THE LOCAL MINIMA IN THE LATTICE-SIMPLEX COVERING PROBLEM

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1. Abstract

The lattice-simplex covering density problem aims to determine the minimal density by which lattice translates of the n-simplex cover n-space. Currently the problem is completely solved in 2 dimensions. A computer search on the problem in three dimensions gives experimental evidence that for the simplex D (the convex hull of the unit basis vectors), the most effective lattice corresponds to the tile known as the 84-shape. The 84-shape tile has been shown to be a local minimum of the density function. We explain the mechanics behind an algorithm which determines the most efficient lattice in the interior of an arbitrary combinatorial type.

2. Introduction

Finding the thinnest covering density by translates of an object C is a well established problem with an extensive and impressive history of research. Covering problems are related to packing problems, wherein one tries to maximize the amount of space covered by translated shapes without overlap. In covering problems the desired translates are required to cover all of $\mathbb{R}^n$ and one attempts to minimize the overlap or redundancy of the covering. This paper restricts to a currently unsolved problem where the covering object is an n-simplex in $\mathbb{R}^n$ and the translates are given by a lattice.

For any n-simplex S, the optimal covering density $\vartheta_n(S)$ in $\mathbb{R}^n$, since linear transformation of S can transform the lattice covering in the same way, preserving the covering density. In two dimensions $\vartheta_2(S) = 1.5$. A proof of this fact can be found in [5], though it is not the earliest known published result. In three dimensions $\vartheta_3(S) \leqq \frac{10^3}{84 \cdot 6} \approx 1.9841...$ an upper bound found independently by Fiduccia, Zito, and Mann [4], and by Dougherty and Faber [2], corresponding to the 84-shape. The authors have run a computer search algorithm in 4 dimensions and currently have established $\vartheta_4(S) \leqq \frac{16^4}{922 \cdot 24} \approx 2.9616775....$

These upper bounds have been found using a computer search on the diameters of abelian Cayley graphs. For an abelian group with n generators, the generators induce a homomorphism from $\mathbb{Z}^n \to G$. The kernel of this homomorphism is a sublattice L of $\mathbb{Z}^n$. If ∂ is the diameter of the Cayley graph on the n generators, $\partial + n$ is the Manhattan diameter of the tile T: the maximum distance in the one-norm from the origin to a point in T. The tile T is canonical and satisfies $T + L = \mathbb{R}^n$. We can explicitly find T by using the constructions mentioned in [2] and [3]. However, for our search algorithm the diameter can be found without explicitly finding the tile. Regardless, this search method only works for rational or integer tiles. We make this concept precise in the following sections. Restricting to integer tiles does not find all efficient tiles.

We proceed to define a combinatorial type and then present an algorithm which finds the local minima within a combinatorial type.

3. Combinatorial Type

Since linear actions on the simplex preserve the optimal density, we will fix the simplex and search for the most efficient lattice. The traditional choice of simplex is the convex hull, D, of the unit vectors and the origin. This will enable us to find the canonical tile corresponding to a given lattice via the subtraction construction [5]. The subtraction construction enables us to create a uniquely determined tile from the lattice. We will not define the subtraction construction, but full details and implementation can be found in [5].

The subtraction construction gives a tile T s.t $T + L$ covers $\mathbb{R}^n$. Furthermore T is uniquely determined and T satisfies a cascading condition from the axis: If $s \in T$ then all points that are entrywise nonnegative and entrywise no greater than s are also in T. T is contained in the smallest dilation of D for the given lattice. Let d denote the Manhattan diameter of T. Then d and the determinant of any basis for the lattice (the volume of the fundamental domain), give us the covering density of the lattice with the simplex D, $\delta(L) = \frac{d^n}{\det(L)\cdot n!}$. The density of the lattice covering is a continuous function on lattices, allowing us to search for local minima.

The tiles found by the subtraction construction will be defined by their *blockers*, lattice points whose coordinates determine the face of the tile. Then each corner of the tile will have at least three blockers, corresponding to the three faces that intersect at the corner. The translation of the tile at the blocker will cover the corresponding face of the tile. We say L is in *general position* if each outer face of the tile has exactly one blocker.

The *combinatorial type* of L is the set of all lattices, L' with the same blocker-to-corner relationships as L.

4. An Algorithm For Finding Local Minima

Previously the process of proving that a tile is a local minimum was done in a few specific cases such as the 84-shape [5]. We generalize that process as follows.

Given a lattice L in general position, with basis given by the rows of the matrix B, we can use the subtraction construction to find a tile T whose Manhattan diameter is minimal for that lattice. We proceed by assuming to L as being a three dimensional lattice, although all of the work generalizes to n-dimensions.

Since L is in general position, each corner is determined by exactly three blockers, one for each of its three faces. This means that L is in the interior of our combinatorial type. The boundary of a combinatorial type will contain lattices whose tiles have at least one face determined by multiple blockers.

Given a particular corner $A = (a_1, a_2, a_3)$ of the tile, we may arrange its three blockers, viewed as row vectors, into a *configuration matrix*

$$C = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

where $x_{11} = a_1$, $x_{22} = a_2$ and $x_{33} = a_3$. In other words, the coordinates of the corner appear on the diagonal, and the one norm distance from the origin to A is equal to the trace of C. Since the rows of C are lattice points, we have $C = NB$ where N is an integer matrix. Thus the Manhattan diameter is $\mathrm{tr}(C) = \langle N^\tau, B\rangle$ (the matrix inner product). Since we assumed that L is in the interior of its combinatorial region, changing L slightly will not change the relationship between blockers and corners, thus preserving the combinatorial type of L. We may write $B' = B(I + \epsilon D)$ where D is a matrix representing the direction of change and ϵ is a small positive real number. Then $C' = NB' = NB(I + \epsilon D)$ will be the corresponding corner of the tile for the new lattice. Suppose the corner A has maximal diameter in the tile. Then the density of the simplex covering is proportional to (leaving out the $\frac{1}{6}$ coefficient)

$$\delta(C) = \frac{(\mathrm{tr}(C))^3}{|\det(B)|}.$$

Noting that $\det(B') = \det(B)\det(I+\epsilon D) = \det(B)\big(1+\epsilon\,\mathrm{tr}(D)+\dots\big)$ and $\mathrm{tr}(C') = \langle C, I + \epsilon D^\tau\rangle = \mathrm{tr}(C) + \epsilon\langle C, D^\tau\rangle$, we may expand δ as a function of ϵ, in series form

$$\delta(C') = \frac{(\mathrm{tr}(C) + \epsilon\,\langle C, D^\tau\rangle)^3}{det(B)(1+\epsilon\,\mathrm{tr}(D)+\dots)} = \delta(C)\Big(1 + \frac{3}{\mathrm{tr}(C)}\langle G, D^\tau\rangle\,\epsilon\, + \,\dots\Big)$$

where $G = C - \frac{\mathrm{tr}(C)}{3} I$ (in other words, G has its diagonal reduced to trace 0). We may think of G as a kind of *gradient* matrix, since the change from $\delta(C)$ to $\delta(C')$ is proportional to the matrix dot product of the direction matrix D with the gradient matrix G.

Now if we have a tile with corners $A_1, A_2, \ldots A_k$ and corresponding configuration matrices C_i and gradient matrices $G_i = C_i - \frac{\mathrm{tr}(C_i)}{3} I$, then the covering density for the given lattice is simply the maximum of the values of $\delta(C_i)$. Thus, if our lattice is a local minimum in the covering density (and in general position, etc.) then any directional change D must result in *increasing* $\delta(C_i)$ for some i. Thus

T1: *At a general position local minimum, the origin lies in the convex hull of the gradient matrices for the corners with maximum diameter.*

That condition can be re-written as follows. If $(0) = \sum_i \alpha_i G_i$ and $C_i = N_i B$, then $\left(\sum_i \alpha_i N_i\right) B = \left(\sum_i \alpha_i \mathrm{tr}(C_i)/3\right) I$ so $\sum_i \alpha_i N_i = \gamma B^{-1}$ where $\gamma > 0$. In other words, B^{-1} is in the convex cone of the integer matrices N_i. Thus

T1a: *At a general position local minimum, the inverse basis matrix B^{-1} lies in the convex cone of the integer matrices N_i corresponding to the maximum diameter corners.*

This means that B^{-1} is a linear combination, with non-negative coefficients, of the N_i matrices, but we will not even use that in its full strength. All we shall actually need is

T1b: *At a general position local minimum, the inverse basis matrix B^{-1} is a linear combination of the integer matrices N_i corresponding to the maximum diameter corners.*

and since $\mathrm{tr}(C_i) = \langle N_i^T, B \rangle$ it follows that

T2: *At a general position local minimum, the basis matrix B has equal matrix scalar products with each of the transposed integer matrices N_i^T corresponding to the maximum diameter corners.*

From those two conditions T1b and T2, we may obtain a full set of equations for finding a local minimum lattice within a given combinatorial type. T1b forces B^{-1} to lie in the subspace spanned by the N_is corresponding to the maximum corners. Let's say that subspace has dimension k. Then T2 forces B to lie in the subspace orthogonal to the *affine* subspace spanned by the transposes of the N_i's – which must be of dimension $8 - k$. Thus, altogether we get 8 equations governing B (and B^{-1}), leaving one variable for scaling. We may describe this in (two) other ways. (We assume, for simplicity, that the affine flat of the N matrices does not contain the origin.)

Algorithm A1: From T1b, we may say that B^{-1} is a linear combination of k matrices (the basis for the space spanned by the N_i). Then $B^{-1} = \sum_{i=1}^{k} \alpha_i D_i$. By formally computing the adjoint of B^{-1}, we obtain a matrix which is quadratic in the α_is, and by subjecting that matrix to the T2 condition that it be orthogonal to the affine space of the N_j^T, we get $k-1$ simultaneous equations in the k variables α_i. (We can set one of the variables equal to 1 to eliminate the scaling factor - leaving $k-1$ simultaneous equations in $k-1$ variables.)

Algorithm A2: Alternatively, we may say that (using condition T2 first) B is a linear combination of the $9-(k-1)$ matrices generating (via linear combinations) the subspace normal (via matrix inner-product) to the *flat* of the N_j^T. Thus $B = \sum_{i=1}^{10-k} \beta_i B_i$. Computing the formal adjoint of that matrix, we get a matrix which is quadratic in the β_i. Instead of subjecting that matrix to the T1a condition, we may simply require that it be orthogonal (matrix inner product) to the space of dimension $9-k$ orthogonal to the N matrices. That gives us $9-k$ simultaneous equations in the $10-k$ variables.

In either case, whether we use algorithm **A1** or algorithm **A2**, we get one fewer equation than the number of variables, which is proper since the solutions are preserved by scaling. Thus, in effect, we obtain (possibly multiple but a finite number of) solutions in 8-dimensional projective space. Among those solutions *must be found* the local minimum (or local minima) for the region if it exists.

Symmetrically, either Algorithm A1 or A2 reduce to the following situation: We are given a subspace W of dimension k in the vector space of all $n \times n$ real matrices (the subspace generated by our special subset of the N-matrices) and we are given a subspace W_1 of dimension $k-1$ in W (the subspace parallel to the *flat* of those matrices.

We seek a matrix X and its cofactor matrix $X^c = (\det(X) \cdot X^{-1})^T$ satisfying two conditions: (i) $X \in W$ and (ii) X^c is orthogonal by matrix inner product to all matrices in W_1. In other words $M \in W_1 \Rightarrow \langle M, X^c \rangle = 0$. This is equivalent to requiring that the trace of $X^{-1}M$ (or the trace of XM^{-1} or the trace of MX^{-1}, etc.) is zero for all M in W_1.

If we write X as a linear combination of k linearly-independent matrices chosen as a basis for W, the resulting expression has k undefined variables. Each entry in X is a linear expression in the k variables. Then the matrix X^c of cofactors has entries which are homogeneous polynomials of degree $n-1$ in those same variables. Computing the matrix inner product of X^c in turn with each of $k-1$ basis elements for W_1 and setting each of those equal to zero, gives $k-1$ homogeneous polynomial equations of degree $n-1$ in k variables. Thus the solution set of matrices X must be a finite set of points in projective (n^2-1)-space (a finite set of rays in n^2-space) but what else might we say?

Note we could just as well require that C^c be contained in $W_1^\perp$ $(n-k+1$ variables) and that C be orthogonal to every vector in $W^\perp$ $(n-k$ equations). Since $(C^c)^c$ is a scalar multiple of C, this dual statement of the problem is equivalent.

5. Concerns

The above algorithm is ideal for finding the local minimum algebraically, but there are technicalities. The strategies **A1** and **A2** both require that we anticipate which set of corners will be on the diameter plane of the tile for the local minimum or require that we eliminate all possible sets of corners, a lengthy procedure. Given that we have a sample lattice within a combinatorial type, how do we determine which of its corners will be maximized at the corresponding local minimum? And even when we know the correct corners to maximize, the resulting equations are typically of degree higher than one – and have multiple roots. Some of those roots give lattice bases which fail to live up to their promise. Some of them give a large number of solution sets (for the α_i or the β_i), most of which are extraneous. How do we tell which is which without using the (very expensive) process of reconstructing the subtraction tile which the solution lattice implies?

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