Some Constructions of Pairwise Orthogonal Diagonal Latin Squares

B. Du

Department of Mathematics
Suzhou University
Suzhou 215006 China (PRC)

Abstract. A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals. In this paper we give some constructions of pairwise orthogonal diagonal Latin squares (PODLS). As an application of such constructions we improve the known result about three PODLS and show that there exist three PODLS of order n whenever $n > 46$; orders $2 \leq n \leq 6$ are impossible, the only orders for which the existence is undecided are: 10, 14, 15, 18, 21, 22, 26, 30, 33, 34 and 46.

1. Introduction

A *Latin square* of order n is $n \times n$ array such that every row and every column is a permutation of an n-set. A *transversal* in a Latin square is a set of positions, one per row and one per column among which the symbols occur precisely once each. A *symmetric transversal* in a Latin square of even order is a transversal which is a set of symmetric positions. A *transversal Latin square* is a Latin square whose main diagonal is a transversal. (It is clear that a transversal square is equivalent to an idempotent square in [11]). A *diagonal Latin square* is a transversal Latin square whose back diagonal also forms a transversal. It is easy to see that the existence of a transversal Latin square with a symmetric transversal implies the existence of a diagonal Latin square.

Two Latin squares of order n are *orthogonal* if each symbol in the first square meets each symbol in the second square exactly once when they are superimposed. *t pairwise orthogonal diagonal (transversal) Latin squares* of order n, denoted briefly by t PODLS (n) (POILS(n)) are t pairwise orthogonal Latin squares each of which is a diagonal (transversal) Latin square of order n. We let $D(n)$ ($I(n)$) denote the maximum number of pairwise orthogonal diagonal (transversal) Latin squares of order n.

For $t = 2$, it has been shown (see [3,4,6,7,10]) that a pair of orthogonal diagonal Latin squares exists for all n with the 3 exceptions: $n \in \{2, 3, 6\}$. For $t = 3$, it has been shown (see [11]) that three pairwise orthogonal diagonal Latin squares of order n exist for all n with the 5 exceptions when $n \in \{2, 3, 4, 5, 6\}$ and 28 possible exceptions.

It is our purpose here to reduce this number of possible exceptions to 11. We also present some constructions of pairwise orthogonal diagonal Latin squares.

For our purpose, let $IA_t(v, k)$ denote t pairwise orthogonal Latin squares of order v (briefly t POLS(v)) with t sub-POLS(k) missing. Usually we leave the size k hole in the lower right corner. Further denote by $IA_t^*(v, k)$ an $IA_t(v, k)$

in which the first $v-k$ elements in the main diagonal of every square are distinct and different from the missing elements. It is easy to see that the existence of an $IA_{t+1}(v,k)$ implies the existence of an $IA_t^*(v,k)$, and that $IA_t^*(v,1)$ exists if there exist t pairwise orthogonal transversal Latin squares of order v. From [2,9] we have

Example 1.1. *There exist* $IA_3^*(9,2)$ *and* $IA_3^*(10,2)$.

We also denote by $IA_t^{**}(v,k)$ an $IA_t^*(v,k)$ in which the elements in the cells $(1,v-k),(2,v-k-1),\ldots,(v-k,1)$ of every square are distinct and different from the missing elements. It is clear that an $IA_t^{**}(v,0)$ exists if $D(v)\geq t$ and that an $IA_t^{**}(v,1)$ exists if v is odd and $D(v)\geq t$.

Notice that if an $IA_t^*(v,k)$ has a common symmetric transversal in the upper left $(v-k)\times(v-k)$ subarray with entries different from the missing elements, then we can obtain an $IA_t^{**}(v,k)$ from it. From [8] we have

Example 1.2. *There exist* $IA_3^{**}(18,4)$, $IA_3^{**}(38,8)$, *and* $IA_3^{**}(42,8)$.

Finally, we denote by $A_t(v,k)$ a set of t POILS(v) in which the cells $\{(v-k+i,v-i+1):1\leq i\leq k\}$ form a common transversal about elements $x_1,x_2,\ldots,x_k$.

Example 1.3. *An* $A_6(9,2)$ *exists.*

Proof: In $GF(9)=\{a_0=0,a_1=1,a_2=-1,a_3,\ldots,a_8\}$, take the 9×9 array

$$L_k=\left(h_{ij}^k\right)\quad 3\leq k\leq 8$$

where $h_{ij}^k=\lambda_k a_i+\mu_k a_j,\lambda_k,\mu_k\in GF(9)\backslash\{0,1,-1\},\lambda_k\neq\mu_k$ and $\lambda_k+\mu_k=1$. It is easy to see that L_k, $3\leq k\leq 8$, are 6 POILS(9) each of which has different elements in the cells $(0,1)$ and $(1,0)$; then we obtain $A_6(9,2)$ by permuting rows and columns and renaming symbols.

2. Some Constructions

We need the following new constructions. For simplicity we shall not state their general form, but only the special case needed for this paper.

Let Q be a Latin square of order n based on the set $I_n=\{0,1,\cdots,n-1\}$ and let S, T be transversals of Q. Form a permutation $\sigma_{S,T}$ on I_n as follows: $\sigma_{S,T}(s)=t$ where s and t are the entries of S and T, respectively, occurring in the same row. We denote by $Q(S,T)$ the Latin square obtained by renaming symbols using $\sigma_{S,T}$. Obviously we have

(a) If U is a transversal of Q, then U is also a transversal of $Q(S,T)$;

(b) If V is a Latin square which is orthogonal to Q, then V is also orthogonal to $Q(S,T)$.

Let Q be a Latin square and let h be a symbol; we denote by Q_h the copy of Q obtained by replacing each entry x of Q with the ordered pair (h,x).

Lemma 2.1. *For a positive integer k, let A, B, C, be three pairwise orthogonal Latin squares of order k which possess four disjoint common transversals T_1, T_2, T_3 and the main diagonal D. Then there exist three pairwise orthogonal diagonal Latin squares of order $4k$.*

Proof: Consider the three pairwise orthogonal Latin squares of order $4k$

$$\bar{A} = \begin{matrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & A_3 & A_0 & A_1 \\ A_3 & A_2 & A_1 & A_0 \end{matrix} \qquad \bar{B} = \begin{matrix} B_0 & B_1 & B_2 & B_3 \\ B_2 & B_3 & B_0 & B_1 \\ B_3 & B_2 & B_1 & B_0 \\ B_1 & B_0 & B_3 & B_2 \end{matrix} \qquad \bar{C} = \begin{matrix} C_0 & C_1 & C_2 & C_3 \\ C_3 & C_2 & C_1 & C_0 \\ C_1 & C_0 & C_3 & C_2 \\ C_2 & C_3 & C_0 & C_1 \end{matrix}$$

Denote the 16 subsquares of $\bar{A}$ by $\bar{A}_{ij}$, $0 \le i, j \le 3$. For $0 \le i \le 3$, $1 \le j \le 3$, replace $\bar{A}_{ij}$ by $\bar{A}_{ij}(D, T_j)$. Then, for $0 \le i \le 3$, $1 \le j \le 3$, and for every cell $(r, c) \in T_j$, interchange the entries in cells $\bar{A}_{10}(r, c)$ and $\bar{A}_{ij}(r, r)$. Call the resulting array $\hat{A}$. From (a) it follows immediately that $\bar{A}$ is a transversal Latin square with a symmetric transversal.

Do the same replacement and exchange of entries for $\bar{B}$ and $\bar{C}$. By (b) the resulting squares $\hat{B}$ and $\hat{C}$ together with $\hat{A}$ form 3 POILS with a common symmetric transversal, which consists of the main diagonals of those blocks appearing in the block back diagonal. By simultaneously permuting rows and columns we have 3 PODLS($4k$).

Corollary 2.2. $D(n) \ge 3$ *for* $n \in \{20, 28, 36, 44, 52\}$.

Proof: Since for each n there exists three pairwise orthogonal Latin squares of order $\frac{n}{4}$ (see [1]) satisfying the hypotheses of the Lemma 2.1, we have $D(n) \ge 3$.

Lemma 2.3. *For a positive integer k, let A, B, C be three pairwise orthogonal diagonal Latin squares of order k which possess three disjoint common transversals T_1, T_2 and the main diagonal D. If the positions of T_1, T_2 are symmetric about the main diagonal, then there exist three pairwise orthogonal diagonal Latin squares of order $5k$.*

Proof: Consider the three pairwise orthogonal Latin squares of order $5k$.

$$\bar{A} = \begin{matrix} A_0 & A_1 & A_2 & A_3 & A_4 \\ A_1 & A_2 & A_3 & A_4 & A_0 \\ A_2 & A_3 & A_4 & A_0 & A_1 \\ A_3 & A_4 & A_0 & A_1 & A_2 \\ A_4 & A_0 & A_1 & A_2 & A_3 \end{matrix} \quad \bar{B} = \begin{matrix} B_0 & B_1 & B_2 & B_3 & B_4 \\ B_2 & B_3 & B_4 & B_0 & B_1 \\ B_4 & B_0 & B_1 & B_2 & B_3 \\ B_1 & B_2 & B_3 & B_4 & B_0 \\ B_3 & B_4 & B_0 & B_1 & B_2 \end{matrix} \quad \bar{C} = \begin{matrix} C_0 & C_1 & C_2 & C_3 & C_4 \\ C_3 & C_4 & C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 & C_4 & C_0 \\ C_4 & C_0 & C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 & C_0 & C_1 \end{matrix}$$

Notice that the set E_i^j of the entries of the j-th column of $\bar{A}$ which lie on the transversals T_i coincides with the set $\bar{E}_i^j$ of the entries of the $(4k + j)$-th column of $\bar{A}$ lying on the transversals T_i, the set F_i^j of the entries of the $(k + j)$-th column of $\bar{A}$ lying on the transversals T_i, the set F_i^j of the entries of the $(k + j)$-th column of $\bar{A}$ which lie on the transversals T_i coincides with the set $\bar{F}_i^j$ of the entries of the $(3k+j)$-th column of $\bar{A}$ lying on the transversals T_i. For each $i = 1, 2$, $j = 1, 2, \ldots, k$, exchange in $\bar{A}$ the elements of K_i^j and $\bar{K}_i^j$ appearing on the same row, $K = E, F$. It is clear that the resulting array $\hat{A}$ is a transversal Latin square with a transversal which consists of an element in the central cell and a set of elements in symmetric positions.

Do the same exchange of entries for $\bar{B}$ and $\bar{C}$. It is easy to see that the resulting squares $\hat{B}$ and $\hat{C}$ together with $\hat{A}$ form 3 POILS with a common transversal, which consists of the back diagonal in the central block and the T_1 in the upper right blocks and the T_2 in the lower left blocks of the block back diagonal. By simultaneously permuting rows and columns we have 3 PODLS($5k$).

Corollary 2.4. *$D(n) \geq 3$ for $n \in \{35, 45\}$.*

Proof: For each n there exist three pairwise orthogonal Latin squares of order $\frac{n}{5}$ (see [1]) satisfying the hypotheses of the Lemma 2.4. Hence $D(n) \geq 3$.

Lemma 2.5. *Suppose there are $t+1$ POLS(q) such that t of them are t PODLS(q).*

(1) Suppose qmk is odd, $D(k) \geq t$ and that $IA_t^*(m+k_i, k_i)$ exist for $0 \leq i \leq q-1$, where $k = k_0 + k_1 + \cdots + k_{q-1}$. Further suppose an $IA_t^{**}(m+k_0, k_0)$ exists if q is odd. Then $D(qm+k) \geq t$.

(2) Suppose qmk is odd, $D(k) \geq t$ and that $IA_t^*(m+k_i, k_i)$ exist for $0 \leq i \leq q-1$, where $k = k_0 + k_1 + \cdots + k_{q-1}$, $k_0 = 1$. Then $D(m+1) \geq t$ implies $D(qm+k) \geq t$.

Proof: Since (1) is Lemma 3.3 in [11], we only prove (2).

Since t PODLS(q) have an extra orthogonal mate, they have q disjoint common transversals each of which is determined by an element in the extra square. Label these transversals as $T_0, T_1, \cdots, T_{q-1}$, provided that T_0 contains the central cells.

Begin with the t PODLS(q) and replace each of its cells with an $m \times m$ array labelled by the elements in the cell. The array is the upper left part of $IA_t^*(m + k_i, k_i)$ if the cell is contained in T_i, $1 \leq i \leq q-1$. But if the cell is in the back diagonal of the t PODLS(q), it will be filled with a modified $IA_t^*(m+k_i, k_i)$, that is, by permuting the first m columns the main diagonal of the upper left part in the $IA_t^*(m+k_i, k_i)$ becomes its back diagonal. The array is the upper left part of $IA_t^*(m+1, 1)$ which is obtained from t PODLS($m+1$) missing an element (say x) in the lower right corner, if the cell is contained in T_0. Suppose every $IA_t^*(m+k_i, k_i)$ is based on certain m elements and k_i new elements, $0 \leq i \leq q-1$, and the new elements remain unchanged when labelling. Then we obtain the upper left part of an $IA_t^{**}(qm+k, k)$ whose right part consists of the columns $C_0, C_1, \cdots, C_{q-1}$ where C_i comes from the right part of the $IA_t^*(m+k_i, k_i)$ in T_i, and the lower part is obtained in a similar fashion (see Figure 1).

Begin with the t PODLS(k) again, and suppose the cell $(\frac{k+1}{2}, \frac{k+1}{2})$ of each Latin square is always x. We permute rows and columns with permutation σ_1.

$$\sigma_1 = \begin{pmatrix} 1 & 2 & \cdots & \frac{k-1}{2} & \frac{k+1}{2} & \frac{k+3}{2} & \cdots & k \\ \frac{k+3}{2} & \frac{k+5}{2} & & k & 1 & 2 & & \frac{k+1}{2} \end{pmatrix}$$

Then we obtain t POLS(k) for which the cell $(1, 1)$ is always x; for each of these, the main diagonal and the cells $\{(i, k+2-i): 2 \le i \le k\}$ is a transversal.

Now fill the size k hole in the lower right corner of the $IA^{**}(qm+k, k)$ with the above t POLS(k), and permute rows and columns with permutation σ_2

$$\sigma_2 = \left(\begin{matrix} 1 & 2 & \cdots & \frac{qm-m}{2} & \frac{qm-m}{2}+1 & \frac{qm-m}{2}+2 & \cdots \\ \frac{k+1}{2} & \frac{k+2}{2} & & \frac{qm+k}{2}-4 & \frac{qm+k}{2}+5 & \frac{qm+k}{2}+6 & \end{matrix} \right.$$

$$\begin{matrix} qm+1 & qm+2 & qm+3 & \cdots & qm+\frac{k+1}{2} & qm+2 & qm+4 & \cdots \\ qm+\frac{k+1}{2} & \frac{qm+k}{2}-3 & \frac{qm+k}{2}-2 & & \frac{qm+k}{2}+4 & 1 & 2 & \end{matrix}$$

$$\left. \begin{matrix} qm+\frac{k+1}{2} & qm+\frac{k+3}{2} & qm+\frac{k+5}{2} & \cdots & qm+k \\ \frac{k-1}{2} & qm+\frac{k+3}{2} & qm+\frac{k+5}{2} & & qm+k \end{matrix} \right)$$

Then we obtain t PODLS($qm+k$).

The remaining verification is a routine matter and the proof is complete.

Corollary 2.6. $D(n) \ge 3$ *for* $n \in \{70, 102, 110, 114, 118\}$.

Proof: The conclusion comes from Lemma 2.3 (2) with the following expressions

$$70 = 9 \times 7 + 7 \qquad 102 = 13 \times 7 + 11$$

and Lemma 2.3 (1) with the following expressions. There exist an $IA_3^{**}(18, 4)$ from Example 1.2

$$110 = 7 \times 14 + 3 \times 4 \qquad 114 = 7 \times 14 + 4 \times 4 \qquad 118 = 7 \times 14 + 5 \times 4$$

Lemma 2.7. *Suppose there are* $t + w + 1$ *POLS(q) such that* t *of them are* t *PODLS(q),* $h_0 = 0$, q *is even, and there exist* $IA_t^*(m + h_i, h_i)$ *and* $IA_t^*(m + 1 + h_i, h_i)$, $0 \le i \le q-1$, $h = h_0 + h_1 + \cdots + h_{q-1}$. *Further suppose there are* $A_t(m+w, w)$. *Then* $D(h) \ge t$ *implies* $D(qm + w + h) \ge t$, *provided* w *or* h *is even.*

Proof: We only prove the Lemma with h even (the proof for w even is similar).

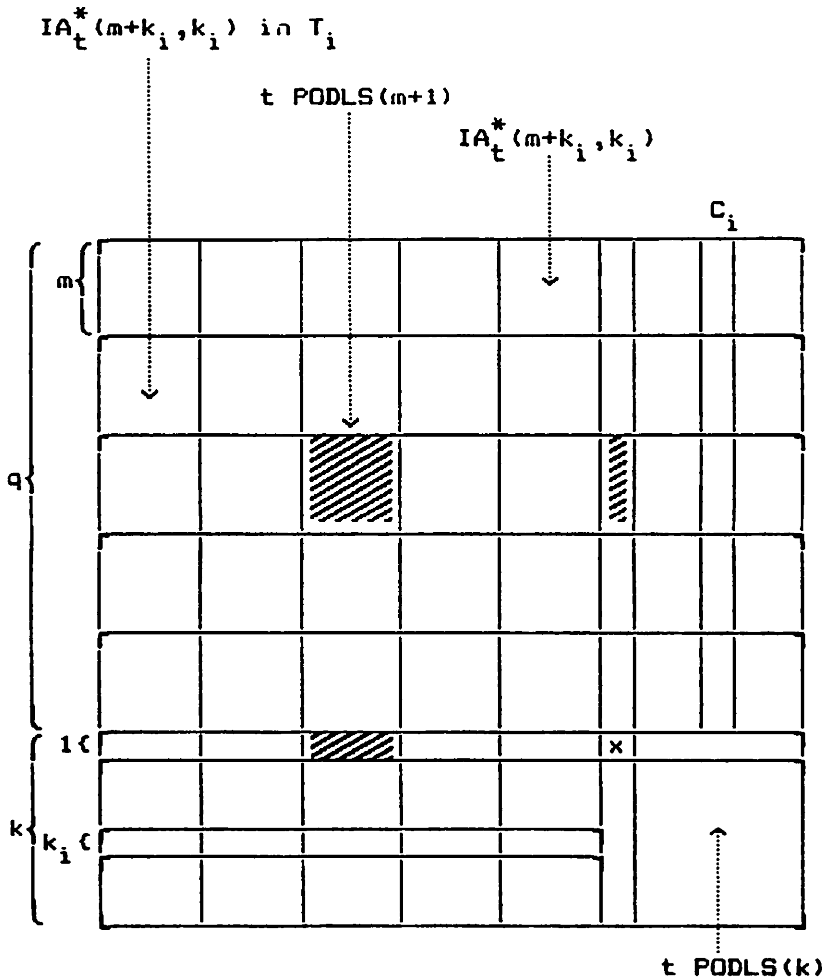

Figure 1.

Since t PODLS(q) have an extra orthogonal mate, they have q disjoint common transversals each of which is determined by an element in the extra square. Label these transversals as $T_0, T_1, \ldots, T_{q-1}$, provided that T_0 contains the (q, q) cells. At the same time, elements in cell (q, q) in each of the w extra orthogonal mates determine a common transversal in the t PODLS(q). Each of such w transversals intersecting in the cell (q, q) just meets T_i in a cell $(0 \leq i \leq q-1)$. We fill its cell with an $IA_t^*(m+1, 1)$, but fill the cell which is also in T_i with $IA_t^*(m+$

$1 + h_i, h_i)$ $(1 \le i \le q-1)$ and leave the cell (q, q) empty. Fill any other cell with $IA_t^*(m + h_i, h_i)$ if it is in T_i $(0 \le i \le q-1)$. Notice that the cells in the back diagonal of t PODLS(q) are filled with some modified $IA_t^*(m + h_i, h_i)$ or $IA_t^*(m + 1 + h_i, h_i)$ whose back diagonal of the order m subarray is occupied by

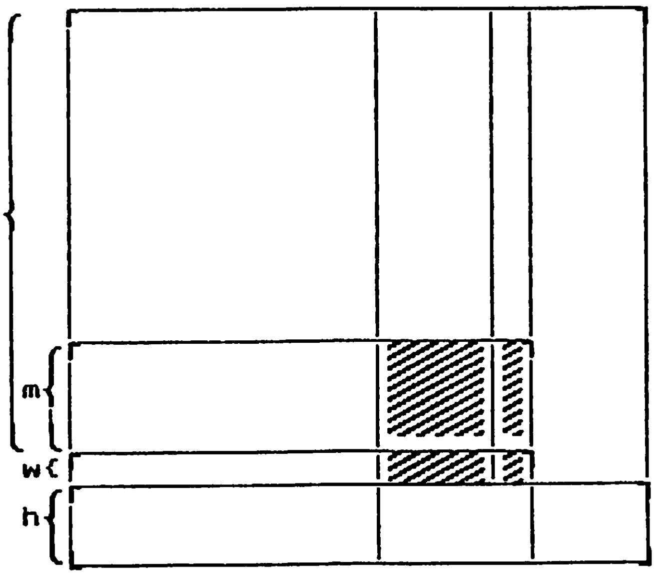

Figure 2.

different elements. Label the elements and get the right and lower parts as we did in Lemma 2.5; it has to be noticed that the cells which fill $IA_t^*(m + 1 + h_i, h_i)$ belongs two transversals. Then we get an array shown in Figure 2.

We fill the size $m + w$ hole with the given $A_t(m + w, w)$ and the size h hole with the given $D(h)$. Finally we permute rows and columns with permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & \frac{qm}{2} & \frac{qm}{2}+1 & \frac{qm}{2}+2 & \cdots \\ \frac{h}{2}+1 & \frac{h}{2}+2 & \cdots & \frac{qm+h}{2} & \frac{qm+h}{2}+w+1 & \frac{qm+h}{2}+w+2 & \cdots \end{pmatrix}$$

$$\begin{matrix} qm & qm+1 & qm+2 & \cdots & qm+w & qm+w+1 & qm+w+2 & \cdots \\ \frac{qm+h}{2}+w & \frac{qm+h}{2}+1 & \frac{qm+h}{2}+2 & \cdots & \frac{qm+h}{2}+w & 1 & 2 & \cdots \end{matrix}$$

$$\begin{matrix} qm+2+\frac{h}{2} & qm+\frac{h}{2}+w+1 & qm+\frac{h}{2}+w+2 & \cdots & qm+w+h \\ \frac{h}{2} & qm+\frac{h}{2}+w+1 & qm+\frac{h}{2}+w+2 & \cdots & qm+w+h \end{matrix}\Bigg)$$

Then we obtain t PODLS($qm + w + h$).

The remaining verification is a routine matter and the proof is complete.

Corollary 2.8. $D(66) \geq 3$.

Proof: Write $66 = 8 \times 7 + 2 \times 1 + 4 \times 2$. Since there are $IA_3^*(9,2)$, $IA_3^*(10,2)$ and $A_3(9,2)$ from Examples 1.1 and 1.3, we have $D(66) \geq 3$ from Lemma 2.7.

Lemma 2.9. $D(74) \geq 4$.

Proof: The central element in each of the 4 extra order 9 orthogonal mates determines a common transversal in the 4 PODLS(9). Label these transversals as T_0, T_1, T_2, T_3. Here T_2 is the main diagonal and T_3 the back diagonal. And they also have 9 disjoint common transversals $T_2, S_1, S_2, \ldots, S_8$.

Begin with the 4 PODLS(9) and for each of such 4 transversals intersecting in the central cell, fill its cells with an $IA_4^*(8,1)$, but leave the central cell empty, as in Lemma 3.7 of [11]. At the same time, do the construction as in Lemma 2.5 for the $S_1, S_2, \ldots, S_8$, that fill the cells of $S_1, S_2, \ldots, S_7$ with an $IA_4^*(8,1)$ or $IA_4^*(9,1)$; fill the cells of S_8 with $IA_4^*(7,0)$ or $IA_4^*(8,0)$. We get an array as in Lemma 2.7.

Now fill the size 11 hole of the above array with the $IA_4^*(11,1)$ which is obtained from 4 PODLS(11) missing an element (say x) in the lower right corner. Then we get an array shown in Figure 3.

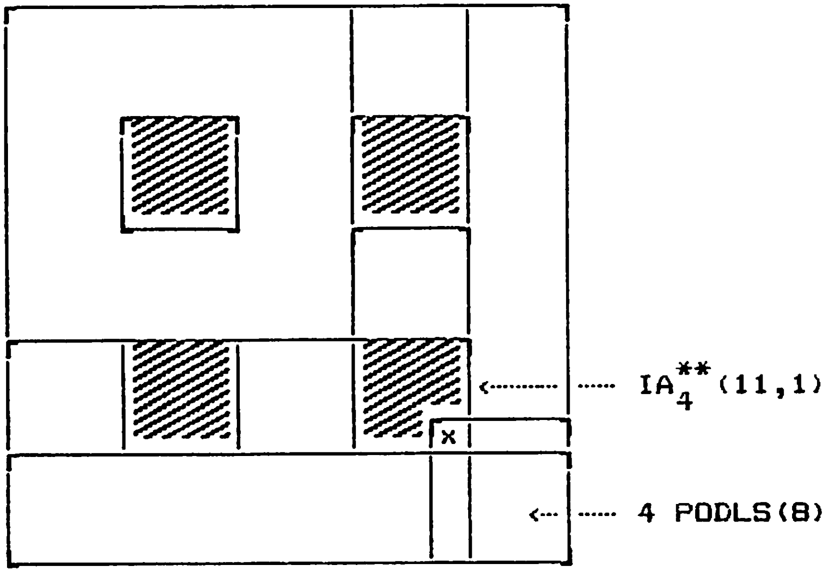

Figure 3.

Finally, fill the size 8 hole of the array with the 4 PODLS(8) for which the cell $(1,1)$ is x. Then we get 4 PODLS(74) by permuting rows and columns as in Lemma 2.7.

Lemma 2.10. $D(n) \geq 3$ *for* $n \in \{38, 42\}$.

Proof: Fill the size 8 hole of the $IA_3^{**}(n, 8)$ with the 3 PODLS(8) where $IA_3^{**}(n, 8)$ from Example 1.2.

Finally, we construct three pairwise orthogonal diagonal Latin squares of order 24 by modifying the four pairwise orthogonal Latin squares of order 24 constructed by Roth and Peters in [5]. Their construction goes as follows. Let G be the group $Z_6 \times Z_2 \times Z_2$ generated by elements a, b and c, where $a^6 = b^2 = c^2 = e$, the identity. Let $S = (s_1, s_2, \ldots, s_{24})$ be some specified ordering of the elements of G, with first element e, and let $P = \{p_1, p_2, \ldots, p_{24}\}$ be any ordering of G. Then $L_S(P)$ is the 24×24 array with (i, j) entry $s_i p_j$ and $L_S(p)$ is clearly a Latin square. Roth and Peters constructed rows P_1, P_2, P_3, and P_4 such that $L_S(P_1), L_S(P_2), L_S(P_3)$, and $L_S(P_4)$ were pairwise orthogonal.

We have modified their solution as Wallis and Zhu did in [7] to obtain the following four rows which generate orthogonal Latin squares:

$$P_1 = e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b, a^5c, a^4c, a^3c, a^2c, ac, c, a^5bc, a^4bc, a^3bc, a^2bc, abc, bc;$$

$$P_2 = e, a^3, a^5b, a^2c, a^2b, a^3c, c, a^4bc, ac, a^4, a^5bc, a, ab, a^5c, a^4b, abc, a^4c, bc, a^3bc, a^2, a^2bc, a^5, a^3b, b;$$

$$P_3 = e, a^2, a^4, a^5c, a^3bc, a^4b, bc, a^2bc, a^5bc, ab, a, a^3c, a^3, ac, abc, a^5b, a^2b, b, a^4bc, a^3b, a^5, a^4c, a^2c, c;$$

$$P_4 = e, a^2b, a^2bc, a^4bc, a^5c, ab, a^4c, b, a^5, bc, a^3c, a^4, a, c, a^3bc, a^2, a^3b, ac, a^4b, a^2c, abc, a^5bc, a^5b, a^3.$$

Now if we have

$$S = e, a^5, a^4, a^3, a^2, a, b, a^5b, a^4b, a^3b, a^2b, ab, ac, a^2c, a^3c, a^4c, a^5c, c, abc, a^2bc, a^3bc, a^4bc, a^5bc, bc.$$

We see that $L_S(P_1)$ has main diagonal and back diagonal constant (e and bc respectively). By orthogonality to $L_S(P_1)$, each $L_S(P_i)$, $i > 1$, must have main diagonal and back diagonal transversals. So they are three pairwise orthogonal Latin squares of order 24. Then we have

Lemma 2.11. $D(24) \geq 3$.

3. Conclusion

It has been shown that $D(n) \geq 3$ for $n \in \{20, 24, 28, 35, 36, 38, 42, 44, 45, 52, 66, 70, 74, 102, 110, 114, 118\}$. Updating the result in [11] we obtain the following theorem.

Theorem 3.1. *There exist three pairwise orthogonal diagonal Latin squares of every order n where $n > 46$. Orders $2 \leq n \leq 6$ are impossible; the only orders for which the existence is undecided are:*

$$10 \quad 14 \quad 15 \quad 18 \quad 21 \quad 22 \quad 26 \quad 30 \quad 33 \quad 34 \quad 46.$$

Acknowledgment

I would like to thank Professor L. Zhu for introducing me to this work. Also, I would like to thank the referees for their valuable suggestions for a previous version of this paper.

References

1. A.E. Brouwer, *The number of mutually orthogonal Latin squares—a table up to order 10, 000*, Math. Centr. report ZW123. June (1979).
2. A.E. Brouwer, *4 MOLS of order 10 with a hole of size 2*, J. Statist. Planning and Infer. **10** (1984), 203–205.
3. J.W. Brown, F. Cherry, L. Most, M. Most, E.T. Parker and W.D. Wallis, *The spectrum of orthogonal diagonal Latin squares*. (to appear).
4. K. Heinrich and A.J.W. Hilton, *Doubly diagonal orthogonal Latin squares*, Discrete Math. **46** (1983), 173–182.
5. R. Roth and M. Peters, *Four pairwise orthogonal Latin squares of order 24*, J. Combinatorial Theory (A) **44** (1987), 152–155.
6. W.D. Wallis, *Three new orthogonal diagonal Latin squares*, in "Enumeration and Design", Academic Press Canada, 1984, pp. 313–315.
7. W.D. Wallis and L. Zhu, *Four pairwise orthogonal diagonal Latin squares of order 12*, Utilitas Math. **21C** (1982), 205–207.
8. S.P. Wang, *On self-orthogonal Latin squares and partial transversals of Latin squares*. Ph.D. thesis, Ohio State University (1978).
9. L. Zhu, *Pairwise orthogonal Latin squares with orthogonal small subsquares*. Research Report CORR 83-19, University of Waterloo, June 1983.
10. L. Zhu, *Orthogonal diagonal Latin squares of order fourteen*, J. Austral. Math. Soc. (A) **36** (1984), 1–3.
11. L. Zhu, *Three pairwise orthogonal diagonal Latin squares*, JCMCC **5** (1989), 27–40.