On the Ramsey Numbers of Certain Graphs of Order Five versus All Connected Graphs of Order Six

ROLAND LORTZ and INGRID MENGERSEN

Technische Universität Braunschweig
Diskrete Mathematik
38092 Braunschweig, Germany
r.lortz@tu-bs.de

Ostfalia Hochschule für angewandte Wissenschaften
Fakultät Informatik
38302 Wolfenbüttel, Germany
i.mengersen@ostfalia.de

Abstract

The Ramsey numbers $r(F, G)$ are investigated, where F is a non-tree graph of order 5 and minimum degree 1 and G is a connected graph of order 6. For all pairs (F, G) where $F \neq K_5 - K_{1,3}$ the exact value of $r(F, G)$ is determined. In order to settle $F = K_5 - K_{1,3}$, we prove $r(K_5 - K_{1,3}, G) = r(K_4, G)$.

MATHEMATICAL SUBJECT CLASSIFICATION: 05C55
KEYWORDS: Ramsey number, small graph, minimum degree

1 Introduction

Ramsey numbers $r(F, G)$ where F and G both are rather small graphs were first studied by Chvátal and Harary [5, 6] in 1972. Clancy [7] extended their results to graphs F with at most four vertices and graphs G with exactly five vertices. In 1989, Hendry [9] gave a table of all but seven Ramsey numbers where F and G both have exactly five vertices. Meanwhile five of the missing numbers have been determined in [1, 3, 17, 20, 22], also cf. [19]. Thus, nearly all Ramsey numbers $r(F, G)$ are known if $p(F), p(G) \leq 5$.

Moreover, there are several results if G is a graph on exactly six vertices. First, $r(P_3, G)$ and $r(2K_2, G)$ may be derived from general theorems [6]. The next attempt was on $F = K_3$. Here, Faudree, Rousseau, and Schelp [8] obtained all Ramsey numbers $r(K_3, G)$ for connected graphs G of order six. Hoeth and Mengersen [10] investigated $r(K_{1,3}+e, G)$ and $r(K_4-e, G)$, only failing with $r(K_4-e, K_6)$. Later McNamara [18] completed their results giving $r(K_4-e, K_6) = 21$. All Ramsey numbers $r(C_4, G)$ were calculated by Jayawardene and Rousseau [13, 21], additional erratum in [2]. The current knowledge on Ramsey numbers $r(F, G)$ where $p(F) \leq 4$ and $p(G) = 6$ was completed by Lortz and Mengersen [16] studying $r(P_4, G)$ and $r(K_{1,3}, G)$. Only for the hardest case $F = K_4$ about half of the Ramsey numbers are still unsettled (cf. Section 3.4).

Hence, it is a considerable further step to approach $r(F, G)$ if $p(F) = 5$ and $p(G) = 6$. Here, the first results were due to Jayawardene and Rousseau [14, 15] who gave $r(C_5, G)$ for all graphs G of order six. Furthermore, Hua Gu, Hongxue Song, and Xiangyang Liu [11] investigated $r(K_{1,4}, G)$ for some special graphs on six vertices. Finally, Lortz and Mengersen [16] obtained $r(P_5, G)$, $r(S_{1,3}, G)$, and $r(K_{1,4}, G)$ for all connected graphs G of order six.

In the present paper we study $r(F, G)$ where F is a non-tree graph of order five additionally satisfying $\delta(F) = 1$. There are eight such graphs as given in Figure 1.

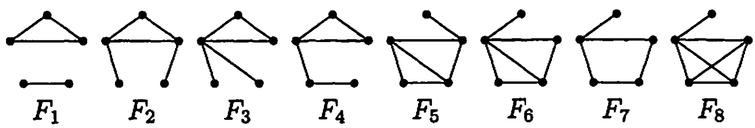


Figure 1: The graphs $F_1, F_2, \ldots, F_8$

Throughout this paper some specialized notation will be used. A 2-coloring of a graph always means a 2-coloring of its edges with colors red and green. A $(G_1, G_2)_p$-coloring is a 2-coloring of the complete graph K_p containing neither a red copy of G_1 nor a green copy of G_2. For the red subgraph of a 2-coloring the degree of a vertex v is denoted by $d_r(v)$ and the set of v's red neighbors is indicated by $N_r(v)$. Moreover, $d_r(v_1, \ldots, v_n) = |N_r(v_1) \cap \ldots \cap N_r(v_n)|$ for any vertices $v_1, \ldots, v_n$. Considering two disjoint subsets U_1 and U_2 of the vertex set V of a 2-colored K_p, $q_r(U_1, U_2)$ means the number of red edges between U_1 and U_2. If U_1 only consists of a single vertex v, then we write $q_r(v, U_2)$ instead. Furthermore, $[U]_r$ denotes the red-edge subgraph induced by the vertex set U. By $U_1 \times U_2$ we refer to the set of all edges u_1u_2 where $u_1 \in U_1$ and $u_2 \in U_2$.

2 Results

In Table 1 we give a survey of already known results on Ramsey numbers $r(F, G)$ where $F \in \{K_3, K_{1,3}+e, K_4-e, C_4\}$ collected from [8, 10, 13, 21] as well as a listing of our new results for $F \in \{F_1, F_2, \ldots, F_8\}$. The respective proofs are presented in Section 3. In several cases we achieve $r(F, G) = r(F-v, G)$ where $v \in V(F)$ and $d_F(v) = \delta(F) = 1$. In other cases both Ramsey numbers differ by one, two, or three.

For the notation, the first column contains the number i of the connected six vertex graph G_i. A drawing of G_i is given in the fifth column. Columns 2, 3, and 4 provide G_i's common name (if any), its number of edges, and its clique number. Most helpful within the subsequent proofs is G_i's complementary graph (ignoring isolates) presented in column 6. All remaining columns contain Ramsey numbers $r(F, G_i)$ with the graph F given in the column's headline. As [10] and Theorem 1 yield $r(K_3, G_i) = r(K_{1,3}+e, G_i) = r(F_1, G_i)$ and our research proves $r(F_2, G_i) = r(F_3, G_i) = r(F_4, G_i)$, $r(F_5, G_i) = r(F_6, G_i)$, and $r(K_4, G_i) = r(F_8, G_i)$, we may use a single column in each of these four cases. In the column concerning K_4 and F_8 there are some gaps where the respective Ramsey number is currently unknown.

Finally, $S_{1,n}$ denotes a tailed star obtained from a star $K_{1,n}$ where an additional vertex is joined to one of the star's outer vertices. Therefore $p(S_{1,n}) = n+2$.

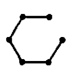
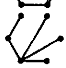
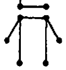
No.	G_i	q	cl	$D(G_i)$	$\overline{G_i}$	K_3 $K_{1,3}+e$ F_1	F_2 F_3 F_4	K_4-e	F_5 F_6	C_4	F_7	K_4 F_8
1	P_6	5	2		G_{87}	11	11	11	11	7	7	16
2		5	2		G_{88}	11	11	11	11	7	7	16
3	$S_{1,4}$	5	2		G_{89}	11	11	11	11	7	7	16
4		5	2		G_{90}	11	11	11	11	7	7	16
5		5	2		G_{91}	11	11	11	11	7	7	16
6	$K_{1,5}$	5	2		K_5	11	11	11	11	8	9	16

Table 1: $r(F, G_i)$ for $i = 1, \ldots, 6$

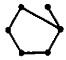
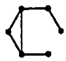
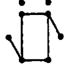
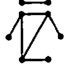
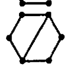
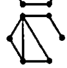
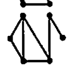
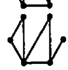
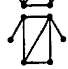
No.	G_i	q	cl	$D(G_i)$	$\overline{G_i}$	K_3 $K_{1,3}+e$ F_1	F_2 F_3 F_4	K_4-e	F_5 F_6	C_4	F_7	K_4 F_8
7	C_6	6	2		G_{67}	11	11	11	11	7	7	16
8		6	2		G_{68}	11	11	11	11	7	9	16
9		6	2		G_{69}	11	11	11	11	7	7	16
10		6	3		G_{70}	11	11	11	11	7	9	16
11		6	2		G_{71}	11	11	11	11	7	7	16
12		6	2		G_{72}	11	11	11	11	7	7	16
13		6	3		G_{73}	11	11	11	11	7	9	16
14		6	3		G_{74}	11	11	11	11	7	9	16
15	$K_{1,5}+e$	6	3		K_5-e	11	11	11	11	8	9	16
16		6	2		G_{75}	11	11	11	11	7	7	16
17		6	3		G_{77}	11	11	11	11	7	9	16
18		6	3		G_{79}	11	11	11	11	7	9	16
19		6	3		G_{80}	11	11	11	11	7	9	16
20		7	2		G_{43}	11	11	11	11	7	7	
21		7	3		G_{44}	11	11	11	11	7	9	
22		7	2		G_{45}	11	11	11	11	7	9	
23		7	3		G_{46}	11	11	11	11	7	9	16
24		7	3		G_{47}	11	11	11	11	7	9	16
25		7	3		G_{48}	11	11	11	11	7	9	16
26		7	3		G_{49}	11	11	11	11	7	9	16

Table 1: $r(F, G_i)$ for $i = 7, \ldots, 26$

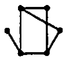
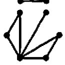
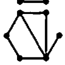
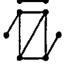
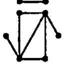
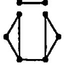
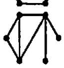
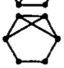
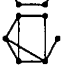
No.	G_i	q	cl	$D(G_i)$	$\overline{G_i}$	K_3 $K_{1,3}+e$ F_1	F_2 F_3 F_4	K_4-e	F_5 F_6	C_4	F_7	K_4 F_8
27		7	3		G_{50}	11	11	11	11	7	9	16
28		7	3		G_{51}	11	11	11	11	7	9	
29		7	2		G_{52}	11	11	11	11	8	8	16
30		7	3		G_{54}	11	11	11	11	7	9	16
31		7	2		G_{55}	11	11	11	11	8	8	16
32		7	3		$K_5 - P_3$	11	11	11	11	8	9	16
33		7	3		$K_5 - 2K_2$	11	11	11	11	8	9	16
34		7	3		G_{56}	11	11	11	11	7	9	16
35		7	3		G_{57}	11	11	11	11	7	9	16
36		7	3		G_{58}	11	11	11	11	7	9	16
37		7	3		G_{59}	11	11	11	11	8	10	
38		7	3		G_{60}	11	11	11	11	7	9	16
39		8	3		$K_5-(P_3 \cup K_2)$	11	11	11	11	8	9	
40		8	3		$K_5 - P_4$	11	11	11	11	8	9	16
41		8	3		$K_5 - K_{1,3}$	11	11	11	11	9	9	16
42		8	4		B_3	11	13	11	13	10	13	18
43		8	3		G_{20}	11	11	11	11	8	10	
44		8	3		G_{21}	11	11	11	11	7	9	
45		8	3		G_{22}	11	11	11	11	8	10	
46		8	3		G_{23}	11	11	11	11	8	9	16

Table 1: $r(F, G_i)$ for $i = 27, \ldots, 46$

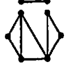
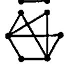
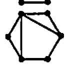
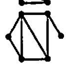
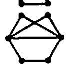
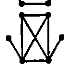
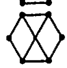
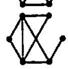
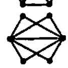
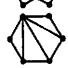
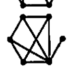
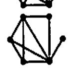
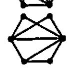
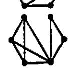
No.	G_i	q	cl	$D(G_i)$	$\overline{G_i}$	K_3 $K_{1,3}+e$ F_1	F_2 F_3 F_4	K_4-e	F_5 F_6	C_4	F_7	K_4 F_8
47		8	3		G_{24}	11	11	11	11	8	9	
48		8	3		G_{25}	11	11	11	11	8	9	16
49		8	3		G_{26}	11	11	11	11	7	9	16
50		8	3		G_{27}	11	11	11	11	7	9	
51		8	3		G_{28}	11	11	11	11	8	9	16
52		8	3		G_{29}	11	11	11	11	8	10	
53	$K_{2,4}$	8	2		$K_4 \cup K_2$	11	11	13	13	9	9	
54		8	3		G_{30}	11	11	11	11	7	9	
55		8	4		G_{31}	11	13	11	13	10	13	18
56		8	3		G_{34}	11	11	11	11	7	9	16
57		8	3		G_{35}	11	11	11	11	8	9	
58		8	4		G_{36}	11	13	11	13	10	13	18
59	$K_{3,3}-e$	8	2		G_{37}	11	11	11	11	8	8	
60		8	3		G_{38}	11	11	11	11	9	9	16
61	B_4	9	3		K_4	11	11	13	13	11	11	
62	K_1+P_5	9	3		C_5+e	11	11	11	11	8	9	
63		9	3		K_1+2K_2	11	11	11	11	9	9	17
64		9	4		F_6	11	13	11	13	10	13	18
65	$K_1+S_{1,3}$	9	3		F_5	11	11	11	11	9	9	
66		9	4		$K_{2,3}$	11	13	11	13	10	13	

Table 1: $r(F, G_i)$ for $i = 47, \ldots, 66$

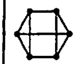
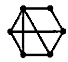
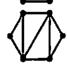
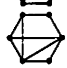
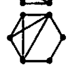
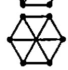
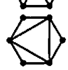
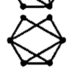
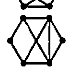
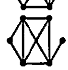
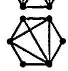
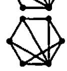
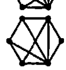
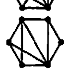
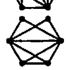
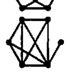
No.	G_i	q	cl	$D(G_i)$	$\overline{G_i}$	K_3, $K_{1,3}+e$, F_1	F_2, F_3, F_4	K_4-e	F_5, F_6	C_4	F_7	K_4, F_8
67		9	3		C_6	11	11	11	11	8	10	
68		9	3		G_8	11	11	11	11	8	10	
69		9	3		G_9	11	11	11	11	8	10	
70		9	3		G_{10}	11	11	11	11	8	9	
71		9	3		G_{11}	11	11	11	11	8	10	
72		9	4		G_{12}	11	13	11	13	10	13	18
73		9	3		G_{13}	11	11	11	11	8	9	
74		9	3		G_{14}	11	11	11	11	9	9	17
75		9	4		G_{16}	11	13	11	13	10	13	
76	$K_{3,3}$	9	2		$2K_3$	12	12	16	16	11	11	
77		9	3		G_{17}	11	11	11	11	8	10	
78		9	3		$(K_4-e)\cup K_2$	11	11	13	13	9	9	
79		9	3		G_{18}	11	11	11	11	9	9	
80		9	4		G_{19}	11	13	11	13	10	13	18
81		10	4		F_2	11	13	11	13	10	13	
82	W_5	10	3		C_5	11	13	11	13	10	13	
83		10	3		F_4	11	11	11	11	9	9	
84		10	4		F_7	11	13	11	13	10	13	
85	B_4+e	10	4		K_4-e	11	13	13	13	11	13	
86		10	4		$K_{1,4}+e$	11	13	13	13	11	13	19

Table 1: $r(F, G_i)$ for $i = 67, \ldots, 86$

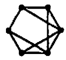
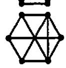
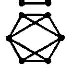
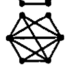
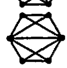
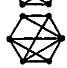
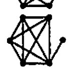
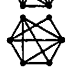
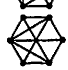
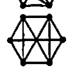
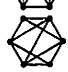
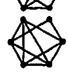
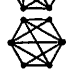
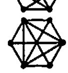
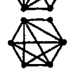
No.	G_i	q	cl	$D(G_i)$	$\overline{G_i}$	K_3 $K_{1,3}+e$ F_1	F_2 F_3 F_4	K_4-e	F_5 F_6	C_4	F_7	K_4 F_8
87		10	3		P_6	11	11	11	11	8	10	
88		10	4		G_2	11	13	11	13	10	13	
89		10	4		$S_{1,4}$	11	13	13	13	11	13	19
90		10	3		G_4	11	11	11	11	9	10	
91		10	4		G_5	11	13	11	13	10	13	
92	$K_{3,3}+e$	10	3		$K_3 \cup P_3$	12	12	16	16	11	11	
93		10	3		$C_4 \cup K_2$	11	11	13	13	9	10	
94		10	3		$(K_{1,3}+e)\cup K_2$	11	11	13	13	9	9	
95		11	4		$K_{1,3}+e$	11	13	13	13	11	13	
96	K_2+2K_2	11	4		C_4	11	13	16	16	11	13	
97		11	4		P_5	11	13	11	13	10	13	
98		11	5		$K_{1,4}$	14	17	16	17	14	17	25
99		11	4		$S_{1,3}$	11	13	13	13	11	13	
100	$P_3+\overline{K_3}$	11	3		$K_3 \cup K_2$	12	12	16	16	11	11	
101		11	4		$2P_3$	12	13	16	16	11	13	
102		11	3		$P_4 \cup K_2$	11	11	13	13	9	10	
103		11	4		$K_{1,3} \cup K_2$	12	13	13	13	11	13	
104		12	4		P_4	11	13	16	16	11	13	
105	$K_3+\overline{K_3}$	12	4		K_3	14	14	16	16	13	13	
106		12	5		$K_{1,3}$	14	17	16	17	14	17	25

Table 1: $r(F, G_i)$ for $i = 87, \ldots, 106$

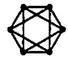
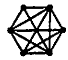
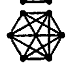
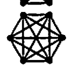
No.	G_i	q	cl	$D(G_i)$	$\overline{G_i}$	K_3 $K_{1,3}+e$ F_1	F_2 F_3 F_4	K_4-e	F_5 F_6	C_4	F_7	K_4 F_8
107		12	4		$P_3 \cup K_2$	12	13	16	16	11	13	
108		12	3		$3K_2$	13	13	14	14	11	11	
109		13	5		P_3	14	17	16	17	14	17	≤ 27
110		13	4		$2K_2$	13	13	16	16	13	13	
111	$K_6 - e$	14	5		K_2	17	17	17	17	16	17	≤ 36
112	K_6	15	6		—	18	21	21	21	18	21	≤ 41

Table 1: The Ramsey numbers $r(F, G_i)$ where F is a non-tree graph of order 5 and minimum degree 1 and G_i is a connected graph of order 6

3 Proofs

The subsequent proofs are sorted by the four vertex graphs $F_i - v$, where $d_{F_i}(v) = 1$, the extended graphs F_i may be derived from. Actually we have the four vertex subgraphs $K_{1,3} + e$ for F_2, F_3, F_4, $K_4 - e$ for F_5, F_6, C_4 for F_7, and K_4 for F_8. Especially, each subsection's results are summarized in a preceding theorem, while the proofs are split in a number of appropriate lemmas. Mind that already known Ramsey numbers used within the proofs are either taken from the known results reminded in Table 1 or from the papers of Chvátal and Harary [6], Clancy [7], Hendry [9], or McKay and Radziszowski [17]. To improve readability we omit the respective references throughout this section's proofs.

In order to settle the only disconnected case $F_1 = K_3 \cup K_2$ we have a short look at the following straightforward result.

Theorem 1 *Let G be any connected graph of order* 6. *Then*

$$r(F_1,G) = \begin{cases} 18 & \textit{if } G = G_{112}, \\ 17 & \textit{if } G = G_{111}, \\ 14 & \textit{if } G = G_{105} \textit{ or } G_{98} \subset G \subset G_{109}, \\ 13 & \textit{if } G = G_{108} \textit{ or } G = G_{110}, \\ 12 & \textit{if } G = G_{103} \textit{ or } G_{76} \subset G \subset G_{107}, \\ 11 & \textit{otherwise, i.e. if } G \subset G_{104}. \end{cases}$$

Proof: Considering the known results on $r(K_3,G)$, it suffices to prove $r(F_1,G) = r(K_3,G)$. Clearly, $K_3 \subset F_1$ implies $r(F_1,G) \geq r(K_3,G)$. In order to obtain $r(F_1,G) \leq r(K_3,G) = r$, we assume that there is a $(F_1,G)_r$-coloring χ by definition producing a red subgraph K_3. Avoiding a red subgraph F_1, the $r-3$ remaining vertices induce a green subgraph K_{r-3}. Since $r-3 \geq 8$, we achieve $K_{r-3} \supset G$ for any connected graph G of order 6, contradicting the initial assumption and completing the proof.

3.1 Extending $K_{1,3}+e$

Theorem 2 *Let $i \in \{2,3,4\}$, and let G be any connected graph of order* 6. *Then*

$$r(F_i,G) = \begin{cases} 21 & \textit{if } G = G_{112}, \\ 17 & \textit{if } G_{98} \subset G \subset G_{111}, \\ 14 & \textit{if } G = G_{105}, \\ 13 & \textit{if } G = G_{82} \textit{ or } G = G_{108} \textit{ or } G \subset G_{110} \textit{ where } cl(G) = 4, \\ 12 & \textit{if } G_{76} \subset G \subset G_{100}, \\ 11 & \textit{otherwise, i.e. if } G = G_{61} \textit{ or } G \subset G_{83} \textit{ or } G \subset G_{102}. \end{cases}$$

Lemma 3.1.1

$$r(F_2,G_{112}) = 21.$$

Proof: As the Turan-type $(F_2,G_{112})_{20}$-coloring given by a red subgraph $5K_4$ and a green subgraph $K_{4,4,4,4,4}$ yields $r(F_2,G_{112}) \geq 21$, we continue assuming that there is a $(F_2,G_{112})_{21}$-coloring χ. Then, $r(K_{1,3}+e,G_{112}) =$

18 demands a red subgraph $K_{1,3}+e$ to exist in χ. Avoiding a red subgraph F_2, $K_{1,3}+e$'s vertices of degree 2 must not have any red neighbors among the 17 remaining vertices. Moreover, these vertices produce a green subgraph K_5 since $r(F_2, K_5) = 17$. Thus, a green subgraph $\overline{K_2} + K_5 \supset G_{112}$ is to occur in χ, and our proof is done. ∎

Lemma 3.1.2

$$r(F_2, G) = \begin{cases} 17 & \text{if } G = G_{111}, \\ 14 & \text{if } G = G_{105}, \\ 13 & \text{if } G = G_{108} \text{ or } G = G_{110}, \\ 12 & \text{if } G_{76} \subset G \subset G_{100}, \\ 11 & \text{if } G \subset G_{83} \text{ or } G \subset G_{102}. \end{cases}$$

Proof: To verify the lemma's results we prove $r(F_2, G) = r(K_{1,3}+e, G) = r$. Obviously, $K_{1,3}+e \subset F_2$ yields $r(F_2, G) \geq r$. Hence, we may assume that there is a $(F_2, G)_r$-coloring χ, by definition containing a red subgraph $K_{1,3}+e$. Avoiding a red subgraph F_2, $K_{1,3}+e$'s vertices of degree 2 and the $r-4$ remaining vertices induce a green subgraph $K_{2,r-4}$. For $r = 17$, $r-4 = 13 = r(F_2, K_5-e)$ forces a green subgraph $K_2+(K_5-e) \supset G_{111}$ to be found in χ. Now, consider $r = 14$. Due to $r-4 > 9 = r(F_2, K_5-K_3)$, a green subgraph $K_2+(K_5-K_3) \supset G_{105}$ may not be avoided in this case. If $r = 13$, then a green subgraph $K_2+(K_4-e) = G_{110} \supset G_{108}$ is to occur in χ by $r-4 = 9 = r(F_2, K_4-e)$. Next, let $r = 12$. Here, $r-4 > 7 = r(F_2, K_{1,3})$ implies the existence of a green subgraph $K_2 + K_{1,3} = G_{100} \supset G_{76}$. We conclude our case analysis with $r = 11$. Because of $r-4 = 7 = r(F_2, K_{1,3}) = r(F_2, C_4)$, χ produces green subgraphs $K_2 + K_{1,3} = K_6 - (K_3 \cup K_2)$ and $K_2+C_4 = K_6 - 3K_2$, and as $r = r(K_{1,3}+e, G) = 11$ requires $G \subset K_6 - P_4$, for $r(F_2, G) = 11$ we have to demand $G \subset G_{83}$ or $G \subset G_{102}$. Thus, the proof is complete.

Lemma 3.1.3 *Let G be a connected graph of order 6 satisfying $cl(G) = 5$ and $G \neq G_{111}$. Then*

$$r(F_2, G) = 17.$$

Proof: The lemma's assertion is a direct consequence of $r(F_2, K_5) = r(F_2, G_{111}) = 17$ (for the latter result cf. Lemma 3.1.2) and $K_5 \subset G \subset G_{111}$.

Lemma 3.1.4 *Let G be a connected graph of order 6 satisfying $cl(G) = 4$ and $G \subsetneq G_{110}$. Then*

$$r(F_2, G) = 13.$$

Proof: From $r(F_2, K_4) = r(F_2, G_{110}) = 13$ (for the latter result cf. Lemma 3.1.2) and $K_4 \subset G \subset G_{110}$ we derive all the Ramsey numbers specified above. ■

Lemma 3.1.5

$$r(F_2, G_{82}) = 13.$$

Proof: Regarding $r(F_2, G_{110}) = 13$ (cf. Lemma 3.1.2) and $G_{82} \subset G_{110}$, we obtain $r(F_2, G_{82}) \leq 13$. The corresponding lower bound may be established by a $(F_2, G_{82})_{12}$-coloring with red subgraph $3K_4$ and green subgraph $K_{4,4,4}$. ■

Lemma 3.1.6

$$r(F_2, G_{61}) = 11.$$

Proof: For $r(F_2, G_{61}) \geq 11$ we consider $r(K_3, G_{61}) = 11$ and $K_3 \subset F_2$. Hence, we assume that there is a $(F_2, G_{61})_{11}$-coloring χ. Since $r(K_{1,3} + e, G_{61}) = 11$, χ contains a red subgraph $K_{1,3}+e$ with edges $v_1v_2, v_2v_3, v_2v_4, v_3v_4$. Avoiding a red subgraph F_2, v_3, v_4, and the seven remaining vertices $u_1, \ldots, u_7$ are to induce a green subgraph $K_{2,7}$. Now, have a look at v_1. If v_1 produces at least four red neighbors in $U = \{u_1, \ldots, u_7\}$, then a green subgraph K_4 is forced in $[U]$, yielding a green subgraph $\overline{K_2}+K_4 \supset G_{61}$ in χ. Therefore, v_1 must have at least four green neighbors in U. So, all possible spines of a green subgraph G_{61}, i.e. edges v_1v_3 and v_1v_4, have to be colored red. Thus, we find $[\{v_1, v_2, v_3, v_4\}]_r = K_4$, demanding v_iu_j to be colored green where $i \in \{1, \ldots, 4\}$ and $j \in \{1, \ldots, 7\}$. As any single green edge in $[U]$ would become the spine of a green subgraph G_{61}, $[U]_r = K_7 \supset F_2$, and the proof is done. ■

Lemma 3.1.7

$$r(F_3, G_{112}) = 21.$$

Proof: The lower bound $r(F_3, G_{112}) \geq 21$ may be derived from a Turan-type $(F_3, G_{112})_{20}$-coloring given by a red subgraph $5K_4$ and a green subgraph $K_{4,4,4,4,4}$. For the proof of the corresponding upper bound we assume that there is a $(F_3, G_{112})_{21}$-coloring χ. Because of $r(K_{1,3}+e, G_{112}) = 18$, χ produces a red subgraph $K_{1,3}+e$. Avoiding a red subgraph F_3, $K_{1,3}+e$'s vertex of degree 3 must not have any red neighbors among the 17 remaining vertices. Since $r(F_3, K_5) = 17$, these vertices yield a green subgraph K_5. So, χ contains a green subgraph $K_1 + K_5 = G_{112}$, and this argument completes the proof. ■

Lemma 3.1.8

$$r(F_3, G) = \begin{cases} 17 & \textit{if } G = G_{111}, \\ 14 & \textit{if } G = G_{105}, \\ 13 & \textit{if } G = G_{108}, \\ 12 & \textit{if } G_{76} \subset G \subset G_{100}, \\ 11 & \textit{if } G \subset G_{83}. \end{cases}$$

Proof: Regarding the known results on $r(K_3, G)$, due to $K_3 \subset F_3$ we only have to prove $r(F_3, G) \leq r(K_3, G) = r$. Thus, we assume that there is a $(F_3, G)_r$-coloring χ, definitely containing a red subgraph K_3. Avoiding a red subgraph F_3, any of K_3's vertices must have at most one red neighbor among the $r - 3$ remaining vertices. Hence, for vertices v_1, v_2, v_3 of a red subgraph K_3 we obtain $d_g(v_i) \geq r-4$, $d_g(v_i, v_j) \geq r-5$, and $d_g(v_1, v_2, v_3) \geq r - 6$.

First, consider $r = 17$. Here, $d_g(v_i) \geq r - 4 = 13 = r(F_3, K_5 - e)$ forces a green subgraph $K_1 + (K_5 - e) = G_{111}$ to exist in χ. Now, let $r = 14$. Then, a green subgraph $K_1 + (K_5 - K_3) = G_{105}$ may not be avoided in χ since $d_g(v_i) \geq r - 4 > 9 = r(F_3, K_5 - K_3)$. For $r = 13$, due to $d_g(v_i, v_j) \geq r - 5 > 7 = r(F_3, C_4)$ a green subgraph $\overline{K_2} + C_4 = G_{108}$ is to be found in χ. If $r = 12$, then we meet a green subgraph $\overline{K_2} + K_{1,3} = G_{100}$, and any of its subgraphs, in χ because $d_g(v_i, v_j) \geq r - 5 = 7 = r(F_3, K_{1,3})$. Finally, we discuss $r = 11$. As $d_g(v_1, v_2, v_3) \geq r - 6 = 5 = r(F_3, P_3)$, a green subgraph $\overline{K_3} + P_3 = K_6 - (K_3 \cup K_2)$ is to occur in χ. However, $r = r(K_3, G) = 11$ holds for graphs G with $G \subset K_6 - P_4$ only, and we must constrain our argument to graphs G with $G \subset G_{83}$ for this reason, completing the proof for $r = 17$ through $r = 11$.

Lemma 3.1.9 *Let G be a connected graph of order* 6 *satisfying $cl(G) = 5$ and $G \neq G_{111}$. Then*

$$r(F_3, G) = 17.$$

Proof: The lemma's assertion immediately follows from $r(F_3, K_5) = r(F_3, G_{111}) = 17$ (for the latter result cf. Lemma 3.1.8) and $K_5 \subset G \subset G_{111}$.

Lemma 3.1.10 *Let G be a connected graph of order* 6 *satisfying $cl(G) =$* 4 *and $G \subset G_{110}$. Then*

$$r(F_3, G) = 13.$$

Proof: Since $K_4 \subset G$, $r(F_3, G) \geq 13$ is settled by $r(F_3, K_4) = 13$. Thus, we are left with $r(F_3, G_{110}) \leq 13$, assuming that there is a $(F_3, G_{110})_{13}$-coloring χ. Regarding $r(F_2, G_{110}) = 13$ (cf. Lemma 3.1.2), χ contains a red subgraph F_2 with edges $v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_4v_5$. Avoiding a red subgraph F_3, v_2, v_4, and the eight remaining vertices $u_1, \ldots, u_8$ yield a green subgraph $K_{2,8}$. Then, $U = \{u_1, \ldots, u_8\}$ must fulfill

$$\text{(P1)} \quad K_{1,4} \not\subset [U]_r$$

because otherwise a green subgraph $\overline{K_2} + K_4 \supset G_{110}$ would occur in χ. Moreover, a green subgraph K_3 in $[U]$ would force each of the five remaining vertices of U to have at least two red neighbors among K_3's vertices. Hence, the vertices of a green subgraph K_3 incide with at least ten red edges, producing a red subgraph $K_{1,4}$ in $[U]$. As this argument contradicts (P1), property

$$\text{(P2)} \quad K_3 \not\subset [U]_g$$

has to hold, too.

Considering (P1), u_4 must have at least four green neighbors among U's vertices. Let $u_5, \ldots, u_8$ be such neighbors. Now, (P2) demands $[\{u_5, \ldots, u_8\}]_r = K_4$, and (P1) implies that u_1u_i, u_2u_i, u_3u_i are colored green where $i \in \{5, \ldots, 8\}$. Next, have a look at u_5. Since u_iu_5 is colored green where $i \in \{1, \ldots, 4\}$, (P2) yields $[\{u_1, \ldots, u_4\}]_r = K_4$, and we obtain $[U]_r = 2K_4$ and $[U]_g = K_{4,4}$. Avoiding a red subgraph F_3, v_1v_4 is to be colored green and v_1 must not have any red neighbors among U's vertices. Thus, the green subgraph $K_2 + C_4 = G_{110}$ in $[\{v_1, v_4, u_3, u_4, u_5, u_6\}]$ contradicts our initial assumption, and the proof is done. ∎

Lemma 3.1.11

$$r(F_3, G_{82}) = 13.$$

Proof: From $r(F_3, G_{110}) = 13$ (cf. Lemma 3.1.10) and $G_{82} \subset G_{110}$ we achieve $r(F_3, G_{82}) \leq 13$. On the other hand, a red subgraph $3K_4$ and a green subgraph $K_{4,4,4}$ determine a $(F_3, G_{82})_{12}$-coloring, and the proof is complete. ∎

Lemma 3.1.12 *Let G be a connected graph of order 6 satisfying $G \subset G_{102}$. Then*

$$r(F_3, G) = 11.$$

Proof: From the known results on $r(K_3, G)$ and by $K_3 \subset F_3$ we derive $r(F_3, G) \geq r(K_3, G) = 11$ for any graph G with $G \subset G_{102}$. Thus, assume that there is a $(F_3, G_{102})_{11}$-coloring χ. Due to $r(F_2, G_{102}) = 11$

(cf. Lemma 3.1.2) χ contains a red subgraph F_2 that may be given by the edges $v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_4v_5$. Then, v_2, v_4, and the six remaining vertices $u_1, \ldots, u_6$ have to induce a green subgraph $K_{2,6}$ since otherwise a red subgraph F_3 would be obtained in χ. Additionally, avoiding a green subgraph $G_{102} = \overline{K_2} + P_4$ demands properties

$$\begin{array}{ll} \text{(P1)} & P_4 \not\subset [\{u_1, \ldots, u_6\}]_g, \\ \text{(P2)} & K_{1,4} \not\subset [\{u_1, \ldots, u_6\}]_r. \end{array}$$

Furthermore, $r(F_3, 2K_2) = 6$ yields two independent green edges, say u_1u_2 and u_3u_4, in $[\{u_1, \ldots, u_6\}]$ and (P1) forces $[\{u_1, \ldots, u_4\}]_r = C_4$. Now, (P2) requires u_4u_5 to be colored green, and as a direct consequence of (P1) u_1u_5, u_2u_5, u_5u_6 must be colored red. By (P2) u_1u_6 and u_2u_6 are green, from (P1) we derive that u_3u_6 and u_4u_6 have to be red, and (P2) implies u_3u_5 to be green, resulting in $[\{u_1, \ldots, u_6\}]_g = 2K_3$ and $[\{u_1, \ldots, u_6\}]_r = K_{3,3}$. Avoiding a red subgraph F_3, v_1 must not have red neighbors in both $\{u_1, u_2, u_6\}$ and $\{u_3, u_4, u_5\}$. Therefore, v_1u_3, v_1u_4, v_1u_5 may be assumed green. Moreover, v_1v_4 has to be green, too, and we obtain a green subgraph $(K_2 \cup K_1) + K_3 \supset G_{102}$ in $[\{v_1, v_2, v_4, u_3, u_4, u_5\}]$. So, the proof is done.

Lemma 3.1.13

$$r(F_3, G_{61}) = 11.$$

Proof: As $r(F_3, G_{61}) \geq 11$ immediately follows from $r(K_3, G_{61}) = 11$ and $K_3 \subset F_3$, we may assume that there is a $(F_3, G_{61})_{11}$-coloring χ. Then, due to $r(F_2, G_{61}) = 11$ (cf. Lemma 3.1.6) a red subgraph F_2 is to exist in χ. Hence, let $v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_4v_5$ be the edges of such a subgraph. Now, neither v_2 nor v_4 may have any red neighbors among the six remaining vertices, say $u_1, \ldots, u_6$, and both v_1v_4 and v_2v_5 must not be red, too. Additionally, $v_3u_1, \ldots, v_3u_5$ may be assumed green.

Since any green subgraph K_3 in $[\{u_1, \ldots, u_6\}]$ would yield a green subgraph G_{61} in χ, we are to demand property

$$\text{(P1)} \quad K_3 \not\subset [\{u_1, \ldots, u_6\}]_g.$$

Considering v_1, we have to discuss three cases. If v_1 produces at least four green neighbors in $\{u_1, \ldots, u_5\}$, then $G_{61} \not\subset [V]_g$ forces $K_4 \subset [\{u_1, \ldots, u_5\}]_r$, and the absence of a red subgraph F_3 implies $|N_g(u_6) \cap \{u_1, \ldots, u_5\}| \geq 4$, making v_4u_6 the spine of a green subgraph G_{61}. Furthermore, v_1 must not have three or more red neighbors among $u_1, \ldots, u_5$ because otherwise we would obtain a red subgraph F_3 in χ or a green subgraph K_3 in $[\{u_1, \ldots, u_5\}]$, the latter contradicting (P1). Thus, we are left with v_1

having exactly three green neighbors in $\{u_1, \ldots, u_5\}$. Without loss of generality assign v_1u_1, v_1u_2, v_1u_3 green and v_1u_4, v_1u_5 red. As v_1v_4 must not become the spine of a green subgraph G_{61}, v_1u_6 has to be colored red, too. Hence, we fail to avoid a red subgraph F_3 in χ as well as a green subgraph K_3 in $[\{u_1, \ldots, u_6\}]$, where (P1) forbids the latter coloring. By this argument both the case analysis and the proof are complete. ■

Lemma 3.1.14

$$r(F_4, G) = \begin{cases} 21 & \text{if } G = G_{112}, \\ 17 & \text{if } G_{98} \subset G \subset G_{111}, \\ 14 & \text{if } G = G_{105}, \\ 13 & \text{if } G = G_{82} \text{ or } G = G_{108} \text{ or } G \subset G_{110} \text{ where } cl(G) = 4, \\ 12 & \text{if } G_{76} \subset G \subset G_{100}, \\ 11 & \text{if } G = G_{61} \text{ or } G \subset G_{83}. \end{cases}$$

Proof: In fact, we prove $r(F_4, G) = r(F_2, G) = r$ for all graphs G mentioned in the lemma's assertion. So, we may assume that there is a $(F_4, G)_r$-coloring χ, containing a red subgraph F_2 by definition. Avoiding a red subgraph F_4, in χ we find a green subgraph B_{r-5} given by F_2's vertices of degree 1 and the $r-5$ remaining vertices. For $r = 21$, $r-5 > 13 = r(F_4, K_4)$ demands the existence of a green subgraph $K_2 + K_4 = G_{112}$ in χ. If $r \geq 14$, then $r-5 \geq 9 = r(F_4, K_4 - e)$ forces a green subgraph $K_2 + (K_4 - e) = G_{111}$ and all of its subgraphs in χ. In case of $r \geq 12$ a green subgraph $K_2 + C_4 = G_{110} \supset G_{82}, G_{100}, G_{108}$ may not be avoided in χ since $r-5 \geq 7 = r(F_4, C_4)$. Finally, we discuss $r = 11$. Due to $r - 5 > 5 = r(F_4, P_3)$, a green subgraph $K_2 + (P_3 \cup K_1) \supset G_{61}, G_{83}$ is to occur in χ. Regarding the known results on $r(F_2, G)$, these arguments determine the respective upper bounds for all graphs G dealt with in this lemma. The corresponding lower bounds may be derived from $r(F_4, G) \geq r(K_3, G)$, $r(F_4, G) \geq r(F_4, K_4) = 13$ if $cl(G) \geq 4$, $r(F_4, G) \geq r(F_4, K_5) = 17$ if $cl(G) \geq 5$, or the Turan-type colorings cited in Lemma 3.1.1 and Lemma 3.1.5, respectively. Thus, the proof is done.

Lemma 3.1.15 *Let G be a connected graph of order 6 satisfying $G \subset G_{102}$. Then*

$$r(F_4, G) = 11.$$

Proof: As $r(F_4, G) \geq r(K_3, G) = 11$ for any graph G with $G \subset G_{102}$ immediately follows from the known results on $r(K_3, G)$ and $K_3 \subset F_4$, we are left with the proof of $r(F_4, G_{102}) \leq 11$. Hence, we assume that there is a $(F_4, G_{102})_{11}$-coloring χ. Considering $r(F_3, G_{102}) = 11$ (cf. Lemma 3.1.12), χ produces a red subgraph F_3 that may be given by the edges $v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_3v_5$. Avoiding a red subgraph F_4, v_4, v_5, and the six remaining vertices $u_1, \ldots, u_6$ have to create a green subgraph B_6, anyway. If there is even a single red edge running from v_1 or v_2 to u_i, say v_1u_1, then u_1 must not have any red neighbors among $u_2, \ldots, u_6$, and any green edge in $[\{u_2, \ldots, u_6\}]$ would yield a green subgraph $K_6 - P_3 \supset G_{102}$ in χ. On the other hand, $[\{u_2, \ldots, u_6\}]_r = K_5 \supset F_4$, and in consequence all edges v_1u_i and v_2u_i may be supposed green. Since $r(F_4, P_3) = 5$, u_1u_2 and u_2u_3 have to be colored green, too, and we obtain a green subgraph $K_6 - (P_3 \cup K_2) \supset G_{102}$ in $[\{v_2, v_4, v_5, u_1, u_2, u_3\}]$, completing our proof. ■

3.2 Extending $K_4 - e$

Theorem 3 *Let $i \in \{5, 6\}$, and let G be any connected graph of order 6. Then*

$$r(F_i, G) = \begin{cases} 21 & \text{if } G = G_{112}, \\ 17 & \text{if } G_{98} \subset G \subset G_{111}, \\ 16 & \text{if } G \in \{G_{96}, G_{104}, G_{105}\} \text{ or } G_{76} \subset G \subset G_{110}, \\ 14 & \text{if } G = G_{108}, \\ 13 & \text{if } G_{61} \subset G \subset G_{95} \text{ or } G \subset G_{99} \text{ where } cl(G) = 4 \\ & \quad \text{or } G_{53} \subset G \subset G_{102} \text{ or } G \in \{G_{82}, G_{86}, G_{97}, G_{103}\}, \\ 11 & \text{otherwise, i.e. if } G \subset G_{83} \text{ or } G \subset G_{87} \text{ or } G \subset G_{90}. \end{cases}$$

Lemma 3.2.1 *Let $i \in \{5, 6\}$. Then*

$$r(F_i, G) = \begin{cases} 21 & \text{if } G = G_{112}, \\ 17 & \text{if } G = G_{111}, \\ 16 & \text{if } G \in \{G_{96}, G_{104}, G_{105}\} \text{ or } G_{76} \subset G \subset G_{110}, \\ 14 & \text{if } G = G_{108}, \\ 13 & \text{if } G_{53} \subset G \subset G_{102} \text{ or } G \in \{G_{86}, G_{99}, G_{103}\}, \\ 11 & \text{if } G \subset G_{83} \text{ or } G \subset G_{87} \text{ or } G \subset G_{90}. \end{cases}$$

Proof: For all graphs G mentioned in the lemma's assertion the lower bound may be derived from the known results on $r(K_4 - e, G) = r$ and $K_4 - e \subset F_5, F_6$. In order to prove the corresponding upper bounds we assume that there is a $(F_i, G)_r$-coloring χ where $i \in \{5,6\}$. Clearly, χ contains a red subgraph $K_4 - e$, and due to the absence of a red subgraph F_i we find a green subgraph $K_{2,r-4}$ in χ. If $r = 21$, then a green subgraph $\overline{K_2} + K_5 \supset G_{112}$ may not be avoided in χ as $r - 4 = 17 = r(F_i, K_5)$. Regarding $r = 17$, $r-4 = 13 = r(F_i, K_4)$ demands a green subgraph $\overline{K_2} + K_4 = G_{111}$ to exist in χ. Now, let $r = 16$. Here, $r-4 > 11 = r(F_i, K_5 - K_3)$ yields a green subgraph $\overline{K_2} + (K_5 - K_3) \supset G_{105}$. Moreover, because of $r - 4 \geq 10 = r(F_i, K_4 - e)$ we obtain a green subgraph $\overline{K_2} + (K_4 - e) = G_{110} \supset G_{96}, G_{104}, G_{108}$ for $r \geq 14$. Next, we discuss $r = 13$. Since $r - 4 = 9 = r(F_i, K_{1,3} + e)$ a green subgraph $\overline{K_2} + (K_{1,3} + e) \supset G_{86}, G_{99}, G_{102}, G_{103}$ is to occur in χ. Finally, we consider $r = 11$, meeting green subgraphs $\overline{K_2} + K_{1,3} \supset G_{83}$ and $\overline{K_2} + C_4 \supset G_{87}, G_{90}$ as $r - 4 = 7 = r(F_i, K_{1,3}) = r(F_i, C_4)$. Thus, the proof is done. ■

Lemma 3.2.2 *Let $i \in \{5,6\}$, and let G be a connected graph of order 6 satisfying $cl(G) = 5$ and $G \neq G_{111}$. Then*

$$r(F_i, G) = 17.$$

Proof: Due to $K_5 \subset G \subset G_{111}$ this result is a direct consequence of $r(F_i, K_5) = r(F_i, G_{111}) = 17$ (for the latter Ramsey number cf. Lemma 3.2.1). ■

Lemma 3.2.3 *Let $i \in \{5,6\}$, and let G be a connected graph of order 6 satisfying $cl(G) = 4$ and $G \subsetneq G_{99}$. Then*

$$r(F_i, G) = 13.$$

Proof: The upper bound $r(F_i, G) \leq 13$ immediately follows from $r(F_i, G_{99}) = 13$ (cf. Lemma 3.2.1) and $G \subset G_{99}$, while the corresponding lower bound is determined by $r(F_i, K_4) = 13$ and $K_4 \subset G$. ■

Lemma 3.2.4 *Let $i \in \{5,6\}$, and let $G \in \{G_{82}, G_{97}\}$. Then*

$$r(F_i, G) = 13.$$

Proof: As $r(F_2, G_{82}) = 13$ (cf. Lemma 3.1.5) and $F_2 \subset F_i$ imply $r(F_i, G_{82}) \geq 13$, we may assume that there is a $(F_i, G_{97})_{13}$-coloring χ where $i \in \{5,6\}$. Then, $r(K_4 - e, G_{97}) = 11$ forces a red subgraph $K_4 - e$ to exist in χ. Avoiding a red subgraph F_i, we obtain a green subgraph $K_{2,9}$, yielding a green subgraph $\overline{K_2} + (C_5 + e) \supset G_{97} \supset G_{82}$ because $r(F_i, C_5 + e) = 9$. Hence, the proof is complete for G_{82} as well as G_{97}. ■

Lemma 3.2.5 *Let G be a connected graph of order 6 satisfying $G_{61} \subset G \subset G_{95}$. Then*

$$r(F_5, G) = 13.$$

Proof: Regarding $r(K_4 - e, G_{61}) = 13$ and $K_4 - e \subset F_5$ we achieve $r(F_5, G_{61}) \geq 13$. Next, we assume that there is a $(F_5, G_{95})_{13}$-coloring χ. Due to $r(K_4 - e, G_{95}) = 13$, χ produces a red subgraph $K_4 - e$, and avoiding a red subgraph F_5 a green subgraph $K_{2,9}$ is to be found in χ, too. If χ actually contains a red subgraph K_4, then the arising green subgraph $K_{4,9}$ and $r(F_5, K_3) = 9$ force a green subgraph $K_3 + K_3 \supset G_{95}$. Thus, a green subgraph B_9 is to exist in χ, and $r(F_5, K_{1,3} + e) = 9$ implies a green subgraph $K_2 + (K_{1,3} + e) \supset G_{95}$. So, we have verified $r(F_5, G_{95}) \leq 13$, and the proof is done for all graphs G with $G_{61} \subset G \subset G_{95}$.

Lemma 3.2.6 *Let G be a connected graph of order 6 satisfying $G_{61} \subset G \subset G_{95}$. Then*

$$r(F_6, G) = 13.$$

Proof: From $r(K_4 - e, G_{61}) = 13$ and $K_4 - e \subset F_6$ we derive $r(F_6, G_{61}) \geq 13$. Additionally, we may assume that there is a $(F_6, G_{95})_{13}$-coloring χ. Since $r(K_4 - e, G_{95}) = 13$, χ yields a red subgraph $K_4 - e$ with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and $|N_r(v_1) \cap V| = |N_r(v_2) \cap V| = 3$. Moreover, the absence of a red subgraph F_6 demands a green subgraph $K_{2,9}$ to occur in χ. Now, consider $K_{2,9}$'s 9-element vertex subset $U = \{u_1, \ldots, u_9\}$.

If any of U's vertices produces at least five green neighbors in U itself, i.e. without loss of generality $u_1u_2, \ldots, u_1u_6$ may be supposed green, then $P_3 \subset [\{u_2, \ldots, u_6\}]_g$ creates a green subgraph $P_3 + P_3 \supset G_{95}$ with vertices v_1, v_2, u_1 determining the second green subgraph P_3. Hence, $[\{u_2, \ldots, u_6\}]_r \supset K_5 - 2K_2 \supset F_6$, yet another contradiction to our initial assumption. Therefore, $|N_g(u_i) \cap U| \leq 4$ where $i \in \{1, \ldots, 9\}$. On the other hand, if any of U's vertices has at least five red neighbors in U itself, then $r(P_3, K_4 - e) = 5$ forces either a red subgraph $K_1 + (P_3 \cup K_1) = F_6$ in $[U]$ or a green subgraph $\overline{K_2} + (K_4 - e) \supset G_{95}$ in $[\{v_1, v_2, u_i, u_j, u_k, u_l\}]$ with an appropriate selection of i, j, k, l. Thus, both $[U]_r$ and $[U]_g$ are regular of degree 4, and we may assume $u_1u_2, \ldots, u_1u_5$ to be colored red and $u_1u_6, \ldots, u_1u_9$ to be colored green. Avoiding a red subgraph F_6 in χ as well as a green subgraph $K_4 - e$ in $[U]$, $P_3 \not\subset [\{u_2, \ldots, u_5\}]_r$ and $P_3 \not\subset [\{u_6, \ldots, u_9\}]_g$, implying $[\{u_2, \ldots, u_5\}]_g = C_4$, $[\{u_2, \ldots, u_5\}]_r = 2K_2$, and $C_4 \subset [\{u_6, \ldots, u_9\}]_r$. Furthermore, from $[\{u_6, \ldots, u_9\}]_r \supset K_4 - e$ and $|N_r(u_i) \cap U| = 4$ we would immediately obtain a red subgraph F_6 in χ. So, $[\{u_6, \ldots, u_9\}]_r = C_4$ and $[\{u_6, \ldots, u_9\}]_g = 2K_2$. Certainly, this situation does not only apply for u_1 but for any of U's vertices.

Having a closer look at one of $K_4 - e$'s vertices of degree 2, say v_3, it must produce either nine red neighbors or at least one green neighbor among U's vertices. In the first case, we easily find a red subgraph $K_1 + (K_1 + 2K_2) \supset F_6$ in χ. In the latter case, we may suppose that v_3u_1 and $u_1u_2, u_1u_3, u_1u_4, u_1u_5, u_2u_3, u_4u_5$ are colored green. As any additional green edge in $\{v_3\} \times \{u_2, \ldots, u_5\}$ would complete a green subgraph G_{95} in $[\{v_1, v_2, v_3, u_1, u_i, u_j\}]$ with appropriately selected $i, j \in \{2, \ldots, 5\}$, all these edges have to be colored red, and regarding the previous paragraph's results a red subgraph $K_1 + C_4 \supset F_6$ may not be avoided in χ. Due to $G_{61} \subset G \subset G_{95}$ we achieve $13 \leq r(F_6, G_{61}) \leq r(F_6, G) \leq r(F_6, G_{95}) \leq 13$, and the proof is done for all graphs from G_{61} through G_{95}.

3.3 Extending C_4

Theorem 4 *Let G be any connected graph of order 6. Then*

$$r(F_7, G) = \begin{cases} 21 & \text{if } G = G_{112}, \\ 17 & \text{if } G_{98} \subset G \subset G_{111}, \\ 13 & \text{if } G = G_{105} \text{ or } G \subset G_{110} \text{ where } G \not\subset G_{61}, G_{100}, G_{108}, \\ 11 & \text{if } G = G_{61} \text{ or } G_{76} \subset G \subset G_{100} \text{ or } G = G_{108}, \\ 10 & \text{if } G \subset G_{102} \text{ where } G \not\subset G_{61}, G_{100}, \\ 9 & \text{if } G \subset G_{62} \text{ where } \Delta(G) = 5 \\ & \quad \text{or } G \in \{G_{41}, G_{53}, G_{60}, G_{63}, G_{65}, G_{79}, G_{83}\} \\ & \quad \text{or } G \subset G_{94} \text{ where } G \text{ is not bipartite}, \\ 8 & \text{if } G \in \{G_{29}, G_{31}, G_{59}\}, \\ 7 & \text{otherwise, i.e. if } G \text{ is a tree where } \Delta(G) \leq 4 \\ & \quad \text{or } G \in \{G_7, G_9, G_{11}, G_{12}, G_{16}, G_{20}\}. \end{cases}$$

Lemma 3.3.1

$$r(F_7, G_{112}) = 21.$$

Proof: The lower bound $r(F_7, G_{112}) \geq 21$ is a direct consequence of the Turan-type $(F_7, G_{112})_{20}$-coloring given by a red subgraph $5K_4$ and a green subgraph $K_{4,4,4,4,4}$. In order to verify the corresponding upper bound we assume that there is a $(F_7, G_{112})_{21}$-coloring χ. Since $r(C_4, G_{112}) = 18$, χ contains a red subgraph C_4, and none of C_4's vertices may have any red neighbors among the 17 remaining vertices. Moreover, a green subgraph

K_5 is forced by these vertices because $r(F_7, K_5) = 17$. Hence, we obtain a green subgraph $\overline{K_4} + K_5 \supset G_{112}$ in χ, and the proof is complete. ■

Lemma 3.3.2 *Let G be a connected graph of order 6 satisfying $cl(G) = 5$. Then*

$$r(F_7, G) = 17.$$

Proof: Assuming that there is a $(F_7, G_{111})_{17}$-coloring χ, due to $r(C_4, G_{111}) = 16$ a red subgraph C_4 may not be avoided in χ. Furthermore, the absence of a red subgraph F_7 demands a green subgraph $K_{4,13}$ to exist in χ, induced by C_4's vertices on one hand and the 13 remaining vertices on the other hand. Then, $r(F_7, K_5 - e) = 13$ yields a green subgraph $\overline{K_4} + (K_5 - e) \supset G_{111}$ in χ, implying $r(F_7, G_{111}) \leq 17$. Thus, the lemma's assertion may be derived from $17 = r(F_7, K_5) \leq r(F_7, G) \leq r(F_7, G_{111}) \leq 17$ where $K_5 \subset G \subset G_{111}$, and the proof is done. ■

Lemma 3.3.3

$$r(F_7, G) = \begin{cases} 13 & \text{if } G = G_{105} \text{ or } G = G_{110}, \\ 11 & \text{if } G = G_{61} \text{ or } G_{76} \subset G \subset G_{100} \text{ or } G = G_{108}, \\ 9 & \text{if } G \in \{G_{41}, G_{53}, G_{60}, G_{63}, G_{65}, G_{79}, G_{83}, G_{94}\}, \\ 8 & \text{if } G \in \{G_{29}, G_{31}, G_{59}\}, \\ 7 & \text{if } G \text{ is a tree where } \Delta(G) \leq 4 \\ & \quad \text{or } G \in \{G_7, G_9, G_{11}, G_{12}, G_{16}, G_{20}\}. \end{cases}$$

Proof: With regard to the known results on $r(C_4, G)$ we prove $r(F_7, G) = r(C_4, G) = r$. As $r(F_7, G) \geq r$ follows from $C_4 \subset F_7$, we may assume that there is a $(F_7, G)_r$-coloring χ, definitely producing a red subgraph C_4. Avoiding a red subgraph F_7, we find a green subgraph $K_{4,r-4}$ in χ, too. For $r = 13$, we achieve green subgraphs $\overline{K_4} + (K_5 - K_3) \supset G_{105}$ and $\overline{K_4} + (K_5 - 2K_2) \supset G_{110}$ since $r - 4 = 9 = r(F_7, K_5 - K_3) = r(F_7, K_5 - 2K_2)$. Next, we discuss $r = 11$. Here, $r - 4 = 7 = r(F_7, K_{1,4})$ forces a green subgraph $\overline{K_4} + K_{1,4} \supset G_{61}, G_{100}$ to occur in χ. Additionally, χ contains a green subgraph $\overline{K_4} + F_7 \supset G_{108}$ because $r - 4 > 6 = r(F_7, F_7)$. Now, let $r = 9$. Due to $r - 4 = 5 = r(F_7, P_3)$ a green subgraph $\overline{K_4} + P_3$ is to exist in χ, along with its subgraphs $G_{41}, \ldots, G_{94}$. Hence, we are left with $r \in \{7, 8\}$ where χ's green subgraph produces any bipartite graph on six vertices except $K_{1,5}$, and the proof is complete. ■

Lemma 3.3.4 *Let G be a connected graph of order 6 satisfying $G \subsetneq G_{110}$ and $G \not\subset G_{61}, G_{100}, G_{108}$. Then*

$$r(F_7, G) = 13.$$

Proof: Considering $r(F_7, G_{110}) = 13$ (cf. Lemma 3.3.3) and $G \subset G_{110}$, we obtain $r(F_7, G) \leq 13$. The corresponding lower bound may be derived from the Turan-type $(F_7, G)_{12}$-coloring χ given by a red subgraph $3K_4$ and a green subgraph $K_{4,4,4}$. As χ's maximum green subgraphs on six vertices are G_{61}, G_{100}, and G_{108}, the proof is done. ■

Lemma 3.3.5 *Let G be a connected graph of order 6 satisfying $G \subset G_{102}$ and $G \not\subset G_{61}, G_{100}$. Then*

$$r(F_7, G) = 10.$$

Proof: First, we assume that there is a $(F_7, G_{102})_{10}$-coloring χ. Since $r(C_4, G_{102}) = 9$, χ contains a red subgraph C_4. Then, a red subgraph F_7 may only be avoided if C_4's vertices and the six remaining vertices yield a green subgraph $K_{4,6}$. Moreover, $r(F_7, F_7) = 6$, and a green subgraph $\overline{K_4} + F_7 \supset G_{102}$ is to be found in χ. Thus, $r(F_7, G_{102}) \leq 10$. Regarding K_9's edge two-coloring given by a red subgraph $2K_4$ and a green subgraph $K_1 + K_{4,4}$, it does neither create a red subgraph F_7 nor produce any green subgraph on six vertices that is no subgraph of G_{61} or G_{100}. So, by $10 \leq r(F_7, G) \leq r(F_7, G_{102}) \leq 10$ the proof is complete. ■

Lemma 3.3.6 *Let G be a connected non-bipartite graph of order 6 satisfying $G \subsetneq G_{94}$. Then*

$$r(F_7, G) = 9.$$

Proof: Clearly, $r(F_7, G_{94}) = 9$ (cf. Lemma 3.3.3) and $G \subset G_{94}$ imply $r(F_7, G) \leq 9$. Verifying the corresponding lower bound, we consider a red subgraph $2K_4$ and a green subgraph $K_{4,4}$ determining a $(F_7, G)_8$-coloring avoiding any non-bipartite green subgraphs at all. Hence, we are done with the proof. ■

Lemma 3.3.7 *Let G be a connected graph of order 6 satisfying $\Delta(G) = 5$ and $G \subset G_{62}$. Then*

$$r(F_7, G) = 9.$$

Proof: For all graphs G mentioned in the lemma's assertion the lower bound $r(F_7, G) \geq 9$ is a direct consequence of the Turan-type $(F_7, G)_8$-coloring given by a red subgraph $2K_4$ and a green subgraph $K_{4,4}$. Now, we assume that there is a $(F_7, G_{62})_9$-coloring χ where $r(C_4, G_{62}) = 8$ forces a red subgraph C_4 to exist in χ. Due to the absence of a red subgraph F_7, C_4's vertices and the five remaining vertices must induce a green subgraph $K_{4,5}$, and we may have a look at an arbitrary vertex v of $K_{4,5}$'s 5-element vertex subset U. If $|N_g(v) \cap U| \geq 2$, then χ contains a green subgraph $K_1 + K_{2,3} \supset G_{62}$. Therefore, $|N_r(u) \cap U| \geq 3$ holds for any vertex $u \in U$, and we achieve $[U]_r \supset K_5 - 2K_2 \supset F_7$, contradicting our initial assumption. Thus, $9 \leq r(F_7, G) \leq r(F_7, G_{62}) \leq 9$, and the proof is complete.

3.4 Extending K_4

Theorem 5 *Let G be a connected graph of order 6 satisfying $\delta(G) = 1$, or let $G = G_7$ or $G = G_{106}$. Then*

$$r(F_8, G) = \begin{cases} 25 & \text{if } G = G_{98} \text{ or } G = G_{106}, \\ 19 & \text{if } G = G_{86} \text{ or } G = G_{89}, \\ 18 & \text{if } cl(G) = 4 \text{ but } K_5 - e \not\subset G, \\ 17 & \text{if } G = G_{63} \text{ or } G = G_{74}, \\ 16 & \text{otherwise, i.e. if } G = G_7 \text{ or } G \subset H \\ & \quad \text{where } H \in \{G_{40}, G_{41}, G_{46}, G_{48}, G_{49}, G_{51}, G_{56}, G_{60}\}. \end{cases}$$

In a first step we reduce the problem of calculating $r(F_8, G)$ to determining $r(K_4, G)$.

Lemma 3.4.1 *Let G be any connected graph of order 6. Then*

$$r(F_8, G) = r(K_4, G).$$

Proof: The inequality $r(F_8, G) \geq r(K_4, G) = r$ is immediately obtained from $K_4 \subset F_8$. So, we may continue assuming that there is a $(F_8, G)_r$-coloring χ, by definition producing a red subgraph K_4. Avoiding a red subgraph F_8, χ yields a green subgraph $K_{4,r-4}$, too. Hence, $r - 4 \geq r(K_4, T) - 4 = 12 > 11 = r(F_8, K_4 - e)$ where $T \subset G$ is a spanning tree demands a green subgraph $\overline{K_4} + (K_4 - e)$ to occur in χ, settling the proof if $G = G_{105}$ or $G \subset G_{110}$. Next, let $cl(G) = 5$. As $r - 4 \geq r(K_4, K_5) - 4 = 21 > 18 = r(F_8, K_4)$, χ contains a green subgraph $\overline{K_4} + K_4 \supset G_{111}$ in this case, and we are done if $G_{98} \subset G \subset G_{111}$. Finally, we are left with $G = G_{112}$. Here, a green subgraph $\overline{K_4} + K_5 \supset G_{112}$ is to exist in χ because $r - 4 \geq 31 > 25 = r(F_8, K_5)$. Thus, our initial assumption fails for all connected graphs on six vertices, and the proof is complete. ■

With regard to Lemma 3.4.1, we may apply already known results on $r(K_4, G)$ obtained by Chvátal [4] and by Jayawardene and Rousseau [12].

Theorem 6 [4] *Let $m, n \geq 2$. Then*

$$r(K_m, T_n) = (m-1)(n-1) + 1.$$

Especially,

$$r(K_4, T_6) = 16.$$

Theorem 7 [12]

$$r(K_4, C_6) = 16.$$

Moreover, we derive some additional results considering known Ramsey numbers $r(K_4, G - v)$ where $d_G(v) = 1$ or precisely counting certain edges if $cl(G) = 5$.

Lemma 3.4.2 *Let G be a connected non-tree graph of order 6 satisfying $\delta(G) = 1$. Then*

$$r(K_4, G) = \begin{cases} 25 & \text{if } G = G_{98}, \\ 19 & \text{if } G = G_{86} \text{ or } G = G_{89}, \\ 18 & \text{if } cl(G) = 4 \text{ but } K_5 - e \not\subset G, \\ 17 & \text{if } G = G_{63} \text{ or } G = G_{74}, \\ 16 & \text{otherwise.} \end{cases}$$

Proof: With regard to the known results on $r(K_4, H)$ where H is an arbitrary graph on five vertices, we define

$$r = \max\Big(\{r(K_4, G - v) : v \in V(G) \text{ and } d_G(v) = 1\} \cup \{r(K_4, T) : T \subset G \text{ is a tree}\}\Big) \geq 16,$$

directly implying $r(K_4, G) \geq r$. In order to prove the corresponding upper bound we assume that there is a $(K_4, G)_r$-coloring χ, however forcing a green subgraph $G - v$. Avoiding a green subgraph G, we are to find a red subgraph $K_{s,r-5}$ in χ with appropriately selected $s \geq 1$. Now, consider $r(K_3, G)$. Since $r - 5 = 20 > 14 = r(K_3, G_{98})$, χ produces a red subgraph $K_s + K_3 \supset K_4$ if $r = 25$. For all the other graphs from the lemma's assertion we may apply a similar argument where $r - 5 \geq 11 = r(K_3, G)$. Hence, the proof is done.

Lemma 3.4.3

$$r(K_4, G_{106}) = 25.$$

Proof: The lower bound $r(K_4, G_{106}) \geq 25$ is a direct consequence of $K_5 \subset G_{106}$ and $r(K_4, K_5) = 25$. For proving the corresponding upper bound we assume that there is a $(K_4, G_{106})_{25}$-coloring χ. As $r(K_4, K_5) = 25$, χ produces a green subgraph K_5, and we divide K_{25}'s vertex set into subsets $V_1 = V(K_5)$ and $V_2 = V(K_{25}) \setminus V(K_5)$. Due to the absence of a green subgraph G_{106} any vertex from V_2 is limitted to at most one green neighbor among V_1's vertices. Thus, we have $q_r(v, V_1) \geq 4$ for any $v \in V_2$, implying $q_r(V_1, V_2) \geq 80$. Moreover, we obtain a vertex $w \in V_1$ satisfying $d_r(w) = q_r(w, V_2) \geq 16$. Hence, $r(K_3, G_{106}) = 14$ forces a red subgraph $K_1 + K_3 = K_4$ to exist in χ, and the proof is complete.

Lemma 3.4.4

$$r(K_4, G_{109}) \leq 27.$$

Proof: Assume there is a $(K_4, G_{109})_{27}$-coloring χ. As $r(K_4, K_5) = 25$, we find a green subgraph K_5 in χ. Now let $V_1 = V(K_5)$ and $V_2 = V(K_{27}) \setminus V(K_5)$. The absence of a green subgraph G_{109} forces $q_g(v, V_1) \leq 2$, i.e. $q_r(v, V_1) \geq 3$, for any vertex $v \in V_2$. So, we achieve $q_r(V_1, V_2) \geq 66$ yielding a vertex $w \in V_1$ where $d_r(w) = q_r(w, V_2) \geq 14$. Because of $r(K_3, G_{109}) = 14$ we obtain a red subgraph $K_1 + K_3 = K_4$ in χ, proving the stated upper bound. ■

Furthermore [19] offers upper bounds for $r(K_4, G_{111})$ and $r(K_4, G_{112})$.

References

[1] A. Babak, S. P. Radziszowski, and Kung-Kuen Tse, *Computation of the Ramsey number $R(B_3, K_5)$*, Bull. Inst. Combin. Appl. **41** (2004), 71–76.

[2] L. Boza, *Erratum to* [13], J. Combin. Math. Combin. Comput., to appear.

[3] J. A. Calvert, M. J. Schuster, and S. P. Radziszowski, *Computing the Ramsey number $R(K_5 - P_3, K_5)$*, J. Combin. Math. Combin. Comput. **82** (2012), 131–140.

[4] V. CHVÁTAL, *Tree-complete graph Ramsey numbers*, J. Graph Theory **1** (1977), 93.

[5] V. CHVÁTAL and F. HARARY, *Generalized Ramsey theory for graphs II: Small diagonal numbers*, Proc. Amer. Math. Soc. **32** (1972), 389–394.

[6] V. CHVÁTAL and F. HARARY, *Generalized Ramsey theory for graphs III: Small off-diagonal numbers*, Pacific J. Math. **41** (1972), 335–345.

[7] M. CLANCY, *Some small Ramsey numbers*, J. Graph Theory **1** (1977), 89–91.

[8] R. J. FAUDREE, C. C. ROUSSEAU, and R. H. SCHELP, *All triangle–graph Ramsey numbers for connected graphs of order six*, J. Graph Theory **4** (1980), 293–300.

[9] G. R. T. HENDRY, *Ramsey numbers for graphs with five vertices*, J. Graph Theory **13** (1989), 245–248.

[10] M. HOETH and I. MENGERSEN, *Ramsey numbers for graphs of order four versus connected graphs of order six*, Util. Math. **57** (2000), 3–19.

[11] HUA GU, HONGXUE SONG, and XIANGYANG LIU, *Ramsey numbers $r(K_{1,4}, G)$ for all three-partite graphs G of order six*, J. Southeast Univ. (English Ed.) **20** (2004), 378–380.

[12] C. J. JAYAWARDENE and C. C. ROUSSEAU, *Ramsey numbers $r(C_6, G)$ for all graphs G of order less than six*, Congr. Numer. **136** (1999), 147–159.

[13] C. J. JAYAWARDENE and C. C. ROUSSEAU, *The Ramsey numbers for a quadrilateral vs. all graphs on six vertices*, J. Combin. Math. Combin. Comput. **35** (2000), 71–87, *Erratum*, J. Combin. Math. Combin. Comput. **51** (2004), 221.

[14] C. J. JAYAWARDENE and C. C. ROUSSEAU, *The Ramsey number for a cycle of length five vs. a complete graph of order six*, J. Graph Theory **35** (2000), 99–108.

[15] C. J. JAYAWARDENE and C. C. ROUSSEAU, *Ramsey numbers $r(C_5, G)$ for all graphs G of order six*, Ars Combin. **57** (2000), 163–173.

[16] R. LORTZ and I. MENGERSEN, *Ramsey numbers for special trees versus all connected graphs of order six*, in preparation.

[17] B. D. MCKAY and S. P. RADZISZOWSKI, $R(4,5) = 25$, J. Graph Theory **19** (1995), 309–322.

[18] J. MCNAMARA, $r(K_4 - e, K_6) = 21$, unpublished.

[19] S. P. RADZISZOWSKI, *Small Ramsey numbers*, Electron. J. Combin. (2011), DS1.

[20] S. P. RADZISZOWSKI, J. STINEHOUR, and KUNG-KUEN TSE, *Computation of the Ramsey number $R(W_5, K_5)$*, Bull. Inst. Combin. Appl. **47** (2006), 53–57.

[21] C. C. ROUSSEAU and C. J. JAYAWARDENE, *The Ramsey number for a quadrilateral vs. a complete graph on six vertices*, Congr. Numer. **123** (1997), 97–108.

[22] YUANSHENG YANG and G. R. T. HENDRY, *The Ramsey number $r(K_1 + C_4, K_5 - e)$*, J. Graph Theory **19** (1995), 13–15.