On Eulerian Irregularities of Prisms, Grids and Powers of Cycles

Eric Andrews, Chira Lumduanhom and Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008-5248, USA

Abstract

For a nontrivial connected graph G, an Eulerian walk in G is a closed walk that contains every edge of G at least once. An Eulerian walk is irregular if it encounters no two edges of G the same number of times and the minimum length of an irregular Eulerian walk in G is the Eulerian irregularity of G. In this work, we determine the Eulerian irregularities of all prisms, grids and powers of cycles.

Key Words: Eulerian walk and irregularity, prism, grid, powers of cycles.

AMS Subject Classification: 05C38, 05C45.

1 Introduction

A closed walk in a nontrivial connected graph G that contains every edge of G exactly once is an *Eulerian circuit.* A graph is *Eulerian* if it contains an Eulerian circuit. It is well known [5] that a connected graph (or multigraph) G is Eulerian if and only if every vertex of G is even. In [1], an *Eulerian walk* in a connected graph G is defined as a closed walk that contains every edge of G at least once. If every edge of a nontrivial connected graph G is replaced by two parallel edges, then the resulting multigraph is Eulerian, which implies that G contains a closed walk in which every edge of G appears exactly twice. Hence if G is a non-Eulerian graph of size $m \geq 1$, then the minimum length of an Eulerian walk in G is more than m but not more than $2m$ and every edge appears once or twice in such an Eulerian walk in G.

Let H be a weighted graph obtained by assigning weights (positive integers) to the edges of a connected graph G. Then the *degree* $\deg_H v$ of a vertex v in H is the sum of the weights of the edges incident with v. Determining the minimum length of an Eulerian walk in G is then equivalent to determining an assignment of the weights 1 or 2 to the edges of G such that the sum of these weights is minimum and the degree of every vertex in H is even. This problem is directly related to a well-known problem called the Chinese Postman Problem named by Alan Goldman for the Chinese mathematician Meigu Guan (often known as Mei-Ko Kwan) who introduced this problem in 1960 [6].

The Chinese Postman Problem *Suppose that a postman starts from the post office and has mail to deliver to the houses along each street on his mail route. Once he has completed delivering the mail, he returns to the post office. Determine the minimum length of a round trip that accomplishes this.*

While the Chinese Postman Problem asks for the minimum length of a closed walk in a connected graph G such that every edge of G appears on the walk once or twice, another problem of interest is that of determining the minimum length of a closed walk in G in which no two edges of G appear the same number of times. Such walks in a graph G distinguish the edges of G by their occurrences on the walk. This gives rise to the concept of irregular Eulerian walks in graphs, which were introduced and studied in [1].

An *irregular Eulerian walk* in a nontrivial connected graph G is an Eulerian walk that encounters no two edges of G the same number of times. Thus, if the size of G is m, then the length of an irregular Eulerian walk in G is at least $1+2+\cdots+m = \binom{m+1}{2}$. Furthermore, if $E(G) = \{e_1, e_2, \ldots, e_m\}$ and each edge e_i $(1 \leq i \leq m)$ of G is replaced by $2i$ parallel edges, then the resulting multigraph M is Eulerian and each Eulerian circuit in M gives rise to an irregular Eulerian walk in which each edge e_i of G appears exactly $2i$ times in the walk. Thus G contains an irregular Eulerian walk of length $2+4+6+\cdots+2m = 2\binom{m+1}{2} = m^2+m$. The length of a walk W is denoted by $L(W)$. If W is an irregular Eulerian walk of minimum length in a connected graph G of size m, then $\binom{m+1}{2} \leq L(W) \leq 2\binom{m+1}{2}$. A problem of interest here is that of determining the minimum length of an irregular Eulerian walk in G, which is defined in [1] as the *Eulerian irregularity* of G and is denoted by $EI(G)$. Therefore, if G is a connected graph of size m, then

$$\binom{m+1}{2} \leq EI(G) \leq 2\binom{m+1}{2}. \tag{1}$$

Both upper and lower bounds in (1) are sharp. In fact, all nontrivial connected graphs of size m having Eulerian irregularity $\binom{m+1}{2}$ and $2\binom{m+1}{2}$ have been characterized in [1]. A subgraph F in a graph G is an *even subgraph* of G if every vertex of F is even.

Theorem 1.1 [1] *If G is a nontrivial connected graph of size m, then*

- $EI(G) = 2\binom{m+1}{2}$ *if and only if G is a tree;*
- $EI(G) = \binom{m+1}{2}$ *if and only if G contains an even subgraph of size* $\lceil m/2 \rceil$.

The concept of Eulerian irregularity has been studied further in [2], where a necessary and sufficient condition has been established for all pairs k, m of positive integers for which there is a nontrivial connected graph G of size m with $EI(G) = k$.

Theorem 1.2 [2] *Let k and m be positive integers with $\binom{m+1}{2} \le k \le 2\binom{m+1}{2}$. Then there exists a nontrivial connected graph G of size m with $EI(G) = k$ if and only if there exists integer x with $0 \le x \le m$ and $x \neq 1, 2$ such that $x^2 + (m-x)(m-x+1) = k$.*

By Theorem 1.2, a pair k, m of positive integers with $\binom{m+1}{2} \le k \le 2\binom{m+1}{2}$ can be realized as the Eulerian irregularity and the size of some nontrivial connected graph if and only if there exists an integer x with $0 \le x \le m$ and $x \neq 1, 2$ such that $x^2+(m-x)(m-x+1) = k$. To determine the possible values of such integers x, we consider the real-valued function

$$L(x) = x^2 + (m-x)(m-x+1) = 2x^2 - (2m+1)x + m^2 + m. \qquad (2)$$

Since $L(x)$ is a concave-up parabola which has the minimum value at $x_0 = \frac{2m+1}{4}$, it follows that the closer x is to x_0, the closer $L(x)$ is to $L(x_0)$. For a positive integer m, let $[0..m]$ be the set of all integers x with $0 \le x \le m$. We list the elements of $[0..m]$ as an ordered sequence s of length $m+1$ where

$$s = (x_1, x_2, \ldots, x_{m+1}) \qquad (3)$$

such that

$$L(x_1) \le L(x_2) \le \cdots \le L(x_{m+1}), \qquad (4)$$

where then $L(x_1) = \binom{m+1}{2}$, $L(x_2) = \binom{m+1}{2} + 1$, $L(x_3) = \binom{m+1}{2} + 3$, ..., $L(x_{m+1}) = 2\binom{m+1}{2}$. The sequence s in (3) that satisfies (2) and (4) is referred to as the *Eulerian irregular sequence of m*. We now state a useful observation on Eulerian irregular sequences.

Observation 1.3 *Let m be a positive integer.*

- *If m is even, then the Eulerian irregular sequence of m is*

$$\left(\left\lceil\frac{m}{2}\right\rceil, \left\lceil\frac{m}{2}\right\rceil + 1, \left\lceil\frac{m}{2}\right\rceil - 1, \left\lceil\frac{m}{2}\right\rceil + 2, \left\lceil\frac{m}{2}\right\rceil - 2, \ldots, m, 0\right). \qquad (5)$$

- *If m is odd, then the Eulerian irregular sequence of m is*

$$\left(\left\lceil\frac{m}{2}\right\rceil, \left\lceil\frac{m}{2}\right\rceil - 1, \left\lceil\frac{m}{2}\right\rceil + 1, \left\lceil\frac{m}{2}\right\rceil - 2, \left\lceil\frac{m}{2}\right\rceil + 2, \ldots, \left\lceil\frac{m}{2}\right\rceil + \left\lfloor\frac{m}{2}\right\rfloor = m, 0\right) \quad (6)$$

An irregular Eulerian walk W in a connected graph G of size m is said to be *optimal* if $L(W) = \binom{m+1}{2}$. If G contains an optimal irregular Eulerian walk, then G is *optimal* and so $EI(G) = \binom{m+1}{2}$. In this case, the edges of G can be ordered as $e_1, e_2, \ldots, e_m$ such that e_i $(1 \le i \le m)$ is encountered exactly i times in W. By Theorem 1.1, no cycle is optimal. On the other hand, in [1, 2] several well-known classes of graphs that are optimal have been determined as well as the Eulerian irregularities of those non-optimal graphs. In this work, we determine the Eulerian irregularities of all prisms, grids and powers of cycles. We refer to [3] for graph theory notation and terminology not described in this paper.

2 Eulerian Irregularities of Prisms and Grids

The *Cartesian product* $G \square H$ of two graphs G and H has vertex set $V(G) = V(G) \times V(H)$ and two distinct vertices (u, v) and (x, y) of $G \square H$ are adjacent if either (1) $u = x$ and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(G)$. The graph $C_n \square K_2$ where $n \ge 3$ is called a *prism* while $P_n \square P_q$ where $n \ge q \ge 2$ is called a *grid*. In this section, we determine the Eulerian irregularities of all prisms and grids. In order to do this, we first present two useful lemmas, the first of which is a consequence of Theorem 1.1 while the second one is a consequence of the proof of Theorem 1.2.

Lemma 2.1 *If G is a connected bipartite graph of size $m \ge 1$ such that $m \equiv 1 \pmod 4$ or $m \equiv 2 \pmod 4$, then G is not optimal.*

Lemma 2.2 *Let G be a nontrivial connected graph of size m. If G contains an even subgraph of size x, then there is an irregular Eulerian walk of length $x^2 + (m - x)(m - x + 1)$ in G and so $EI(G) \le x^2 + (m - x)(m - x + 1)$.*

Theorem 2.3 *For each integer $n \ge 3$, the prism $C_n \square K_2$ is optimal if and only if $n \not\equiv 2 \pmod 4$. Furthermore, if $C_n \square K_2$ is not optimal, then $EI(C_n \square K_2) = \binom{3n+1}{2} + 1$.*

Proof. For $G = C_n \square K_2$, let $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ be two disjoint copies of C_n in G such that $u_i v_i \in E(G)$ for $1 \le i \le n$. If $n \equiv 2 \pmod 4$, then G is a cubic bipartite graph of size $m = 3n$. Since m is congruent to 2 modular 4, it follows by Lemma 2.1 that G is not optimal.

For the converse, suppose that $n \not\equiv 2 \pmod 4$. Then $n \equiv r \pmod 4$, where $r = 0, 1, 3$. We consider these three cases. In each case, we show that G contains a 2-regular subgraph of size $\lceil m/2 \rceil$, where m is the size of G.

Case 1. $n \equiv 0 \pmod 4$. Then $n = 4k$ for some positive integer k and so the size of G is $m = 3n = 12k$. Thus $m/2 = 6k$. The $6k$-cycle

$$C_{6k} = (v_1, v_2, \ldots, v_{3k}, u_{3k}, u_{3k-1}, \ldots, u_1, v_1).$$

is a 2-regular subgraph of size $6k$ in G.

Case 2. $n \equiv 1 \pmod 4$. Then $n = 4k+1$ for some positive integer k and so the size of G is $m = 3n = 12k+3$. Thus $\lceil m/2 \rceil = 6k+2$. The $(6k+2)$-cycle

$$C_{6k+2} = (u_1, u_2, \ldots, u_{3k+1}, v_{3k+1}, v_{3k}, \ldots, v_1, u_1)$$

is a 2-regular subgraph of size $6k+2$ in G.

Case 3. $n \equiv 3 \pmod 4$. Then $n = 4k+3$ for some integer $k \geq 0$ and the size of G is $m = 3n = 12k+9$ and so $\lceil m/2 \rceil = 6k+5$. The $(6k+5)$-cycle

$$\begin{aligned} C_{6k+5} = {} & (v_1, v_2, \ldots, v_{2k+1}, v_{2k+2}, u_{2k+2}, u_{2k+3}, v_{2k+3}, \\ & v_{2k+4}, u_{2k+4}, u_{2k+5}, v_{2k+5}, v_{2k+6}, u_{2k+6}, u_{2k+7}, \ldots, \\ & v_{4k+1}, v_{4k+2}, u_{4k+2}, u_{4k+3}, v_{4k+3}, v_1) \end{aligned}$$

is a 2-regular subgraph of size $6k+5$ in G. In each case, G contains a 2-regular subgraph of size $\lceil m/2 \rceil$, By Theorem 1.1, G is optimal.

We now assume that $n \equiv 2 \pmod 4$ and so $EI(G) \geq \binom{m+1}{2} + 1$. Since m is even, to show that $EI(G) \leq \binom{m+1}{2} + 1$, it suffices to show that G contains an even subgraph of size $\frac{m}{2} + 1$. Let $n = 4k+2$ for some positive integer k and so $\frac{m}{2} + 1 = 6k+4$. Note that G contains vertex-disjoint $H_1 = C_4$ and $H_2 = C_{6k}$ as subgraphs. Thus, G has an even subgraph of size $\frac{m}{2} + 1 = 6k+4$ and so $EI(G) \leq \binom{m+1}{2} + 1$ by Lemma 2.2, giving the desired result. ∎

We next determine the Eulerian irregularities of all grids $P_n \square P_q$ where $n \geq q \geq 2$, beginning with the case when $q = 2$.

Theorem 2.4 *For each integer $n \geq 3$, the Cartesian product $P_n \square K_2$ of P_n and K_2 is optimal if and only if $n \equiv 2, 3 \pmod 4$. Furthermore, if $P_n \square K_2$ is not optimal, then $EI(P_n \square K_2) = \binom{3n-1}{2} + 1$.*

Proof. If $n \equiv 0, 1 \pmod 4$, then the size $m = 3n - 2$ of $P_n \square K_2$ is congruent to 2 or 1 modulo 4. It then follows by Lemma 2.1 that $P_n \square K_2$ is not optimal. For the converse, suppose that $n \equiv 2, 3 \pmod 4$. Let $G = P_n \square K_2$, where $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are the two copies of P_n in G such that $u_i v_i \in E(G)$ for $1 \leq i \leq n$. The size m of G is $3n - 2$. For $n \equiv 2 \pmod 4$, let $n = 4k+2$ for some positive integer k and $\lceil m/2 \rceil = 6k+2$. The cycle

$$C_{6k+2} = (u_1, u_2, \ldots, u_{3k+1}, v_{3k+1}, v_{3k-1}, \ldots, v_2, v_1, u_1)$$

is a 2-regular subgraph of size $\lceil m/2 \rceil$ in G. For $n \equiv 3 \pmod 4$, let $n = 4k+3$ for some nonnegative integer k and $\lceil m/2 \rceil = 6k+4$. The cycle

$$C_{6k+4} = (u_1, u_2, \ldots, u_{3k+2}, v_{3k+2}, v_{3k+1}, \ldots, v_2, v_1, u_1)$$

is a 2-regular subgraph of size $\lceil m/2 \rceil$ in G. By Theorem 1.1, G is optimal if $n \equiv 2, 3 \pmod 4$.

We now assume that G is not optimal. Then either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ and $EI(G) \geq \binom{m+1}{2} + 1$. It remains to show that $EI(G) \leq \binom{m+1}{2} + 1$. For $n \equiv 0 \pmod 4$, let $n = 4k$ for some positive integer k. Since $m = 3n - 2$ is even, it suffices to show that G contains an even subgraph of size $\frac{m}{2} + 1 = 6k$. This is true as G contains C_{6k} as a subgraph and so $EI(G) = \binom{m+1}{2} + 1$ if $n \equiv 0 \pmod 4$. For $n \equiv 1 \pmod 4$, let $n = 4k+1$ for some positive integer k. Since $m = 3n - 2 = 12k + 1$ is odd, it suffices to show that G contains an even subgraph of size $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2} = 6k$. This is true as G contains C_{6k} as a subgraph and so $EI(G) = \binom{m+1}{2} + 1$ if $n \equiv 1 \pmod 4$. ■

We now consider grids $P_n \,\square\, P_q$ for $n \geq q \geq 3$ in general.

Theorem 2.5 *For each pair (n, q) of integers with $n \geq q \geq 3$, the grid $P_n \,\square\, P_q$ is optimal if and only if (n, q) satisfies one of the following conditions:*

- *(i) If n and q are even, then either both n and q are congruent to 0 modulo 4 or both n and q are congruent to 2 modulo 4;*
- *(ii) If n is even and q is odd, then $n \equiv 0 \pmod 4$ and $q \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$;*
- *(iii) If n is odd and q is even, then $n \equiv 1 \pmod 4$ and $q \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ and $q \equiv 2 \pmod 4$.*

Proof. Suppose that $G = P_n \,\square\, P_q$ consists of q paths of order n, which we denote by $P_{n,i} = (v_{1,i}, v_{2,i}, \ldots, v_{n,i})$ for $1 \leq i \leq q$ such that $v_{t,i}$ is adjacent to $v_{t,j}$ $(1 \leq t \leq n)$ when $|i - j| = 1$. The size of G is $m = n(q-1) + (n-1)q$ and G is a bipartite graph. Write $n = 4k + r_n$ and $q = 4\ell + r_q$, where $r_n, r_q \in \{0, 1, 2, 3\}$. Let $G' = P_{4k} \,\square\, P_{4\ell}$ be the induced subgraph of G with

$$V(G') = \{v_{a,b} : 1 \leq a \leq 4k, 1 \leq b \leq 4\ell\}. \tag{7}$$

That is, G' is the induced subgraph in G consisting of the 4ℓ paths $P_{4k,i}$ of order $4k$ where

$$P_{4k,i} = (v_{1,i}, v_{2,i}, \ldots, v_{4k,i}) \text{ for } 1 \leq i \leq 4\ell \tag{8}$$

such that $v_{t,i}$ is adjacent to $v_{t,j}$ $(1 \leq t \leq 4k)$ when $|i - j| = 1$. Then G' contains $k\ell$ vertex-disjoint copies (or blocks) of $P_4 \,\square\, P_4$, denoted by $B_1, B_2, \ldots, B_{k\ell}$ as shown in Figure 1.

B_1	B_{k+1}	$\cdots$	$B_{(\ell-1)k+1}$
B_2	B_{k+2}	$\cdots$	$B_{(\ell-1)k+2}$
B_3	B_{k+3}	$\cdots$	$B_{(\ell-1)k+3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
B_k	B_{2k}	$\cdots$	$B_{k\ell}$

Figure 1: The subgraph $G' = P_{4k} \,\Box\, P_{4\ell}$ in G

In particular, for $1 \le i \le k$, the vertices of B_i appear in the way as shown in Figure 2. Note that $B_i = P_4 \,\Box\, P_4$ contains each of the even cycles $C_4, C_6, C_8, C_{10}, C_{12}, C_{14}, C_{16}$ as a subgraph.

$v_{4i-3,1}$	$v_{4i-3,2}$	$v_{4i-3,3}$	$v_{4i-3,4}$
$v_{4i-2,1}$	$v_{4i-2,2}$	$v_{4i-2,3}$	$v_{4i-2,4}$
$v_{4i-1,1}$	$v_{4i-1,2}$	$v_{4i-1,3}$	$v_{4i-1,4}$
$v_{4i,1}$	$v_{4i,2}$	$v_{4i,3}$	$v_{4i,4}$

Figure 2: The block $B_i = P_4 \,\Box\, P_4$ in G' for $1 \le i \le k$

We consider three cases, according to the parities of n and q.

Case 1. *n and q are even.* If one of n and q is congruent to 0 modulo 4 and the other is congruent to 2 modulo 4, then $m \equiv 2 \pmod 4$ and so G is not optimal by Lemma 2.1. For the converse, suppose that both n and q are congruent to 0 modulo 4 or both n and q are congruent to 2 modulo 4. We consider two subcases. In each subcase, we construct an even subgraph H of size $\lceil m/2 \rceil$.

Subcase 1.1. $n \equiv 0 \pmod 4$ *and* $q \equiv 0 \pmod 4$. Then $n = 4k$ and $q = 4\ell$ for some positive integers k and ℓ with $k \ge \ell$. In this case, the size m_H of a graph H with the desired properties is

$$m_H = \frac{m}{2} = 16k\ell - 2k - 2\ell.$$

The graph $G' = P_{4k} \,\Box\, P_{4\ell}$ contains $k\ell$ vertex-disjoint copies $B_1, B_2, \dots, B_{k\ell}$ of $P_4 \,\Box\, P_4$ as shown in Figure 1.

- For $\ell = 1$, there are k vertex-disjoint blocks $B_1, B_2, \dots, B_k$ of $P_4 \,\Box\, P_4$ in G. Let $H_1 = C_{12}$ in B_1 and $H_i = C_{14}$ in B_i for $2 \le i \le k$. Now let H be the union of these vertex-disjoint subgraphs H_i $(1 \le i \le k)$ in G. Then H is a 2-regular graph of the size $m_H = 12 + 14(k-1) = 14k-2$.
- For $\ell = 2$, there are $2k$ vertex-disjoint blocks $B_1, B_2, \dots, B_{2k}$ of $P_4 \,\Box\, P_4$ in G. If $k = 2$, let $H_1 = C_8$ in B_1 and $H_i = C_{16}$ in B_i

for $2 \le i \le 4$. Now let H be the union of these vertex-disjoint subgraphs H_i ($1 \le i \le 4$) in G. Then H is a 2-regular graph of the size $m_H = 3 \cdot 16 + 8 = 56 = m/2$. If $k \ge 3$, let $H_i = C_{16}$ in B_i for $1 \le i \le k-2$ and let $H_i = C_{14}$ in B_i for $k-1 \le i \le 2k$. Now let H be the union of these vertex-disjoint subgraphs H_i ($1 \le i \le 2k$) in G. Then H is a 2-regular graph of the size $m_H = 16(k-2)+14(k+2) = 30k-4$.

- For $\ell \ge 3$, let $H_i = C_{16}$ in B_i for $1 \le i \le k\ell - k - \ell$ and let $H_i = C_{14}$ in B_i for $k\ell - k - \ell + 1 \le i \le k\ell$. Now let H be the union of these vertex-disjoint subgraphs H_i ($1 \le i \le k\ell$) in G. Then H is a 2-regular graph of the size $m_H = 16(k\ell - k - \ell) + 14(k+\ell) = 16k\ell - 2k - 2\ell$.

Subcase 1.2. $n \equiv 2 \pmod 4$ *and* $q \equiv 2 \pmod 4$. Then $n = 4k+2$ and $q = 4\ell + 2$ for some integers k and ℓ with $k \ge \ell \ge 1$. In this case, the size m_H of a graph H with the desired properties is

$$\begin{aligned} m_H &= \frac{m}{2} = (4k+2)(4\ell+2) - (2k+1) - (2\ell+1) \\ &= (4k+2)(4\ell+2) - 2(k+\ell+1) = 16k\ell + 6k + 6\ell + 2. \end{aligned}$$

Let $G' = P_{4k} \,\Box\, P_{4\ell}$ be the induced subgraph of G as defined in (7) or (8) that contains the $k\ell$ vertex-disjoint blocks $B_1, B_2, \ldots, B_{k\ell}$ of $P_4 \,\Box\, P_4$ as shown in Figure 1. Let $H_i = C_{16}$ in B_i for $1 \le i \le k\ell$ and let $C = C_{6k}$ and $C' = C_{6\ell+2}$ be two vertex-disjoint cycles of orders $6k$ and $6\ell+2$ respectively in $G - E(G')$, where

$$\begin{aligned} C &= (v_{1,4\ell+1}, v_{2,4\ell+1}, \ldots, v_{3k,4\ell+1}, v_{3k,4\ell+2}, v_{3k-1,4\ell+2}, \ldots, \\ &\quad v_{1,4\ell+2}, v_{1,4\ell+1}) \\ C' &= (v_{4k+1,1}, v_{4k+1,2}, \ldots, v_{4k+1,3\ell+1}, v_{4k+2,3\ell+1}, v_{4k+2,3\ell}, \ldots, \\ &\quad v_{4k+2,1}, v_{4k+1,1}). \end{aligned}$$

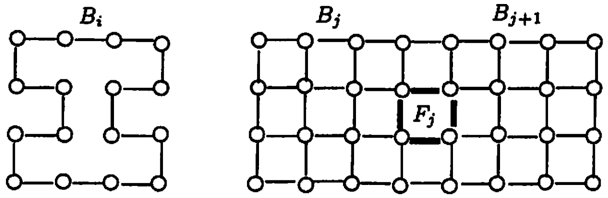
Now let H be the union of these vertex-disjoint subgraphs H_i ($1 \le i \le k\ell$), C and C' in G. Then H is a 2-regular graph of the size $m_H = 16k\ell + 6k + 6\ell + 2$.

Case 2. *n is even and q is odd.* If $n \equiv 0 \pmod 4$ and $q \equiv 3 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 1 \pmod 4$, then $m \equiv 1 \pmod 4$ and so G is not optimal by Lemma 2.1. For the converse, suppose that either $n \equiv 0 \pmod 4$ and $q \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$. We consider two subcases. In each subcase, we construct an even subgraph H of size $\lceil m/2 \rceil$ in G.

Subcase 2.1. $n \equiv 0 \pmod 4$ *and* $q \equiv 1 \pmod 4$. Then $n = 4k$ and $q = 4\ell + 1$ for some positive integers k and ℓ with $k \ge \ell + 1$. In this case, the size m_H of a graph H with the desired properties is

$$m_H = \left\lceil \frac{m}{2} \right\rceil = 4k(4\ell+1) - 2k - 2\ell = 16k\ell + 2k - 2\ell.$$

For each i with $1 \le i \le k\ell$, let $H_i = C_{16}$ in B_i as shown in Figure 3(a), where the edges not belonging to C_{16} are not drawn; while for each j with $1 \le j \le k-1$, let $F_j = C_4$ lying between B_j and B_{j+1} as shown in Figure 3(b). Then H_i $(1 \le i \le k\ell)$ and F_j $(1 \le j \le k-1)$ are edge-disjoint subgraphs of G.



(a) C_{16} in B_i

(b) C_4 betweenB_j and B_{j+1}

Figure 3: The cycles C_{16} and C_4

- If $k-\ell$ is even, then $k-\ell = 2p$ for some integer $p \ge 1$. Let $H_i = C_{16}$ in B_i for $1 \le i \le k\ell$, where H_i is shown as in Figure 3(a). For each j with $1 \le j \le p \le k-1$, let $F_j = C_4$ as defined in Figure 3(b) lying between B_j and B_{j+1}. Now let H consist of these edge-disjoint subgraphs H_i $(1 \le i \le k\ell)$ and F_j $(1 \le j \le p)$. That is,

$$\begin{aligned} V(H) &= \left(\cup_{i=1}^{k\ell} V(H_i)\right) \cup \left(\cup_{j=1}^{p} V(F_j)\right) \\ E(H) &= \left(\cup_{i=1}^{k\ell} E(H_i)\right) \cup \left(\cup_{j=1}^{p} E(F_j)\right) \end{aligned}$$

  Then H is a graph of the size $m_H = 16k\ell + 4p = 16k\ell + 2(k-\ell)$ and each vertex of H has degree 2 or 4.

- If $k-\ell$ is odd, then $k-\ell = 2p+1$ for some integer $p \ge 0$. Then $p+1 \le k-1$. Let $H_i = C_{16}$ in B_i as shown in Figure 3(a) for $1 \le i \le k\ell - 1$ and $H_{k\ell} = C_{14}$ in $B_{k\ell}$. For each j with $1 \le j \le p+1 \le k-1$, let $F_j = C_4$ that lies between B_j and B_{j+1} as shown in Figure 3(b). Now let H consist of these edge-disjoint subgraphs H_i $(1 \le i \le k\ell)$ and F_j $(1 \le j \le p+1)$. Then H is a graph of the size $m_H = 16(k\ell-1) + 14 + 4(p+1) = 16k\ell + 2(k-\ell)$ and each vertex of H has degree 2 or 4.

Subcase 2.2. $n \equiv 2 \pmod 4$ *and* $q \equiv 3 \pmod 4$. Then $n = 4k+2$ and $q = 4\ell+3$ for some positive integers k and ℓ with $k \ge \ell+1$. In this case, the size m_H of a graph H with the desired properties is

$$\begin{aligned} m_H &= \left\lceil \frac{m}{2} \right\rceil = (4k+2)(4\ell+3) - (2k+1) - (2\ell+1) \\ &= 16k\ell + 10k + 6\ell + 4. \end{aligned}$$

For each i with $1 \le i \le k\ell$, let $H_i = C_{16}$ in B_i where for $1 \le i \le k$, the graphs B_i and H_i are defined as shown in Figure 3(a). For each j with $1 \le j \le k-1$, let $F_j = C_4$ between B_j and B_{j+1} are defined in Figure 3(b). Furthermore, let $C = C_{6\ell}$ and $C' = C_{6k+8}$ where

$$\begin{aligned} C &= (v_{4k+1,1}, v_{4k+1,2}, \ldots, v_{4k+1,3\ell}, v_{4k+2,3\ell}, v_{4k+2,3\ell-1}, \ldots, \\ &\quad v_{4k+2,1}, v_{4k+1,1}), \\ C' &= (v_{1,4\ell+2}, v_{2,4\ell+2}, \ldots, v_{3k+4,4\ell+2}, v_{3k+4,4\ell+3}, v_{3k+3,4\ell+3}, \ldots, \\ &\quad v_{1,4\ell+3}, v_{1,4\ell+2}). \end{aligned}$$

Since $k \ge \ell + 1$, it follows that $3k+4 \le 4k+2$ and so such a cycle C' of order $6k+8$ exists. Now let H consist of these edge-disjoint subgraphs H_i $(1 \le i \le k\ell)$, F_j $(1 \le j \le k-1)$, C and C'. Then H is a graph of the size $m_H = 16k\ell + 4(k-1) + 6\ell + 6k + 8 = 16k\ell + 10k + 6\ell + 4$ and each vertex of H has degree 2 or 4.

Case 3. *n is odd and q is even.* If $n \equiv 1 \pmod 4$ and $q \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$ and $q \equiv 0 \pmod 4$, then $m \equiv 1 \pmod 4$ and so G is not optimal by Lemma 2.1. For the converse, suppose that either $n \equiv 1 \pmod 4$ and $q \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ and $q \equiv 2 \pmod 4$. We consider these two subcases. In each subcase, we construct an even subgraph H of size $\lceil m/2 \rceil$ in G. Let $G' = P_{4k} \,\Box\, P_{4\ell}$ be the induced subgraph in G consisting of the 4ℓ paths of order $4k$ as defined in (8) and let $B_1, B_2, \ldots, B_{k\ell}$ are the $k\ell$ vertex-disjoint blocks of $P_4 \,\Box\, P_4$ in G' as shown in Figure 1.

Subcase 3.1. *$n \equiv 1 \pmod 4$ and $q \equiv 0 \pmod 4$.* Then $n = 4k+1$ and $q = 4\ell$ for some positive integers k and ℓ with $k \ge \ell$. In this case, the size m_H of a graph H with the desired properties is

$$m_H = \left\lceil \frac{m}{2} \right\rceil = (4k+1)4\ell - 2k - 2\ell = 16k\ell - 2k + 2\ell.$$

Let $H_i = C_{14}$ in B_i if $1 \le i \le k - \ell$ and let $H_i = C_{16}$ in B_i if $k - \ell + 1 \le i \le k\ell$. Let H be the union of these vertex-disjoint subgraphs H_i for $1 \le i \le k\ell$. Then H is a 2-regular subgraph of G and the size of H is $14(k-\ell) + 16[k\ell - (k-\ell)] = 16k\ell - 2k + 2\ell$.

Subcase 3.2. *$n \equiv 3 \pmod 4$ and $q \equiv 2 \pmod 4$.* Then $n = 4k+3$ and $q = 4\ell + 2$ for some positive integers k and ℓ with $k \ge \ell$. In this case, the size m_H of a graph H with the desired properties is

$$m_H = \left\lceil \frac{m}{2} \right\rceil = (4k+3)(4\ell+2) - (2k+1) - (2\ell+1) = 16k\ell + 6k + 10\ell + 4.$$

Let $H_i = C_{16}$ in B_i if $1 \le i \le k\ell$ which are defined as shown in Figure 3(a) and for each j with $1 \le j \le \ell - 1 \le k - 1$, let $F_j = C_4$ between B_j and B_{j+1} as defined in Figure 3(b). Furthermore, let $C = C_{6\ell}$ and $C' = C_{6k+8}$ where

$$\begin{aligned} C &= (v_{4k+2,1}, v_{4k+2,2}, \ldots, v_{4k+2,3\ell}, v_{4k+3,3\ell}, v_{4k+3,3\ell-1}, \ldots, \\ &\quad v_{4k+3,1}, v_{4k+2,1}), \\ C' &= (v_{1,4\ell+1}, v_{2,4\ell+1}, \ldots, v_{3k+4,4\ell+1}, v_{3k+4,4\ell+2}, v_{3k+3,4\ell+2}, \ldots, \\ &\quad v_{1,4\ell+2}, v_{1,4\ell+1}). \end{aligned}$$

Since $3k + 4 \le 4k + 3$, such a cycle C' of order $6k + 8$ exists. Now let H consist of these edge-disjoint subgraphs H_i $(1 \le i \le k\ell)$, F_j $(1 \le j \le \ell - 1)$, C and C'. Then H is a graph of the size $m_H = 16k\ell + 4(\ell - 1) + 6\ell + 6k + 8 = 16k\ell + 10\ell + 6k + 4$ and each vertex of H has degree 2 or 4. ∎

Theorem 2.6 *For integers n, p with $n \ge p \ge 3$, if $P_n \square P_q$ is not optimal, then*

$$EI(P_n \square P_q) = \binom{n(q-1) + (n-1)q + 1}{2} + 1.$$

Proof. By Theorem 2.5, if $P_n \square P_q$ is not optimal, then n and q satisfy one of the following:

(*i*) If n and q are even, then either $n \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$;

(*ii*) If n is even and q is odd, then either $n \equiv 0 \pmod 4$ and $q \equiv 3 \pmod 4$ or $n \equiv 2 \pmod 4$ and $q \equiv 1 \pmod 4$;

(*iii*) If n is odd and q is even, then $n \equiv 1 \pmod 4$ and $q \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$ and $q \equiv 0 \pmod 4$.

Suppose that $G = P_n \square P_q$ consists of q paths of order n, which we denote by

$$P_{n,i} = (v_{1,i}, v_{2,i}, \ldots, v_{n,i}) \text{ for } 1 \le i \le q \tag{9}$$

such that $v_{t,i}$ is adjacent to $v_{t,j}$ $(1 \le t \le n)$ when $|i - j| = 1$. The size of G is $m = n(q - 1) + (n - 1)q$. Then $n = 4k + r_n$ and $q = 4\ell + r_q$, where $r_n, r_q \in \{0, 1, 2, 3\}$. Let $G' = P_{4k} \square P_{4\ell}$ be the subgraph of G with

$$V(G') = \{v_{a,b} : 1 \le a \le 4k, 1 \le b \le 4\ell\}. \tag{10}$$

The graph G' is else defined in (7) and (8). Then G' contains $k\ell$ vertex-disjoint copies (or blocks) of $P_4 \square P_4$, denoted by $B_{i,j}$ where $1 \le i \le k$ and $1 \le j \le \ell$. These blocks $B_{i,j}$ appear in G' in the way as shown in Figure 4.

$B_{1,1}$	$B_{1,2}$	$B_{1,3}$	$\cdots$	$B_{1,\ell}$
$B_{2,1}$	$B_{2,2}$	$B_{2,3}$	$\cdots$	$B_{2,\ell}$
$B_{3,1}$	$B_{3,2}$	$B_{3,3}$	$\cdots$	$B_{3,\ell}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{k,1}$	$B_{k,2}$	$B_{k,3}$	$\cdots$	$B_{k,\ell}$

Figure 4: The subgraph $G' = P_{4k} \,\square\, P_{4\ell}$ in G

In each $B_{i,j} = P_4 \,\square\, P_4$, the vertices of $B_{i,j} = P_4 \,\square\, P_4$ appear in the way as shown in Figure 5.

$v_{4i-3,4j-3}$	$v_{4i-3,4j-2}$	$v_{4i-3,4j-1}$	$v_{4i-3,4j}$
$v_{4i-2,4j-3}$	$v_{4i-2,4j-2}$	$v_{4i-2,4j-1}$	$v_{4i-2,4j}$
$v_{4i-1,4j-3}$	$v_{4i-1,4j-2}$	$v_{4i-1,4j-1}$	$v_{4i-1,4j}$
$v_{4i,4j-3}$	$v_{4i,4j-2}$	$v_{4i,4j-1}$	$v_{4i,4j}$

Figure 5: The block $B_{i,j} = P_4 \,\square\, P_4$ in G'

Note that $B_{i,j}$ contains five edge-disjoint copies of C_4, namely

$$\begin{aligned}
Q_1 &= (v_{4i-3,4j-3}, v_{4i-3,4j-2}, v_{4i-2,4j-2}, v_{4i-2,4j-3}, v_{4i-3,4j-3})\\
Q_2 &= (v_{4i-3,4j-1}, v_{4i-3,4j}, v_{4i-2,4j}, v_{4i-2,4j-1}, v_{4i-3,4j-1})\\
Q_3 &= (v_{4i-1,4j-1}, v_{4i-1,4j}, v_{4i,4j}, v_{4i,4j-1}, v_{4i-1,4j-1})\\
Q_4 &= (v_{4i-1,4j-3}, v_{4i-1,4j-2}, v_{4i,4j-2}, v_{4i,4j-3}, v_{4i-1,4j-3})\\
Q_5 &= (v_{4i-2,4j-2}, v_{4i-2,4j-1}, v_{4i-1,4j-1}, v_{4i-1,4j-2}, v_{4i-2,4j-2})
\end{aligned}$$

where Q_5 is at the center of $B_{i,j}$ and surrounded clockwise by Q_1, Q_2, Q_3, Q_4. For each pair i, j with $1 \le i \le k$ and $1 \le j \le \ell$, let $F_{i,j}$ be the even subgraph of $B_{i,j}$ consisting of the five edge-disjoint subgraphs Q_1, Q_2, Q_3, Q_4, Q_5, each of which is a copy of C_4 and let $F'_{i,j}$ be the even subgraph of $B_{i,j}$ consisting of the four edge-disjoint subgraphs Q_1, Q_2, Q_3, Q_4. Thus, the size of $F_{i,j}$ is 20 and the size of $F'_{i,j}$ is 16 for all i, j with $1 \le i \le k$ and $1 \le j \le \ell$. We consider three cases.

Case 1. *n and q are even.* Since m is even, it suffices to show that G has an even subgraph of size $\frac{m}{2} + 1$. There are two subcases.

Subcase 1.1. $n \equiv 0 \pmod 4$ *and* $q \equiv 2 \pmod 4$. Let $n = 4k$ and $q = 4\ell+2$, where then $k > \ell \ge 1$. Note that $\frac{m}{2}+1 = 16k\ell+4k+2(k-\ell)$. Let $G' = P_{4k} \,\square\, P_{4\ell}$ be the subgraph of G as described in (10) and $G^* = P_n \,\square\, P_2$ be the subgraph of G which is the Cartesian product of $P_{n,4\ell+1}$ and $P_{n,4\ell+2}$ as described in (9).

- If $k = \ell + 1$, then for $1 \le i \le k-1$ and $j = 1$, let $H_{i,1} = F_{i,1}$ and $H_{k,1} = F'_{k,1}$, while for $1 \le i \le k$ and $2 \le j \le \ell$, let $H_{i,j} = F'_{i,j}$.

- If $k \geq \ell+2$, then for $1 \leq i \leq k$ and $j = 1$, let $H_{i,1} = F_{i,1}$, for $1 \leq i \leq k$ and $2 \leq j \leq \ell$, let $H_{i,j} = F'_{i,j}$ and let $H_{k,\ell+1} = C_{2(k-\ell)}$ be a cycle of order $2(k-\ell)$ in G^* (which is possible since $2 \leq k - \ell \leq 4\ell + 1$).

In each case, let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of m_H is $\frac{m}{2} + 1$.

Subcase 1.2. $n \equiv 2 \pmod 4$ *and* $q \equiv 0 \pmod 4$. Let $n = 4k + 2$ and $q = 4\ell$, where then $k \geq \ell \geq 1$. Note that $\frac{m}{2} + 1 = 16k\ell - 2k + 6\ell$. In this case, we consider the subgraph $G'' = P_{4k+1} \square P_{4\ell}$ of G with vertex set

$$V(G'') = \{v_{a,b} : 1 \leq a \leq 4k+1, 1 \leq b \leq 4\ell\}. \tag{11}$$

Then G'' contains $(k-1)\ell$ vertex-disjoint copies (or blocks) $P_4 \square P_4$, which are denoted by $B_{i,j}$ where $1 \leq i \leq k-1$ and $1 \leq j \leq \ell$ and ℓ vertex-disjoint copies of $P_5 \square P_4$, which are denoted by B'_j where $1 \leq j \leq \ell$. These blocks $B_{i,j}$ and B'_j appear in G'' in the way as shown in Figure 6.

$B_{1,1}$	$B_{1,2}$	$B_{1,3}$	$\cdots$	$B_{1,\ell}$
$B_{2,1}$	$B_{2,2}$	$B_{2,3}$	$\cdots$	$B_{2,\ell}$
$B_{3,1}$	$B_{3,2}$	$B_{3,3}$	$\cdots$	$B_{3,\ell}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{(k-1),1}$	$B_{(k-1),2}$	$B_{(k-1),3}$	$\cdots$	$B_{(k-1),\ell}$
B'_1	B'_2	B'_3	$\cdots$	B'_ℓ

Figure 6: The subgraph $G'' = P_{4k+1} \square P_{4\ell}$ in G

For each j with $1 \leq j \leq \ell$, the vertices of $B'_j = P_5 \square P_4$ appear in the way as shown in Figure 7.

$v_{4k-3,4j-3}$	$v_{4k-3,4j-2}$	$v_{4k-3,4j-1}$	$v_{4k-3,4j}$
$v_{4k-2,4j-3}$	$v_{4k-2,4j-2}$	$v_{4k-2,4j-1}$	$v_{4k-2,4j}$
$v_{4k-1,4j-3}$	$v_{4k-1,4j-2}$	$v_{4k-1,4j-1}$	$v_{4k-1,4j}$
$v_{4k,4j-3}$	$v_{4k,4j-2}$	$v_{4k,4j-1}$	$v_{4k,4j}$
$v_{4k+1,4j-3}$	$v_{4k+1,4j-2}$	$v_{4k+1,4j-1}$	$v_{4k+1,4j}$

Figure 7: The block $B'_j = P_5 \square P_4$ in G''

Note that each B'_j $(1 \leq j \leq \ell)$ contains each of C_{14} and C_{18} as a subgraph. For $i = 1$ and $1 \leq j \leq \ell$, let $H_{1,j} = F_{1,j}$ in $B_{1,j}$, for each pair i, j with $2 \leq i \leq k-1$ and $1 \leq j \leq \ell - 1$, let $H_{i,j} = F'_{i,j}$ in $B_{i,j}$, for $2 \leq i \leq k$ and $j = \ell$, let $H_{i,\ell} = C_{14}$ (in $B_{i,\ell}$ if $1 \leq i \leq k-1$ and in B'_k if $i = k$) and for $i = k$ and $1 \leq j \leq \ell - 1$, let $H_{k,j} = C_{18}$ in B'_j. Let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of m_H is $\frac{m}{2} + 1$.

Case 2. *n is even and q is odd.* Since m is odd, it suffices to show that G has an even subgraph of size $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$. Let $G' = P_{4k} \square P_{4\ell}$ be the subgraph of G as described in (10). There are two cases.

Subcase 2.1. $n \equiv 0 \pmod 4$ *and* $q \equiv 3 \pmod 4$. Let $n = 4k$ and $q = 4\ell + 3$, where then $k > \ell \geq 1$. Note that $\frac{m-1}{2} = 16k\ell + 4k + 2(3k - \ell - 1)$ and $2 \leq 3k - \ell - 1 \leq 4k$. Let $G^* = P_n \square P_2$ be the subgraph of G which is the Cartesian product of $P_{n,4\ell+1}$ and $P_{n,4\ell+2}$ as described in Subcase 1.1. For $1 \leq i \leq k$ and $j = 1$, let $H_{i,1} = F_{i,1}$ in $B_{i,1}$, for $1 \leq i \leq k$ and $2 \leq j \leq \ell$, let $H_{i,j} = F'_{i,j}$ in $B_{i,j}$ and let $H_{k,\ell+1} = C_{2(3k-\ell-1)}$ be a subgraph in G^*. Let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of H_m is $\frac{m-1}{2}$.

Subcase 2.2. $n \equiv 2 \pmod 4$ *and* $q \equiv 1 \pmod 4$. Let $n = 4k + 2$ and $q = 4\ell + 1$, where then $k \geq \ell \geq 1$. Note that $\frac{m-1}{2} = 16k\ell + 2(k + 3\ell)$ and $2 \leq k + 3\ell \leq 4\ell + 1 = q$. Let $F^* = P_2 \square P_q$ be the subgraph of G which is the Cartesian product of the two paths

$$(v_{n-1,1}, v_{n-1,2}, \ldots, v_{n-1,q}) \text{ and } (v_{n,1}, v_{n,2}, \ldots, v_{n,q}). \tag{12}$$

For each pair i, j with $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let $H_{i,j} = F'_{i,j}$ in $B_{i,j}$ and let $H_{k,\ell+1} = C_{2(k+3\ell))}$ be a subgraph in F^*. Let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of H_m is $\frac{m-1}{2}$.

Case 3. *n is odd and q is even.* Since m is odd, we are seeking for an even subgraph of size $\lceil \frac{m}{2} \rceil - 1 = \frac{m-1}{2}$ in G. Let $G' = P_{4k} \square P_{4\ell}$ and $G'' = P_{4k+1} \square P_{4\ell}$ be the subgraphs of G as described in (10) and (11), respectively. There are two cases.

Subcase 3.1. $n \equiv 1 \pmod 4$ *and* $q \equiv 2 \pmod 4$. Let $n = 4k + 1$ and $q = 4\ell + 2$, where then $k > \ell \geq 1$. Note that $\frac{m-1}{2} = 16k\ell + 4k + 2(k + \ell)$ and $2 \leq k + \ell \leq 4k$. Let $G^* = P_n \square P_2$ be the subgraph of G as described in Subcase 1.1. For $1 \leq i \leq k$ and $j = 1$, let $H_{i,1} = F_{i,1}$ in $B_{i,1}$, for $1 \leq i \leq k$ and $2 \leq j \leq \ell$, let $H_{i,j} = F'_{i,j}$ in $B_{i,j}$ and let $H_{k,\ell+1} = C_{2(k+\ell)}$ be a subgraph in G^*. Let H be the even subgraph of G consisting of edge-disjoint subgraphs $H_{i,j}$ and then the size of H_m is $\frac{m-1}{2}$.

Subcase 3.2. $n \equiv 3 \pmod 4$ *and* $q \equiv 0 \pmod 4$. Let $n = 4k + 3$ and $q = 4\ell$, where then $k \geq \ell \geq 1$. Let $F^* = P_2 \square P_q$ be the subgraph of G which is the Cartesian product of the two paths described in (12).

- If $\ell = 1$, then $G = P_{4k+3} \square P_4$ and $\frac{m-1}{2} = 14k + 8$. For $1 \leq i \leq k$, let $H_i = C_{14}$ in $B_{i,1}$ and let H_{k+1} be a cycle C_8 of order 8 where

$$\begin{aligned} H_{k+1} = {} & (v_{4k+1,1}, v_{4k+1,2}, v_{4k+1,3}, v_{4k+1,4}, v_{4k+2,4}, v_{4k+2,3}, \\ & v_{4k+2,2}, v_{4k+2,1}, v_{4k+1,1}). \end{aligned}$$

Let H be the even subgraph of G consisting of edge-disjoint subgraphs H_i and then the size of H_m is $\frac{m-1}{2}$.

- If $\ell \geq 2$, then $\frac{m-1}{2} = 16k\ell - 2k + 6\ell + 2(2\ell - 1)$. Let H_1 be the even subgraph of size $16k\ell - 2k + 6\ell$ in $P_{4k+1} \square P_{4\ell}$ (which is described in Subcase 1.2) and let $H_2 = C_{2(2\ell-1)}$ be a subgraph of F^*. Let H be the even subgraph of G consisting of edge-disjoint subgraphs H_1 and H_2 and then the size of H_m is $\frac{m-1}{2}$. ■

3 Optimal Powers of Cycles

For a connected graph G and a positive integer k, the *kth power* G^k of G is that graph whose vertex set is $V(G)$ such that uv is an edge of G^k if $1 \leq d_G(u,v) \leq k$. The graph G^2 is called the *square* of G and G^3 is the *cube* of G. If $k \geq \text{diam}(G)$, then G^k is a complete graph. It is known that all complete graphs of order at least 4 is optimal, as we state next.

Theorem 3.1 [1] *The complete graph K_n of order n is optimal if and only if $n \geq 4$.*

By Theorem 1.1, the n-cycle C_n is not optimal; while by Theorem 3.1, the complete graph K_n is. Thus if $k = 1$, then $C_n^1 = C_n$ is not optimal; while if $k \geq \lfloor n/2 \rfloor$, then C_n^k is. We show, in fact, that C_n^k is optimal for each integer $k \geq 2$. In order to do this, we introduce an additional definition. For a positive integer t, the *t-step* $G^{[t]}$ of a connected graph G is that graph whose vertex set is $V(G)$ such that uv is an edge of $G^{[t]}$ if $d_G(u,v) = t$. In particular, $G^{[1]} = G$. Furthermore, if $t \leq k$, then $G^{[t]}$ is a subgraph of G^k and

$$E(G^k) = E(G^{[1]}) \cup E(G^{[2]}) \cup \cdots \cup E(G^{[k]}).$$

For the n-cycle $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $n \geq 3$ and each integer i with $1 \leq i \leq n$, the vertex v_i is adjacent to v_{i+t} and v_{i-t} in $G^{[t]}$, where the subscripts are expressed as integers modulo n. Thus $C_n^{[t]}$ is a 2-regular graph of order n if $t \neq n/2$ and $C_n^{[t]} = \frac{n}{2}K_2$ if $t = n/2$ where then n is even. The kth power C_n^k of C_n is then a $2k$-regular graph of order n and size kn if $k < n/2$

Theorem 3.2 *For each pair k, n of integers, where $2 \leq k \leq \lfloor n/2 \rfloor$ and $n \geq 4$, the kth power C_n^k of the n-cycle is optimal.*

Proof. If $k = \lfloor n/2 \rfloor$, then $C_n^k = K_n$, which is optimal by Theorem 3.1. Thus, we now assume that $k < \lfloor n/2 \rfloor$. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $n \geq 4$. The size of C_n^k is $m = kn$. If k is even, say $k = 2p$ for some positive integer p, then the subgraph C_n^p is a $(2p)$-regular graph of size $pn = \lceil m/2 \rceil$. It then follows by Theorem 1.1 that C_n^k is optimal if k is even. Thus, it remains to consider the case when $k \geq 3$ is odd. Since $k < \lfloor n/2 \rfloor$, it follows that $n \geq 8$. We show that C_n^k contains a subgraph H_k of size $\lceil m/2 \rceil$, each of whose vertex is even. We begin with the cube C_n^3 of C_n. There are two cases, according to whether n is even or n is odd.

Case 1. *n is even.* Let $C^* = (v_1, v_3, v_5, \ldots, v_{n-1}, v_1)$ be the cycle of order $n/2$ in C_n^3 and let H_3 be the spanning subgraph of G with $E(H_3) = E(C_n) \cup E(C^*)$. Then the size of H_3 is $3n/2$ and each vertex of H_3 has degree 2 or 4. By Theorem 1.1, C_n^3 is optimal if n is even.

Case 2. *n is odd.* Let $n = 2\ell + 1$ for some integer $\ell \geq 4$. Then $\lceil m/2 \rceil = \lceil 3n/2 \rceil = 3\ell + 2$. First, suppose that ℓ is even. Let C' be the cycle of order $n - 4$ in G defined by

$$C' = (v_2, v_3, \ldots, v_{\ell-1}, v_{\ell+2}, v_{\ell+1}, v_{\ell+4}, v_{\ell+5}, \ldots, v_{n-1}, v_2)$$

and let C'' be the circuit in G defined by

$$C'' = (v_1, v_3, \ldots, v_{\ell-1}, v_\ell, v_{\ell+2}, v_{\ell+3}, v_{\ell+1}, v_\ell, v_{\ell+3}, v_{\ell+4}, v_{\ell+6}, \ldots, v_{n-1}, v_1).$$

Figure 8(a) shows C' and C'' for $n = 9$ and $n = 13$, where the edges of C' are drawn in solid lines and the edges of C'' are drawn in dashed lines. Let H_3 be the subgraph of G induced by $E(C') \cup E(C'')$. Then the size of H_3 is $|E(C')| + |E(C'')| = (n-4) + 7 + (n-5)/2 = 3\ell + 2 = \lceil 3n/2 \rceil$ and each vertex of H_3 has degree 2 or 4.

Next suppose that ℓ is odd. Let C' be the cycle of order $n - 4$ in G defined by

$$C' = (v_1, v_3, v_4, \ldots, v_{\ell-1}, v_{\ell+2}, v_{\ell+1}, v_{\ell+4}, v_{\ell+5}, \ldots, v_{n-1}, v_1)$$

and let C'' be the circuit in G defined by

$$C'' = (v_2, v_4, \ldots, v_{\ell-1}, v_\ell, v_{\ell+2}, v_{\ell+3}, v_{\ell+1}, v_\ell, v_{\ell+3}, v_{\ell+4}, v_{\ell+6}, \ldots, v_n, v_2).$$

Figure 8(b) shows C' and C'' for $n = 11$ and $n = 15$, where the edges of C' are drawn in solid lines and the edges of C'' are drawn in dashed lines. Let H_3 be the subgraph of G induced by $E(C') \cup E(C'')$. Then the size of H_3 is $|E(C')| + |E(C'')| = (n-4) + 7 + (n-5)/2 = 3\ell + 2 = \lceil 3n/2 \rceil$ and each vertex of H_3 has degree 2 or 4.

In general, if $k \geq 5$ is odd and $n = 2\ell + 1$, then $\lceil m/2 \rceil = \lceil kn/2 \rceil = k\ell + \lceil k/2 \rceil$. For $k = 5$, let H_5 consists of H_3 and $C_n^{[4]}$. Since each vertex in H_3 and $C_n^{[4]}$ is even, every vertex of H_5 is even and the size of H_5 is $|E(H_3)| + n = (3\ell + 2) + (2\ell + 1) = 5\ell + 3 = 5\ell + \lceil 5/2 \rceil$. More generally then, for an odd integer $k \geq 7$ with $k \leq \lfloor n/2 \rfloor - 3$, the subgraph H_{k+2}

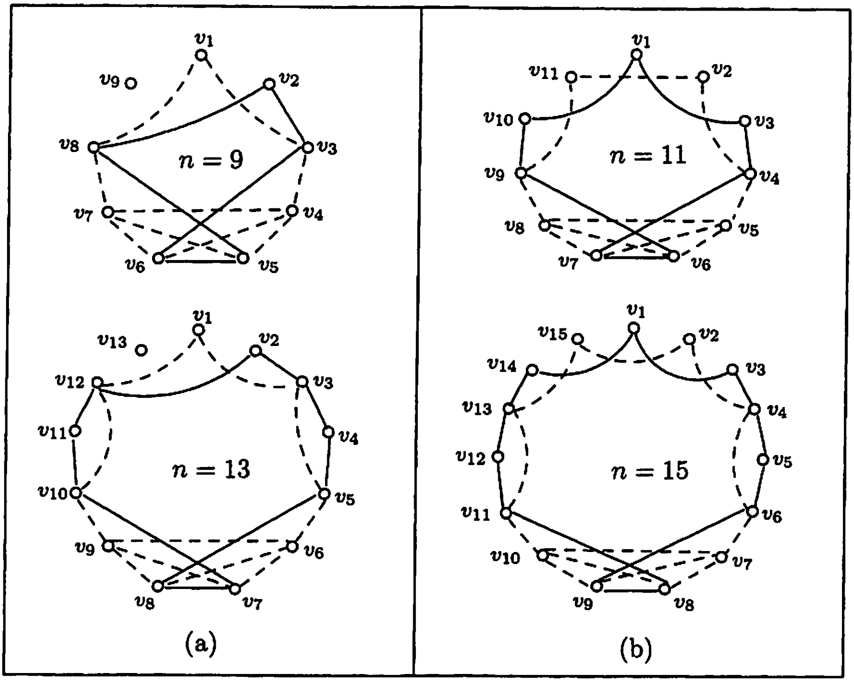


Figure 8: Subgraphs C' and C'' in C_n^3 for $n = 9, 11, 13, 15$

consists of H_k and $C_n^{[k+1]}$, where every vertex of H_k and $C_n^{[k+1]}$ is even and the size of H_k is $k\ell + \lceil k/2 \rceil$. Hence, every vertex of H_{k+2} is even and the size of H_{k+2} is $|E(H_k)| + n = (k\ell + \lceil k/2 \rceil) + (2\ell + 1) = (k+2)\ell + \lceil (k+2)/2 \rceil$.

Therefore, C_n^k is optimal for each integer odd integer k with $3 \le k \le \lfloor n/2 \rfloor$. ■

4 Acknowledgments

We are grateful to Professor Gary Chartrand for suggesting problems to us and kindly providing useful information on this topic.

References

[1] E. Andrews, G. Chartrand, C. Lumduanhom and P. Zhang, On Eulerian walks in graphs. *Bull. Inst. Combin. Appl.* To appear.

[2] E. Andrews, C. Lumduanhom and P. Zhang, On Eulerian irregularity in graphs. Preprint.

[3] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs: 5th Edition*, Chapman & Hall/CRC, Boca Raton, FL (2010).

[4] G. Chartrand and P. Zhang, *Chromatic Graph Theory*. Chapman & Hall/CRC, Boca Raton, FL (2008).

[5] L. Euler, Solutio problematis ad geometriam situs pertinentis. *Comment. Academiae Sci. I. Petropolitanae* **8** (1736) 128-140.

[6] M. K. Kwan, Graphic programming using odd or even points, *Acta Math. Sinica* **10** (1960) 264-266 (Chinese); translated as *Chinese Math.* **1** (1960) 273-277.