Decompositions of λK_n using Stanton–type graphs

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ABSTRACT. A *Stanton–type graph* $S(n, m)$ is a connected multigraph on n vertices such that for a fixed m with $n - 1 \leq m \leq \binom{n}{2}$, there is exactly one edge of multiplicity i (and no others) for each $i = 1, 2, \ldots, m$. In this note, we show how to decompose λK_n (for the appropriate minimal values of λ) into Stanton–type graphs $S(4, 3)$ of the LOE and OLE types.

1. Introduction

A *simple graph* G is an ordered pair (V, E) where V is an n–set (of *points*), and E is a subset of the set of the $\binom{n}{2}$ pairs of distinct elements of V (the *edges*). This definition can be generalized to that of a *multigraph* (without loops) by allowing E to be a multiset, where edges can occur with *frequencies* greater than 1.

For an excellent survey on (simple) graph decompositions, see [**2**]. Chan and Sarvate [4] introduced *Stanton graphs*:

DEFINITION 1. *A* Stanton graph S_n *is a multigraph on* n *vertices such that for each* $i = 1, 2, \ldots, \binom{n}{2}$, *there is exactly one edge of multiplicity* i *(and no others).*

EXAMPLE 1. *The unique (up to isomorphism) Stanton graph* S_3 *on* $V = \{1, 2, 3\}$ *with edge set* $E = \{\{1, 2\}, \{1, 3\}, \{1, 3\}, \{1, 3\}, \{2, 3\}, \{2, 3\}\}$ *can be drawn as*

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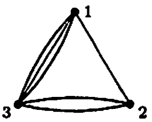


DEFINITION 2. *Given an integer $n \geq 2$ and an integer m such that $n-1 \leq m \leq \binom{n}{2}$, a* Stanton–type graph $S(n,m)$ *is a connected multigraph on n vertices such that for $i = 1, 2, \ldots, m$, there is exactly one edge of multiplicity i (and no others).*

EXAMPLE 2. *A Stanton–type graph on $V = \{1,2,3,4,5\}$ with $m = 4$ and edge set $E = \{\{1,2\},\{2,3\},\{2,3\},\{3,4\},\{3,4\},\{3,4\}, \{4,5\},\{4,5\},\{4,5\},\{4,5\}\}$ can be drawn as*

NOTE 1. *It should be noted that an S_n is the same as an $S(n, \binom{n}{2})$.*

Chan and Sarvate [4] decomposed λK_n into Stanton graphs S_3 for some minimal number λ. Recently, El–Zanati, Lapchinda, Tangsupphathawat and Wannasit [5] have proved that the necessary conditions are sufficient for the decomposition of λK_n (for any λ) into Stanton graphs. There are other types of connected multigraphs with 6 edges having edge frequencies only 1, 2 and 3. In this paper, we show how to decompose λK_n into Stanton–type graphs $S(4,3)$ of the LOE and OLE types.

2. Preliminaries

DEFINITION 3. *Let $V = \{a,b,c,d\}$. An* LOE graph $\langle a,b,c,d\rangle$ *on V is a graph with 6 edges where the frequencies of edges $\{a,b\}$, $\{b,c\}$ and $\{c,d\}$ are 1, 2 and 3 (respectively).*

EXAMPLE 3. *Consider $G_1 = (V,E)$ where $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{2,3\},\{2,3\},\{3,4\},\{3,4\},\{3,4\}\}$. Then G_1 is an LOE graph $\langle 1,2,3,4\rangle$, drawn as*

DEFINITION 4. *Let $V = \{a,b,c,d\}$. An* OLE graph $[a,b,c,d]$ *on V is a graph with 6 edges where the frequencies of edges $\{a,b\}$, $\{b,c\}$ and $\{c,d\}$ are 2, 1 and 3 (respectively).*

EXAMPLE 4. *Consider $G_2 = (V,E)$ where $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{1,2\},\{2,3\},\{3,4\},\{3,4\},\{3,4\}\}$. Then G_2 is an OLE graph $[1,2,3,4]$, drawn as*

We have the following two results, which can be obtained easily by two–way counting and divisibility requirements:

LEMMA 1. *The graph λK_n can be decomposed by LOE or OLE graphs only if 6 divides $\lambda\binom{n}{2}$.*

This result (and also the next theorem) agrees numerically with [4, Theorem 5], where the minimum number of copies λ of the complete graph K_n that can be decomposed into Stanton graphs S_3 was given:

THEOREM 1. *The minimum number of copies λ of the complete graph K_n that can be decomposed into LOE or OLE graphs is*

a) $\lambda = 3$, *when* $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$,
b) $\lambda = 4$, *when* $n \equiv 3, 6, 7, 10 \pmod{12}$, *and*
c) $\lambda = 6$, *when* $n \equiv 2, 11 \pmod{12}$.

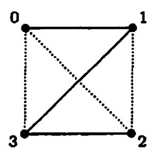
3. Path Decompositions

DEFINITION 5. *The graph P_n is a path on n vertices.*

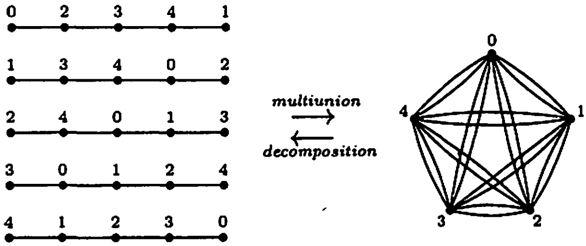
NOTE 2. *Note that P_n has $n - 1$ edges.*

DEFINITION 6. *For any positive integers $\lambda \geq 1$, k and n such that $n \geq k$, a P_k*–decomposition *of λK_n is a set of paths P_k that partition the edge set of λK_n (so that the multiunion of them is λK_n).*

EXAMPLE 5. *Let $V = \mathbb{Z}_4 = \{0, 1, 2, 3\}$. A P_4–decomposition of K_4 is given by the set of paths* 0-1-3-2 *and* 1-2-0-3.



EXAMPLE 6. *We present a P_5–decomposition of $2K_5$ that is generated by developing a base graph. Considering the point set to be $V = \mathbb{Z}_5$, we begin with the base graph* 0-2-3-4-1 *and develop it (modulo 5). The next paths will be* 1-3-4-0-2, 2-4-0-1-3, 3-0-1-2-4 *and* 4-1-2-3-0. *We see that the multiunion of all of these P_5 will be $2K_5$:*

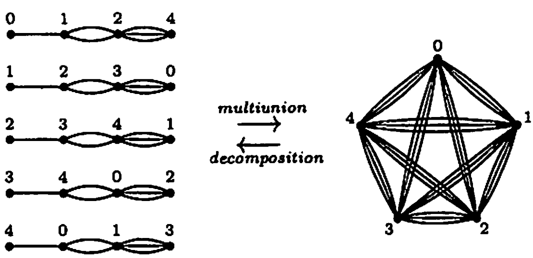


This notion of base graphs being developed to produce a decomposition of λK_n will be extremely important in the sequel.

4. LOE–Decompositions

DEFINITION 7. *For positive integers n and λ, an* LOE–decomposition *of λK_n is a collection of LOE graphs such that the multiunion of their edge sets contains λ copies of all edges in a K_n.*

EXAMPLE 7. *We present an LOE–decomposition of $3K_5$ that is generated by developing a base graph. Considering the set of points to be $V = \mathbb{Z}_5$, the LOE base graph $\langle 0, 1, 2, 4\rangle$ (when developed modulo 5) constitutes an LOE–decomposition of $3K_5$.*



THEOREM 2. *If a P_4-decomposition of λK_n exists, then an LOE–decomposition of $4\lambda K_n$ exists.*

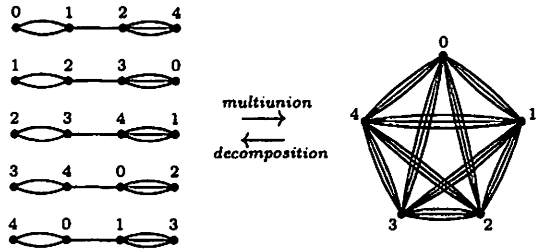
PROOF. Replace each P_4 a-b-c-d with the pair of LOE graphs $\langle a, b, c, d\rangle$ and $\langle d, c, b, a\rangle$. Then each of the edges $\{a, b\}$, $\{b, c\}$, and $\{c, d\}$ will have multiplicity 4. ■

NOTE 3. *As stated in* [3], *there exists a P_k-decomposition of λK_n if and only if $n \geq k$ and $\lambda n(n-1) \equiv 0 \pmod{2k-2}$. Thus, a P_4-decomposition of $4K_n$ exists when $n \geq 4$ and $n(n-1) \equiv 0 \pmod 3$.*

5. OLE–Decompositions

DEFINITION 8. *For positive integers n and λ, an* OLE–decomposition *of λK_n is a collection of OLE graphs such that the multiunion of their edge sets contains λ copies of all edges in a K_n.*

EXAMPLE 8. *Considering the set of points to be $V = \mathbb{Z}_5$, the OLE base graph* [0, 1, 2, 4] *(when developed modulo 5) constitutes an OLE–decomposition of* $3K_5$.



Examples 7 and 8 are very important, for together they demonstrate how a "base graph"-style construction of LOE–decompositions can be carried out for OLE–decompositions, and vice versa. Additionally, Theorem 2 presents a construction method that cannot be duplicated for OLE graphs. Hence, in general, one has to prove existence of both decompositions separately.

6. Decompositions of λK_n

We are now in a position to prove the main results of the paper. We remind the reader that Theorem 1 gives the minimum number λ of copies of K_n under discussion in each case. Also, since LOE and OLE graphs have 4 vertices, we consider $n \geq 4$.

THEOREM 3. *The minimum number of copies of K_n can be decomposed into LOE graphs.*

PROOF. We note that in this proof, we use difference sets to achieve our decompositions of λK_n. In general, we exhibit the base graphs, which can be developed (modulo either n or $n-1$) to obtain the decomposition. We also note that special attention is needed with the base graphs containing the "dummy element" ∞; the non-∞ elements are developed, while ∞ is simply rewritten each time. We further note that the multiplicity of the edges is fixed by position, as per the LOE graph.

case 1: $n \equiv 1, 4 \pmod{12}$

If a $BIBD(n, 4, 1)$ exists, we replace each block $\{a, b, c, d\}$ in the design by the LOE graphs $\langle a, b, c, d\rangle$, $\langle b, c, a, d\rangle$ and $\langle c, a, b, d\rangle$. We note from [1] that a $BIBD(n, 4, 1)$ exists exactly when $n \equiv 1, 4 \pmod{12}$. Hence, in this case, an LOE–decomposition of $3K_n$ exists. ▲

case 2: $n \equiv 3, 6, 7, 10 \pmod{12}$

In all these cases, n is equivalent to either 0 or 1 modulo 3.

Subcase: $n \equiv 0 \pmod 3$

First note that LOE–decompositions of $4K_3$ do not exist since LOE graphs have 4 vertices. We consider two cases (t even and t odd):

If $n = 3t$ and $t = 2s$, then $n = 6s$. We consider the set V as $\mathbb{Z}_{6s-1} \cup \{\infty\}$. Then the differences we must achieve (modulo $6s-1$) are $1, 2, \ldots, 3s-1$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{6s(6s-1)}{3} = 2s(6s-1)$. Thus, we need $2s$ base graphs (modulo $6s-1$). For the first $2s-2$ base graphs, we use $\langle 1, 3x-1, 0, 3x\rangle$ and $\langle 3x, 0, 3x-1, 1\rangle$ for $x = 1, 2, \ldots, s-1$. For the last two base graphs, we use $\langle 3s-2, 0, 3s-1, \infty\rangle$ and $\langle \infty, 3s-1, 0, 3s-2\rangle$. Hence, in this subcase, an LOE–decomposition of $4K_n$ exists. ♦

If $n = 3t$ and $t = 2s+1$, then $n = 6s+3$. We consider the set V as $\mathbb{Z}_{6s+2} \cup \{\infty\}$. Then the differences we must achieve (modulo $6s+2$) are $1, 2, \ldots, 3s+1$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s+3)(6s+2)}{3} = (2s+1)(6s+2)$. Thus, we need $2s+1$ base graphs (modulo $6s+2$). For the first $2s-2$ base graphs, we use $\langle 1, 3x-1, 0, 3x\rangle$ and $\langle 3x, 0, 3x-1, 1\rangle$ for $x = 1, 2, \ldots, s-1$. For the last three base graphs, we use $\langle 1, 3s+1, 0, 3s\rangle$, $\langle 3s-2, 0, 3s-1, \infty\rangle$ and $\langle \infty, 3s-1, 0, 3s-2\rangle$. Hence, in this subcase, an LOE–decomposition of $4K_n$ exists. ▲

Subcase: $n \equiv 1 \pmod 3$

We again consider two cases (t even and t odd):

If $n = 3t + 1$ and $t = 2s$, then $n = 6s + 1$. We consider the set V as $\mathbb{Z}_{6s+1}$. Then the differences we must achieve (modulo $6s + 1$) are $1, 2, \ldots, 3s$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s+1)(6s)}{3} = 2s(6s + 1)$. Thus, we need $2s$ base graphs (modulo $6s + 1$). We use $\langle 3x - 2, 0, 3x - 1, 6x - 1\rangle$ and $\langle 6x - 1, 3x - 1, 0, 3x - 2\rangle$ for $x = 1, 2, \ldots, s$. Hence, in this subcase, an LOE–decomposition of $4K_n$ exists. ♦

If $n = 3t + 1$ and $t = 2s + 1$, then $n = 6s + 4$. We consider the set V as $\mathbb{Z}_{6s+4}$. Then the differences we must achieve (modulo $6s + 4$) are $1, 2, \ldots, 3s + 2$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s+4)(6s+3)}{3} = (2s + 1)(6s + 4)$. Thus, we need $2s + 1$ base graphs (modulo $6s + 4$). For the first $2s$ base graphs, we use $\langle 3x - 2, 0, 3x - 1, 6x - 1\rangle$ and $\langle 6x - 1, 3x - 1, 0, 3x - 2\rangle$ for $x = 1, 2, \ldots, s$. For the last base graph, we use $\langle 6s + 3, 3s + 2, 0, 3s + 1\rangle$. Hence, in this subcase, an LOE–decomposition of $4K_n$ exists. ▲

case 3: $n \equiv 0 \pmod{12}$

Let $n = 12t$. We consider the set V as $\mathbb{Z}_{12t-1} \cup \{\infty\}$. Then, the differences we must achieve (modulo $12t - 1$) are $1, 2, \ldots, 6t - 1$. The number of graphs required is $\frac{3n(n-1)}{12} = \frac{12t(12t-1)}{4} = 3t(12t - 1)$. Thus, we need $3t$ base graphs (modulo $12t - 1$). We use $\langle 0, 3t, 6t + 1, 9t\rangle$, $\langle 0, 3t + 1, 6t + 3, 9t + 1\rangle$, $\langle 0, 3t + 2, 6t + 5, 9t + 2\rangle, \ldots, \langle 0, 6t - 3, 12t - 5, 12t - 3\rangle$, $\langle 0, 6t - 2, 12t - 3, 12t - 2\rangle$ and $\langle 0, 6t - 1, 9t - 1, \infty\rangle$. Hence, in this case, an LOE–decomposition of $3K_n$ exists. ▲

case 4: $n \equiv 2 \pmod{12}$

Let $n = 12t + 2$. We consider the set V as $\mathbb{Z}_{12t+1} \cup \{\infty\}$. Then, the differences we must achieve (modulo $12t + 1$) are $1, 2, \ldots, 6t$. The number of graphs required is $\frac{6n(n-1)}{12} = \frac{(12t+2)(12t+1)}{2} = (6t + 1)(12t + 1)$. Thus, we need $6t + 1$ base graphs (modulo $12t + 1$). For the first $6t - 1$ base graphs, we use $\langle 0,1,3,6t+2\rangle$, $\langle 0,2,5,6t+3\rangle$, $\langle 0,3,7,6t+4\rangle,\ldots,\langle 0,6t-3,12t-5,12t-2\rangle$, $\langle 0,6t-2,12t-3,12t-1\rangle$ and $\langle 0,6t-1,6t,6t+1\rangle$. For the last two base graphs, we use $\langle 0,6t,12t,\infty\rangle$ twice. Hence, in this case, an LOE–decomposition of $6K_n$ exists. ▲

case 5: <u>$n \equiv 5 \pmod{12}$</u>

Let $n = 12t+5$. We consider the set V as $\mathbb{Z}_{12t+5}$. Then, the differences we must achieve (modulo $12t+5$) are $1,2,\ldots,6t+2$. The number of graphs required is $\frac{3n(n-1)}{12} = \frac{(12t+5)(12t+4)}{4} = (3t+1)(12t+5)$. Thus, we need $3t+1$ base graphs (modulo $12t+5$). We use $\langle 0,1,3,6t+5\rangle$, $\langle 0,2,5,6t+6\rangle$, $\langle 0,3,7,6t+7\rangle,\ldots,\langle 0,3t-1,6t-1,9t+3\rangle$, $\langle 0,3t,6t+1,9t+4\rangle$ and $\langle 0,3t+1,3t+2,6t+4\rangle$. Hence, in this case, an LOE–decomposition of $3K_n$ exists. ▲

case 6: <u>$n \equiv 8 \pmod{12}$</u>

Let $n = 12t+8$. We consider the set V as $\mathbb{Z}_{12t+7} \cup \{\infty\}$. Then, the differences we must achieve (modulo $12t+7$) are $1,2,\ldots,6t+3$. The number of graphs required is $\frac{3n(n-1)}{12} = \frac{(12t+8)(12t+7)}{4} = (3t+2)(12t+7)$. Thus, we need $3t+2$ base graphs (modulo $12t+7$). For the first $3t+1$ base graphs, we use $\langle 0,1,3,6t+5\rangle$, $\langle 0,2,5,6t+6\rangle$, $\langle 0,3,7,6t+7\rangle,\ldots,\langle 0,3t-1,6t-1,9t+3\rangle$, $\langle 0,3t,6t+1,9t+4\rangle$ and $\langle 0,3t+1,3t+2,6t+4\rangle$. For the last base graph, we use $\langle 0,6t+3,12t+6,\infty\rangle$. Hence, in this case, an LOE–decomposition of $3K_n$ exists. ▲

case 7: <u>$n \equiv 9 \pmod{12}$</u>

Let $n = 12t+9$. We consider the set V as $\mathbb{Z}_{12t+9}$. Then, the differences we must achieve (modulo $12t+9$) are $1,2,\ldots,6t+4$. The number of graphs required is $\frac{3n(n-1)}{12} = \frac{(12t+9)(12t+8)}{4} = (3t+2)(12t+9)$. Thus, we need $3t+2$ base graphs (modulo $12t+9$). We use $\langle 0,1,3,3t+6\rangle$, $\langle 0,2,5,3t+9\rangle$, $\langle 0,3,7,3t+12\rangle,\ldots,\langle 0,3t,6t+1,12t+3\rangle$, $\langle 0,3t+1,6t+3,12t+6\rangle$ and $\langle 0,3t+2,3t+3,9t+7\rangle$. Hence, in this case, an LOE–decomposition of $3K_n$ exists. ▲

case 8: $n \equiv 11 \pmod{12}$

Let $n = 12t + 11$. We consider the set V as $\mathbb{Z}_{12t+11}$. Then, the differences we must achieve (modulo $12t+11$) are $1, 2, \ldots, 6t + 5$. The total number of graphs required is $\frac{6n(n-1)}{12} = \frac{(12t+11)(12t+10)}{2} = (12t + 11)(6t + 5)$. Thus, we need $6t + 5$ base graphs (modulo $12t + 11$). We use $\langle 0, 1, 3, 6t + 8\rangle$, $\langle 0, 2, 5, 6t+9\rangle$, $\langle 0, 3, 7, 6t+10\rangle, \ldots, \langle 0, 6t+2, 12t+5, 12t+9\rangle$, $\langle 0, 6t + 3, 12t + 7, 12t + 10\rangle$, $\langle 0, 6t + 4, 12t + 9, 12t + 10\rangle$ and $\langle 0, 6t + 5, 6t + 6, 6t + 8\rangle$. Hence, in this case, an LOE–decomposition of $6K_n$ exists.

■

THEOREM 4. *The minimum number of copies of K_n can be decomposed into OLE graphs.*

PROOF. We first note that several cases of this proof (OLE–decompositions) are exactly the same as the proofs of the corresponding cases of the proof of Theorem 3 (LOE–decompositions), with "LOE" replaced by "OLE", and "$\langle a, b, c, d\rangle$" replaced by "$[a, b, c, d]$" (including the base graphs using "∞"); specifically, cases 1 and 3–8, corresponding to $n \equiv 0, 1, 2, 4, 5, 8, 9, 11 \pmod{12}$. This is true because of the way the index λ is achieved for LOE–decompositions. All we must yet address are the cases $n \equiv 3, 6, 7, 10 \pmod{12}$. In all these cases, n is equivalent to either 0 or 1 modulo 3.

Subcase: $n \equiv 0 \pmod 3$

First note that OLE–decompositions of $4K_3$ do not exist since OLE graphs have 4 vertices. We consider two cases (t even and t odd):

If $n = 3t$ and $t = 2s$, then $n = 6s$. We consider the set V as $\mathbb{Z}_{6s-1} \cup \{\infty\}$. Then the differences we must achieve (modulo $6s - 1$) are $1, 2, \ldots, 3s - 1$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{6s(6s-1)}{3} = 2s(6s - 1)$. Thus, we need $2s$ base graphs (modulo $6s - 1$). For the first $2s - 2$ base graphs, we use $[3x - 1, 0, 3x - 2, 6x - 3]$ and $[3x - 1, 0, 3x, 6x - 2]$ for $x = 1, 2, \ldots, s - 1$. For the last two base graphs, we use $[\infty, 3s - 2, 0, 3s - 1]$ and $[\infty, 3s - 1, 0, 3s - 2]$. Hence, in this subcase, an OLE–decomposition of $4K_n$ exists. ♦

If $n = 3t$ and $t = 2s + 1$, then $n = 6s + 3$. We consider the set V as $\mathbb{Z}_{6s+2} \cup \{\infty\}$. Then the differences we must achieve (modulo $6s + 2$) are $1, 2, \ldots, 3s + 1$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s+3)(6s+2)}{3} = (2s + 1)(6s + 2)$. Thus, we need $2s + 1$ base graphs (modulo $6s + 2$). For the first $2s - 2$ base graphs, we use $[3x - 1, 0, 3x - 2, 6x - 2]$ and $[3x - 1, 0, 3x, 6x - 2]$ for $x = 1, 2, \ldots, s - 1$. For the last three base graphs, we use $[3s + 1, 0, 3s, 6s]$, $[\infty, 3s - 2, 0, 3s - 1]$ and $[\infty, 3s - 1, 0, 3s - 2]$. Hence, in this subcase, an OLE–decomposition of $4K_n$ exists. ▲

Subcase: <u>$n \equiv 1 \pmod 3$</u>

We again consider two cases (t even and t odd):

If $n = 3t + 1$ and $t = 2s$, then $n = 6s + 1$. We consider the set V as $\mathbb{Z}_{6s+1}$. Then the differences we must achieve (modulo $6s + 1$) are $1, 2, \ldots, 3s$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s+1)(6s)}{3} = 2s(6s + 1)$. Thus, we need $2s$ base graphs (modulo $6s + 1$). We use $[3x - 1, 0, 3x, 6x - 2]$ and $[3x - 1, 0, 3x - 2, 6x - 2]$ for $x = 1, 2, \ldots, s$. Hence, in this subcase, an OLE–decomposition of $4K_n$ exists. ♦

If $n = 3t + 1$ and $t = 2s + 1$, then $n = 6s + 4$. We consider the set V as $\mathbb{Z}_{6s+4}$. Then the differences we must achieve (modulo $6s + 4$) are $1, 2, \ldots, 3s + 2$. The number of graphs required is $\frac{4n(n-1)}{12} = \frac{(6s+4)(6s+3)}{3} = (2s + 1)(6s + 4)$. Thus, we need $2s + 1$ base graphs (modulo $6s + 4$). For the first $2s$ base graphs, we use $[3x - 1, 0, 3x, 6x - 2]$ and $[3x - 1, 0, 3x - 2, 6x - 2]$ for $x = 1, 2, \ldots, s$. For the last base graph, we use $[3s + 2, 0, 3s + 1, 6s + 2]$. Hence, in this subcase, an OLE–decomposition of $4K_n$ exists.

■

7. Conclusion

There are other types of (connected) multigraphs with six edges on four vertices having edge frequencies only 1, 2 and 3. We continue to search for constructions of the so-called *LEO-decompositions* of λK_n, corresponding to the graph  and so-called *ELO-decompositions* of λK_n, corresponding to the graph 

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