimage of each vertex of P is either K_s or $K_s - e$ for some edge e.

Proof: The inequalities $n \geq 41$ and (7) imply that $\delta(G) \geq 4$ and (6) hold in Corollary 1, and so Corollary 2 follows. ∎

The following example shows that Corollary 1 and Corollary 2 are best possible in some sense.

Example. Let G be the graph constructed by taking the union of K_{23} and the Blanuša snark [3], and by identifying a pair of vertices, one from each component. Thus, G is a 3-edge-connected graph of order $n = 40$, and

$$\delta(G) = 3 \geq \frac{n}{10} - 1,$$

and so for every $uv \in E(G)$ (or $uv \notin E(G)$),

$$d(u) + d(v) \geq 6 \geq \frac{n}{5} - 2.$$

But the reduction of G is the Blanuša snark, which is neither supereulerian nor contractible to the Petersen graph P, and so G does not satisfy any conclusions of Corollaries 1 and 2. One can see that some other 3-edge-connected nonsupereulerian reduced graphs of order $n \leq 40$ can also be used to construct such a graph G. This shows that $\delta(G) \geq 4$ in Corollary 1 is necessary, and $n \geq 41$ in Corollary 2 is best possible in some sense. ∎

We close by mentioning a result of Catlin [7] which is analogous to Corollary 1.

Theorem I. (Catlin [7]) *Let G be a 3-edge-connected simple graph of order n. If n is sufficiently large and if*

$$d(u) + d(v) > \frac{n}{5} - 2$$

whenever $uv \notin E(G)$, then G has a spanning eulerian subgraph. ∎

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