On k-step Hamiltonian Graphs

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Abstract For integers $k \geq 1$, a (p, q)-graph $G = (V, E)$ is said to admit an $AL(k)$-traversal if there exists a sequence of vertices $(v_1, v_2, \ldots, v_p)$ such that for each $i = 1, 2, \ldots, p - 1$, the distance between v_i and v_{i+1} is k. We call a graph k-step Hamiltonian (or say it admits a k-step Hamiltonian tour) if it has an $AL(k)$-traversal and $d(v_1, v_p) = k$. In this paper, we investigate the k-step Hamiltonicity of graphs. In particular, we show that every graph is an induced subgraph of a k-step Hamiltonian graph for all $k \geq 2$.

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1 Introduction

In 1856, Kirkman wrote a paper [12] in which he considered graphs with a cycle which passes through every vertex exactly once. The dodecahedron (see Figure 1) is a graph with this property on which Hamilton played cycle games. Hence, such a graph is said to be Hamiltonian. The Hamiltonicity of a graph is the problem of determining for a given graph whether it contains a path/cycle that visits every vertex exactly once.

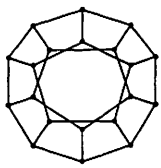

Figure 1: Dodecahedron.

There is no simple characterization of Hamiltonian graphs. However, Hamiltonian graphs are related to the traveling salesman problem, so there are potential practical applications. In general we know very little about Hamiltonian graphs though their properties have been widely studied. A good reference for recent developments and open problems is [7]. For terms used but not defined, we refer to [3].

In this paper we consider simple graphs with no loops. For integers $k \geq 1$, a (p, q)-graph $G = (V, E)$ is said to admit an $AL(k)$-traversal if there exists a sequence $(v_1, v_2, \ldots, v_p)$ such that for each $i = 1, 2, \ldots, p-1$, the distance $d(v_i, v_{i+1}) = k$. We call a graph

k-step Hamiltonian (or say it admits a k-step Hamiltonian tour) if it has an $AL(k)$-traversal and $d(v_1, v_p) = k$.

For example, the cubic graph in Figure 2 is 2-step Hamiltonian and the other two admit an $AL(2)$-traversal but are not 2-step Hamiltonian.

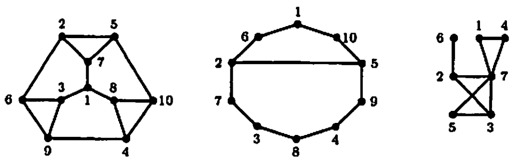

Figure 2: Examples on 2-step Hamiltonicity.

There has been much research on Hamiltonicity of bipartite graphs [1, 2, 5, 8, 9, 13]. Clearly, 1-step Hamiltonian is Hamiltonian. In this paper we consider graphs which are k-step Hamiltonian.

Definition 1.1. *For a graph G, let $D_k(G)$ denote the graph generated from G such that $V(D_k(G)) = V(G)$ and $E(D_k(G)) = \{uv | d(u, v) = k \text{ in } G.\}$.*

Lemma 1.1. *A graph G is k-step Hamiltonian or admits an $AL(k)$-traversal if and only if $D_k(G)$ is Hamiltonian or has a Hamiltonian path, respectively.*

Proof. This follows directly from Definition 1.1.

For graphs G and H, the vertex-gluing (respectively, edge-gluing) of G and H is the identifying of a vertex (respectively, an edge) of G and of H.

Theorem 1.1. *The vertex-gluing of a graph G and an end-vertex of a path of length $n \geq k$ is not k-step Hamiltonian.*

Proof. Let $G(P_n)$ denote the graph such obtained. Observe that $D_k(G(P_n))$ has a cut-vertex and is not Hamiltonian. □

Theorem 1.2. *If graphs G and H are both 2-step Hamiltonian, then so is $G \times H$.*

Proof. It is well known that if G and H are Hamiltonian then so is $G \times H$ (see [4]). By Lemma 1.1, G is 2-step Hamiltonian if and only if $D_2(G)$ is Hamiltonian. We show that $D_2(G) \times D_2(H)$ is a subgraph of $D_2(G \times H)$. Then any Hamiltonian cycle in $D_2(G) \times D_2(H)$ will also exist in $D_2(G \times H)$ and implies that $G \times H$ is also 2-step Hamiltonian. Suppose that edge $e = (u, v_1)(u, v_2)$ is an edge of $D_2(G) \times D_2(H)$. Then (v_1, v_2) must be an edge of $D_2(H)$, so the distance between v_1 and v_2 in H is 2. Let v_1, w, v_2 be a length 2 path from v_1 to v_2 in H. Then $(u, v_1), (u, w), (u, v_2)$ is a length 2 path from (u, v_1) to (u, v_2) in $G \times H$. The only way for (u, v_1) and (u, v_2) to be adjacent in $G \times H$ is if $v_1 v_2$ is an edge of H, which is not the case. Therefore $e = (u, v_1)(u, v_2)$ is also an edge of $D_2(G \times H)$; the argument for edges of the form $e = (u_1, v)(u_2, v)$ is identical. Since all edges and vertices of $D_2(G) \times D_2(H)$ are also in $D_2(G \times H)$, $D_2(G) \times D_2(H)$ is a subgraph of $D_2(G \times H)$. Since G and H are 2-step Hamiltonian, $D_2(G) \times D_2(H)$ is Hamiltonian, and so is $D_2(G \times H)$, implying that $G \times H$ is 2-step Hamiltonian.

2 Bipartite Graphs

A Hamiltonian graph need not be 2-step Hamiltonian. The simplest example is the complete bipartite graph $K(2,2)$ (or the 4-cycle C_4) that does not admit an $AL(2)$-traversal, and hence cannot be 2-step Hamiltonian.

Theorem 2.1. *All bipartite graphs are not k-step Hamiltonian for even $k \geq 2$.*

Proof. Suppose $G = (V, E)$ is bipartite graph with bipartition (X, Y). If $k \geq 2$ is even, the vertex in X cannot connect with vertex in Y, vice versa, in $D_k(G)$. Thus $D_k(G)$ is a disconnected graph with two components X and Y. Hence $D_k(G)$ cannot have a Hamiltonian path. By Lemma 1.1, G is not k-step Hamiltonian. □

Corollary 2.2. *The even cycle C_n is not k-step Hamiltonian for all even $k \geq 2$.*

In a complete multipartite graph, the distance between any pair of veertices is either 1 or 2, therefore we have the following:

Corollary 2.3. *All complete multipartite graphs are not k-step Hamiltonian for $k \geq 2$.*

Corollary 2.4. *The complete bipartite graph $K(m, n)$ is not k-step Hamiltonian for all m, n and even $k \geq 2$.*

Corollary 2.5. *The grid graph $P_n \times P_m$ is not k-step Hamiltonian for all n, m and even $k \geq 2$.*

Corollary 2.6. *All polyomino graphs are bipartite and are not k-step Hamiltonian for even $k \geq 2$.*

Corollary 2.7. *The cylinder graph $C_n \times P_m$ is not 2-step Hamiltonian for all even $n \geq 4$. Also the torus graph $C_m \times C_n$ is not 2-step Hamiltonian for all even $m, n \geq 4$.*

Not all regular graphs admit 2-step Hamiltonian tours. The simplest example is K_4.

Corollary 2.8. *The 3-regular cylinder graph $C_n \times P_2$ is not 2-step Hamiltonian for all even $n \geq 4$.*

Consider the cubic graph $X(n)$ with vertex set $V(X(n)) = \{x_1, x_2, \ldots, x_{2n}\} \cup \{y_1, y_2, \ldots, y_{2n}\}$ and edge set $E(X(n)) = \{x_i x_{i+1} \pmod{2n} \mid i = 1, 2, \ldots, 2n\} \cup \{y_i y_{i+1} \pmod{2n} \mid i = 1, 2, \ldots, 2n\} \cup \{x_i y_{i+1}, y_i x_{i+1})| \; i = 1, 3, 5, \ldots, 2n-1\}$.

Corollary 2.9. *The graph $X(n)$ is bipartite and is not 2-step Hamiltonian.*

The following result shows that there exists n-regular graphs that are not 2-step Hamiltonian for $n \geq 2$.

Corollary 2.10. *Any n-dimensinal hypercube Q_n is not 2-step Hamiltonian.*

3 Tripartite Graphs

In this section, we investigate 2-step Hamiltonicity of tripartite graphs. First we have the following obvious 2-step Hamiltonian graphs.

Theorem 3.1. *All odd cycles are 2-step Hamiltonian.*

Proof. Let C_{2k+1}, $k \geq 2$ be an odd cycle with consecutive vertices $v_1, v_2, v_3, \ldots, v_{2k+1}$. A 2-step Hamiltonian tour is given by the sequence $v_1, v_3, v+5, \ldots, v_{2k+1}, v_2, v_4, \ldots, v_{2k}, v_1$.

Theorem 3.2. *The cylinder graph $C_n \times P_2$ is 2-step Hamiltonian for odd $n \geq 3$.*

Proof. Let the consecutive vertices of the 2 copies of C_{2k+1} for $n = 2k+1$ in the graph be $u_1, u_2, u_3, \ldots, u_{2k+1}$ and $v_1, v_2, v_3, \ldots, v_{2k+1}$ respectively such that u_i is adjacent to v_i for $1 \leq i \leq 2k+1$. If $k = 1$, a 2-step Hamiltonian tour is given by $u_1, v_2, u_3, v_1, u_2, v_3, u_1$. For $k \geq 2$, we begin our tour at vertex u_1 as described in the proof of Theorem 3.1 followed by visiting vertex v_{2k+1} in a similar way but in the opposite direction to end up at vertex v_2, where $d(u_1, v_2) = 2$.

Theorem 3.3. *The vertex-gluing of two cycles is not k-step Hamiltonian for all $k \geq 2$.*

Proof. Let G be the vertex-gluing of two cycles, C and C', at w. We consider three cases.

Case (i). $\text{diam}(C) \leq \text{diam}(C') < k$. In this case, vertex w is distance at most $k-1$ from all other vertices of G. Hence, G is not k-step Hamiltonian.

Case (ii). Without loss of generality, assume that $\text{diam}(C) \geq k$ and $\text{diam}(C') < k$. Let u' and v' be vertices adjacent to w in C'. Note that both u' and v' must be a distance k from exactly two vertices, say u and v, in C since w is a distance $k-1$ from u and v only. Clearly, u' and v' are degree 2 in $D_k(G)$. A Hamiltonian tour of $D_k(G)$ must consist of the sequence u, u', v, v', u, a contradiction. Hence, G is not k-step Hamiltonian.

Case (iii). $\text{diam}(C) \geq \text{diam}(C') \geq k$. Clearly, there are two vertices, say u and v in C, and two vertices, say u' and v' in C' such that $d(w,u) = d(w,v) = d(w,u') = d(w,v') = k$. A Hamiltonian tour of $D_k(G)$ must consist of the two sequences u, w, v and u', w, v'; or else u, w, u' and v, w, v'; or else u, w, v' and v, w, u', a contradiction. Hence, G is not k-step Hamiltonian. □

Theorem 3.4. *The edge-gluing of two cycles, C_a and C_b, is not k-step Hamiltonian for $k \geq 2$ if*

(i) $diam(C_a) \leq diam(C_b) < k$;

(ii) $a = 2k$ and $diam(C_b) < k$;

(iii) $a \geq 7$ is odd with $diam(C_a) = k \geq 2$ and $b = 3$;

(iv) both a and b are odd with $diam(C_a) = k+1$ and $diam(C_b) = k-1 \geq 2$.

Proof. Let G be the edge-gluing of two cycles, C_a and C_b, at edge wx.

(i). Clearly, both vertices w and x are distance at most $k-1$ from all other vertices of G. Hence, G is not k-step Hamiltonian.

(ii). If $a = 2k$ and $\text{diam}(C_b) < k$, then every vertex in C_a is a distance k to exactly one other vertex $V(G)$. Hence, G is not k-step Hamiltonian.

(iii) Note that if $a = 5$, then $k = 2$ and G is 2-step Hamiltonian. If $a \geq 7$ is odd, then $a = 2k + 1$. Let u, z, v be three consecutive vertices in C_a such that $d(z, w) = d(z, x) = k$. Then both u and v are a distance k from the only vertex, say y, in $G \backslash C_a$. Moreover, we can assume that $d(u, x) = d(v, w) = k$. Now, any Hamiltonian tour in $D_k(G)$ necessarily contains the sequence u, y, v, w, z, x, u, a contradiction.

(iv). Let u be adjacent to w, and v be adjacent to x in C_a. Observe that C_a and C_b have exactly one vertex, say y and z, respectively, that is a distance k from both u and v. Hence, a Hamiltonian tour in $D_k(G)$ necessarily contains the sequence y, u, z, v, y, a contradiction.

□

4 Constructions of k-step Hamiltonian Graphs

Suppose G_1 (respectively G_2) is a k-step Hamiltonian graph of order n (respectively m) with a k-step Hamiltonian tour given by $u_1, u_2, \ldots, u_n, u_1$ (respectively $v_1, v_2, \ldots, v_m, v_1$). Let $u_1, a, \ldots, b, u_n$ be a $u_1 - u_n$ path in G_1, and $v_1, c, \ldots, d, v_m$ be a $v_1 - v_m$ path in G_2 such that $d(u_1, u_n) = d(v_1, v_m) = k$. Note that $d(a, b) = d(c, d) = k - 2$. Construct a new graph G from G_1 and G_2 by adding 2 vertices x and y and 4 edges ax, by, cy and dx (see Figure 3).

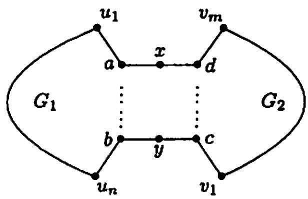

Figure 3: A graph constructed from 2 known k-step Hamiltonian graphs

Theorem 4.1. *For $k \geq 2$, the graph G in Figure 3 is k-step Hamiltonian.*

Proof. Observe that $d(u_1, y) = d(v_m, y) = d(u_n, x) = d(v_1, x) = k$. A k-step Hamiltonian tour in G is given by $u_1, u_2, \ldots, u_n$, x, v_1, v_2, $\ldots$, v_m, y, u_1. ⊔

Let G be a graph of order p with a k-step Hamiltonian tour given by $v_1, v_2, \ldots, v_p, v_1$. Suppose $v_1, u_1, u_2, \ldots, u_{k-1}, v_2$ is a $v_1 - v_2$ path in G such that $d(v_1, v_2) = k$. Denote by $G_{u,v}$ the 1-connected graph obtained from G by joining a vertex u to u_1, and a vertex v to u_{k-1}. Observe that $v_1, v, u, v_2, v_3, \ldots, v_p, v_1$ is a k-step Hamiltonian tour of $G_{u,v}$.

Theorem 4.2. *If there exists a k-step Hamiltonian graph of order p, then there exists a 1-connected k-step Hamiltonian graph of order $p + 2n, n \geq 1$.*

We denote by **Gph** the class of all undirected graphs. Let $\mathcal{AL}(k)$ be the class of all graphs that are $AL(k)$-traversal and we denote $\mathcal{H}(k)$ the class of all k-step Hamiltonian graphs. We denote the composition of G with H by $G[H]$. In this section, we show that

Theorem 4.3. *If $G \in \mathcal{H}(k)$, then for any $H \in$ **Gph**, the composition graph $G[H] \in \mathcal{H}(k)$.*

Proof. Assume G has p vertices and $(v_1, v_2, \ldots, v_p, v_1)$ is a k-step Hamiltonian tour. Assume $V(H) = \{h_1, h_2, \ldots, h_t\}$. It is obvious that the following vertices in $G[H]$ forms a k-step Hamiltonian tour:

$$(v_1, h_1), (v_2, h_1), \ldots, (v_p, h_1), (v_1, h_2), (v_2, h_2), \ldots, (v_p, h_2), \ldots,$$

$$(v_1, h_t), (v_2, h_t), \ldots, (v_p, h_t), (v_1, h_1).$$

Thus, $G[H]$ is k-step Hamiltonian. □

Example 4.1. *The following unicyclic graph G is 2-step Hamiltonian with a tour given by vertex sequence (1, 2, 3, 4, 5, 6). Let $H = N_2$, then we see that $G[H]$ has a 2-step Hamiltonian tour given by* $(1, h_1), (2, h_1), (3, h_1), (4, h_1), (5, h_1), (6, h_1), (1, h_2), (2, h_2), (3, h_2), (4, h_2), (5, h_2), (6, h_2), (1, h_1)$.

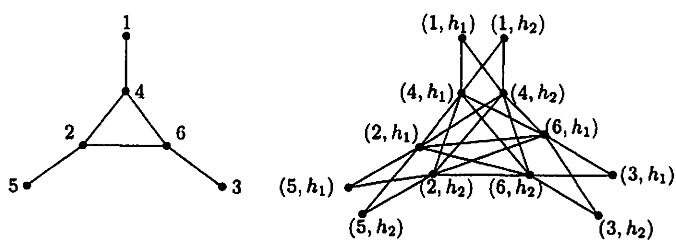

Figure 4: Graphs G and $G[H]$ with a 2-step Hamiltonian Tour

By Theorem 2.5 in [11], we have

Corollary 4.4. *For any graph $H \in$ **Gph**, the graph $C_{2k+1}[H]$ is k-step Hamiltonian.*

Corollary 4.5. *Any graph H is an induced subgraph of a k-step Hamiltonian graph.*

Kuratowski discovered his famous mathematical forbidden graph characterization of planar graphs in 1930. From Corollary 4.5, we have the following result.

Theorem 4.6. *It is impossible to have a Kuratowski type of characterization of k-step Hamiltonian graphs.*

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