# A Note on Discrete Factorial Designs of Resolution Five and Seven and Balanced Arrays

D.V. Chopra
Department of Mathematics and Statistics
Wichita State University
Wichita, KS 67260-0033, USA
dharam.chopra@wichita.edu

Richard M. Low
Department of Mathematics
San Jose State University
San Jose, CA 95192, USA
richard.low@sjsu.edu

R. Dios
Department of Mathematical Sciences
New Jersey Institute of Technology
Newark, NJ 07102-1982, USA
dios@adm.njit.edu

**Abstract**

In this paper, we consider the use of balanced arrays (B-arrays) in constructing discrete fractional factorial designs (FFD) of resolution ($2u + 1$, with $u = 2$ and 3) in which each of the $m$ factors is at two levels (say, 0 and 1), denoted by factorial designs of $2^m$ series. We make use of the well-known fact that such designs can be realized under certain conditions, by using balanced arrays of strength four and six (with two symbols), respectively. Here, we consider the existence of B-arrays of strength $t = 4$ and $t = 6$, and discuss how the results presented can be used to obtain the maximum value of $m$ for a given set of treatment-combinations. Also, we provide some

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illustrative examples in which the current available max($m$) values have been improved upon.

# 1 Introduction and Preliminaries

Design of experiments, founded by R.A. Fisher, has played a very important role in numerous fields of scientific investigation such as medicine, agriculture, education, chemical and hi-tech industries, social and behavioral sciences, etc. Factorial experiments form a very important component of experimental designs. The introduction of orthogonal arrays and block designs and the work done on them (to solve real-life problems) have enriched combinatorics, information theory, and coding theory. Consider an experiment in which the final outcome (i.e. *response*) is influenced by $m$ experimental conditions (called *factors*) where each factor has two or more settings (called *levels*). One of the advantages of using factorial designs is to economize on the cost of the experiment. In this paper, we will consider $m$ factors, each at two levels (say, 0 and 1) and is denoted by $2^m$ series.

For the sake of completeness, we first state some basic concepts and definitions.

**Definition.** A *B-array* $T$ with $m$ factors (constraints, rows), $N$ treatment-combinations (runs, columns), two levels (say, 0 and 1), and of strength $t$ ($1 \leq t \leq m$) is an ($m \times N$)-matrix $T$ with elements 0 and 1 satisfying the following condition: in every ($t \times m$)-submatrix $T^*$ of $T$, each ($t \times 1$) vector $\underline{\alpha}$ of weight $i$ ($0 \leq i \leq t$; the weight of $\underline{\alpha}$ refers to the number of 1s in it) occurs with the same frequency $\mu_i$ (say).

The vector $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \ldots, \mu_t)$ and $m$ are called the parameters of the array $\underline{T}$. For a given $\underline{\mu}'$, the number of runs $N$ is known. Clearly, $N = \sum_{i=0}^{t} \binom{t}{i} \mu_i$.

**Definition.** If $\mu = \mu$ for each $i$, then the B-array $T$ is called an *orthogonal array* (O-array), and $N = 2^t \cdot \mu$ in this case.

Thus, O-arrays form a special case of B-arrays. Also, the incidence matrix of a balanced incomplete block design (BIBD) is a special case of a B-array with $t = 2$. In addition, B-arrays are related to other combinatorial structures such as group divisible designs, nested balanced incomplete block designs, etc. Obviously, O-arrays do not exist for each $N$ ($N = 2^t \cdot \mu$). For example, with $t = 4$, we must have $N$ to be a multiple of 16 (i.e. $N = 16, 32, 48$, etc.), and if $m = 6$ then $N$ must equal 80 (for an O-array to exist). It is a well-known fact that a B-array of strength $t$, under certain conditions, gives rise to a balanced factorial design of resolution $(t+1)$. If $t$ is even ($= 2u$, say), then it allows us to estimate all the effects up to and including $u$-factor interactions under the assumption that all higher order interactions are negligible. In this paper, we restrict ourselves for $t = 4$ and 6. Constructing such combinatorial arrays, for a given index set $\underline{\mu}'$ (i.e. $N$ is given) and with the maximum possible value of $m$, is a very non-trivial and difficult problem in combinatorics and design of experiments. Such problems for O-arrays have been investigated, among others, by Bose and Bush [1], Rao [17, 18, 19], Seiden and Zemach [21], etc. while the corresponding problem for B-arrays has been studied, among others, by Chopra, Low and Dios [7, 8], Rafter and Seiden [16], Saha, Mukerjee and Kageyama [20], etc. To gain further insight into the importance of O-arrays and B-arrays to combinatorics and design of experiments, the interested reader is referred to the list of references at the end (by no means, an exhaustive and complete list) of this paper, as well as further references listed therein.

In this paper, we state some inequalities involving the parameters $m$ and $\underline{\mu}'$ for B-arrays with $t = 4$ and 6. These inequalities are necessary existence conditions for these B-arrays, and we describe how these can be used to obtain, for a given $\underline{\mu}'$, the max($m$). We compare the results given here (for $l = 1, 2, \dots$) with those obtained earlier under $l = 0$.

## 2 Statements of Main Results with Illustrative Examples

*Result 1.* A B-array $T$ with index set $\underline{\mu}' = (\mu_0, \mu_1, \dots, \mu_4)$ and $m = t = 4$ always exists. Similarly, a B-array with $m = t = 6$ exists.

*Result 2.* A B-array $T$ of strength $t$ is also of strength $k$, where $0 \le k \le t$. Viewed as an array of strength $k$, the $j$th element $(0 \le j \le k)$ of $T$ is given by

$$A(j,k) = \sum_{i=0}^{t-k} \binom{t-k}{i} \mu_{i+j}, \quad \text{where } j = 0, 1, 2, \dots, k, \ (k \le t).$$

It is clear that $A(t,t) = \mu_t$, $A(j,t) = \mu_j$, and $A(j,0) = A(0,0) = N$.

**Definition.** Two columns of a B-array are said to have $j$ *coincidences* if the symbols appearing in these two columns in $j$ of the rows are the same $(0 \le j \le m)$.

Next, we quote some results, without proof, connecting $m$ and the elements of $\underline{\mu}'$ with $L_k = \sum_{j=0}^{m} j^k x_j$ ($0 \le k \le 4$ for $t = 4$, $0 \le k \le 6$ for $t = 6$), where $x_j$ denotes the number of columns in $T$ having $j$ coincidences with a column of $T$ (say, the first column) having a weight of $l$.

*Result 3.* For a B-array $T$ of strength $t$ (which is also of strength $k$, where $k \le t$), we have the following result:

$$L_k = \sum_{j=0}^{m} j^k x_j = \sum_{p=1}^{k-1} (-1)^{k-p+1} C_{k-p} L_{k-p} + k! \sum_{i=0}^{k} \binom{l}{i} \binom{m-l}{k-i} [A(i,k) - 1], \text{if } k \text{ is even,}$$

and

$$= \sum_{p=1}^{k-1} (-1)^{k-p} C_{k-p} L_{k-p} + k! \sum_{i=0}^{k} \binom{l}{i} \binom{m-l}{k-i} [A(i,k) - 1], \text{if } k \text{ is odd.}$$

*Note 1.* The above result expresses the moments of order $k$ in terms of moments of lower orders and involving parameters of the array $T$ as well as $l$. For $t = 4$, we obtain five inequalities for each $k$ satisfying $0 \le k \le 4$, and for $t = 6$, we obtain seven inequalities.

By using the non-negative definiteness of the moment matrix (for $t = 4$ and $t = 6$), we obtain the following two theorems.

**Theorem 1.** *For $t = 4$, the following inequalities must be satisfied:*

$$L_0 L_2 \ge L_1^2, \tag{2.1}$$

$$L_0 L_2 L_4 + 2 L_1 L_2 L_3 \ge L_0 L_3^2 + L_1^2 L_4 + L_2^3. \tag{2.2}$$

**Theorem 2.** *For a B-array $T$ with $m$ rows, $t = 6$ and index set $\underline{\mu}'$, the following inequalities must hold:*

$$L_0 L_2 \ge L_1^2, \tag{2.3}$$

$$L_0 L_2 L_4 + 2 L_1 L_2 L_3 \ge L_0 L_3^2 + L_1^2 L_4 + L_2^3, \tag{2.4}$$

$$
\begin{aligned}
&L_0L_2L_4L_6 + 2L_0L_3L_4L_5 + L_1^2L_5^2 + 2L_1L_2L_3L_6 + 2L_1L_3L_4^2 \\
&\quad +2L_2^2L_3L_5 + L_2^2L_4^2 + L_3^4 \geq \\
&\quad L_0L_2L_5^2 + L_0L_3^2L_6 + L_0L_4^3 + L_1^2L_4L_6 + 2L_1L_3^2L_5 \\
&\quad\quad +2L_1L_2L_4L_5 + L_2^3L_6 + 3L_2L_3^2L_4,
\end{aligned}
\tag{2.5}
$$

*where* $L_k = \sum_{j=0}^{m} j^k x_j$, $0 \leq k \leq 6$.

*Note 2.* For Result 3, we always have the following for $k = 0$ and $k = 1$: $L_0 = N - 1 = \sum_{j=0}^{m} x_j$ and $L_1 = \sum_{j=0}^{m} jx_j = \sum_{i=0}^{1} \binom{l}{i}\binom{m-l}{1-i}[A(i,1) - 1]$. For $k \geq 2$, the constants $C_{k-p}$ will be known in the process of obtaining these equalities for $t = 4$ and $t = 6$. For the convenience of the reader, we list these constants for $t = 4$ and $t = 6$. For $t = 4$: $k = 2, C_1 = 1$; $k = 3$, we have $C_2 = 3, C_1 = 2$; and for $k = 4$, we have $C_3 = 6, C_2 = 11, C_1 = 6$. For $t = 6$: $k = 2, C_1 = 1$; $k = 3$, we have $C_2 = 3, C_1 = 2$; $k = 4$, we get $C_3 = 6, C_2 = 11, C_1 = 6$; $k = 5$ gives $C_4 = 10, C_3 = 35, C_2 = 50, C_1 = 24$; and for $k = 6$, we obtain $C_5 = 15, C_4 = 85, C_3 = 225, C_2 = 274, C_1 = 120$.

# 3 Discussion with Illustrative Examples for $t = 4$ and $t = 6$ B-arrays

For a given $m$ and $\underline{\mu}'$, the B-array for $t = 4$ must satisfy conditions (2.1) and (2.2) for it to exist, while for $t = 6$, it must satisfy conditions (2.3), (2.4) and (2.5). In all instances, we used the value $l = 0$ in [7] because of the simplicity it provided us in computations. Here, we consider values of $l = 1, 2, 3, 4$ for $t = 4$ and values $l = 1, 2, 3, 4, 5, 6$ for $t = 6$, and we compare the max($m$) for each of these $l$ values with that of $l = 0$. We also observed that values of $l \geq t + 1$ will not provide us any new max($m$) value.

*Example 1.* ($t = 4$). Consider the arrays with $\underline{\mu}' = (3,2,3,3,3), (4,3,2,3,4), (8,8,8,1,4), (4,1,6,4,1), (2,1,5,3,1), (14,6,\overline{1},1,4), (6,4,1,3,12), (1,2,8,3,1)$ and $(4,4,6,4,4)$. For these arrays, the max($m$) for $l = 0$ are found to be, respectively, 31, 7, 10, 6, 6, 8, 7, 5, and 15, where as the max($m$) for values of $l$ ($1 \leq l \leq 4$) are found to be, respectively, 7 (with $l = 2$), 7 (with $l = 4$), 7 (with $l = 2$), 4 (with $l = 1$), 4 (with $l = 1$), 5 (with $l = 3$), 5 (with $l = 3$), 4 (with $l = 1$), and 8 (with $l = 2$). This clearly demonstrates that there are many arrays where values of $l$ (other than 0) provide us with sharper values of max($m$).

*Example 2.* ($t = 6$). Consider the arrays with $\underline{\mu}'$ to be (3,2,3,3,3,3,3), (8,7,7,5,6,6,8), (20,10,5,4,6,7,12), (5,3,3,2,3,3,5), ($\overline{17}$,10,8,6,7,9,16), (5,4,4,4,4,4,5), and (8,5,5,5,5,5,10). For $l = 0$, the max($m$) were found to be, respectively, 18, 19, 21, 30, NT, NT, and NT (here, NT means the program never gave us a contradiction for values of $m$ up to 1000). For other values of $l$, the max($m$) are found to be, respectively, 8 ($l = 1$), 11 ($l = 2$), 11 ($l = 5, 6$), 8 ($l = 1$), 13 ($l = 6$), 11 ($l = 2, 3, 4, 5, 6$), and 12 ($l = 3, 4, 5, 6$). This indicates that there are numerous arrays where other values of $l$ ($\neq 0$) show significant improvement over those obtained by setting $l = 0$.

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