Length Two Path Centered Surface Area for Bipartite Graphs

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Abstract

After introducing and discussing the notion of length two path centered surface area for general graphs, particularly for bipartite graphs, we derive a closed-form expression and an explicit expression for the length two path centered surface areas of the hypercube and the star graph, respectively.

Keywords

Length two-centered surface area, bipartite graph, hypercube, star graph, combinatorics related to computing

1 Introduction

Given a graph G, and a vertex $v \in G$, a question one may ask is how many vertices are at distance i from v, $i \in [0, D(G)]$, where $D(G)$ stands for the diameter of G. This quantity has been referred to as the Whitney numbers of the second kind of the poset [10], and the surface area with radius i centered at v [5]. The surface area of a (di)graph can find several applications in network performance evaluation, e.g., in computing various bounds for the problem of k-neighborhood broadcasting [4], and identifying spanning trees [12]. As a result, this surface area problem has been studied for a variety of graphs, including the rotator graph, the star graph, the k-ary n-cube, the (n, k)-star graph, and the arrangement graph. (For the solution to this problem for the aforementioned and other graphs, readers are referred to [5, 13] and the references cited within.)

At first glance, this seems an easy problem from the computational complexity point of view as it is clear that, via standard graph search algorithms, such a surface area can be computed in polynomial time with respect to the size of the graph. However, in interconnection networks, the number of vertices is often exponential, or factorial, in n, a network parameter. For example, the number of vertices in a star graph of n dimensions is $n!$. What is needed is either an algorithm that computes this surface area in time polynomial in n or an explicit formula solution with polynomially many terms in n. As a matter of fact, it took a number of years for this problem to be solved satisfactorily for the well-studied star graph, that is, having a correct explicit formula [2, 5, 10].

In this paper, we study an extension of this vertex centered surface area problem: given a length two path $p_2 = (v, w, x)$ in a graph G, referred to as the *reference path* henceforth, *how many vertices are at distance i from* p_2, $i \in [0, D(G)]$? We refer to this quantity as the *p_2-centered surface area with radius i*, denoted as $B_G^{p_2}(i)$ in this paper. In general, we have the notion of *H-centered surface area of G*, where $H \subset G$, such that $B_G^H(0) = |V(H)|$, and for all $i \in [1, D(G)]$, $B_G^H(i) = |\{u \notin V(H) | \min_{v \in H} \{d(u, v)\} = i\}|$. We find it convenient to refer to $(B_G^H(0), B_G^H(1), \ldots, B_G^H(D(G)))$ as the *H-centered surface area sequence of G*.

The above generalization might seem unnecessary as one can simply identify all the vertices in a subgraph H to a single vertex, thus reducing to the above regular surface area problem. Nevertheless, the reason for the existence of nice explicit formulas for interconnection networks such as the one for the star graph is due to symmetry, which will be destroyed once we identify H to a single vertex.

The solution to this p_2-centered surface area problem is certainly an interesting combinatorial result in its own right. Moreover, this notion might also serve practical purposes. As one application, the diagnosability of an interconnection network is defined to be the maximum number of faulty processors that can necessarily be diagnosed within a particular diagnostic model, and a considerable amount of research has been done to determine this important measurement for various networks [6, 9]. It is recently suggested in [14] that, under the popular comparison diagnosis model, the conditional diagnosability of G is at most, and often equal to, $\min_{p_2 \in G} B_G^{p_2}(1)$. Thus, this notion of p_2-centered surface area is also related to a valuable fault-tolerance measurement of interconnection networks, although $B_G^{p_2}(1)$ itself can be directly calculated [3]. This application motivated us to study this p_2-centered surface area problem.

The rest of this paper is organized as follows: In the next section, we will define and discuss this notion of p_2-centered surface area for the general graphs, and particularly that of bipartite graphs. We then derive a closed-form result for p_2-centered surface area for the hypercube in Section 3, and an explicit expression result for the star graph in Section 5, after characterizing the relevant vertex structures of the star graph in Section 4. We conclude this paper in Section 6 with some remarks.

2 p_2-centered surface area for general graphs

Let $G(V, E)$ be a simple and connected graph, $(v, w), (w, x) \in E$, we study the surface area of G, centered at the length-2 path $p_2 = (v, w, x)$, with radius $i \in [0, D(G)]$. It is clear that $B_G^{p_2}(0) = 3$. In general, let $u \in V \setminus \{v, w, x\}$ such that $d(u, p_2) = i \geq 1$, by definition, for some $z \in \{v, w, x\}, d(u, z) = i$.

- If $d(u, w) = i$ then $d(u, v) \geq i$, otherwise, we would have $d(u, p_2) < i$; and $d(u, v) \leq i + 1$, since $(v, w) \in E$. Thus, $i \leq d(u, v) \leq i + 1$. By the same token, $i \leq d(u, x) \leq i + 1$.
- If $d(u, v) = i$, then, $i \leq d(u, w) \leq i + 1$ and $i \leq d(u, x) \leq i + 2$. We note that, if $(v, x) \in E$, we would have $i \leq d(u, x) \leq i + 1$.
- By symmetry, if $d(u, x) = i$, then $i \leq d(u, w) \leq i+1$ and $i \leq d(u, v) \leq i + 2$. Again, if $(v, x) \in E$, then $i \leq d(u, v) \leq i + 1$.

We use Table 1 to summarize the above analysis on the relationship among $d(u, v), d(u, w)$, and $d(u, x)$, after removing redundancy:

For all $i \in [1, D(G)]$, let

$$
\begin{aligned}
B_G^1(v, w, x, i) &= |\{u | d(u, v) = d(u, w) = d(u, x) = i\}|, \\
B_G^2(v, w, x, i) &= |\{u | d(u, v) = d(u, w) = i, d(u, x) = i + 1\}|, \\
B_G^3(v, w, x, i) &= |\{u | d(u, v) = i + 1, d(u, w) = d(u, x) = i\}|, \\
B_G^4(v, w, x, i) &= |\{u | d(u, v) = d(u, x) = i + 1, d(u, w) = i\}|, \\
B_G^5(v, w, x, i) &= |\{u | d(u, w) = i + 1, d(u, v) = d(u, x) = i\}|, \\
B_G^6(v, w, x, i) &= |\{u | d(u, v) = i, d(u, w) = d(u, x) = i + 1\}|, \\
B_G^7(v, w, x, i) &= |\{u | d(u, v) = i, d(u, w) = i + 1, d(u, x) = i + 2\}|, \\
B_G^8(v, w, x, i) &= |\{u | d(u, v) = d(u, w) = i + 1, d(u, x) = i\}|, \text{ and,} \\
B_G^9(v, w, x, i) &= |\{u | d(u, v) = i + 2, d(u, w) = i + 1, d(u, x) = i\}|,
\end{aligned}
$$

Table 1: Cases among $d(u,v), d(u,w)$, and $d(u,x)$

Case	$d(u,v)$	$d(u,w)$	$d(u,x)$
1	i	i	i
2	i	i	$i+1$
3	$i+1$	i	i
4	$i+1$	i	$i+1$
5	i	$i+1$	i
6	i	$i+1$	$i+1$
7	i	$i+1$	$i+2$
8	$i+1$	$i+1$	i
9	$i+2$	$i+1$	i

we have the following general expression for p_2-centered surface area of G.

Proposition 2.1 *Let G be a simple and connected graph, and let $p_2 = (v,w,x)$ be a length 2 path of G. Then, for all $i \in [1, D(G)]$,*

$$\begin{aligned} B_G^{p_2}(i) &= B_G^1(v,w,x,i) + B_G^2(v,w,x,i) + B_G^3(v,w,x,i) + B_G^4(v,w,x,i) \\ &\quad + B_G^5(v,w,x,i) + B_G^6(v,w,x,i) + B_G^7(v,w,x,i) + B_G^8(v,w,x,i) \\ &\quad + B_G^9(v,w,x,i). \end{aligned} \tag{1}$$

Note: for $k \in [2,9]$, $B_G^k(v,w,x,D(G)) = 0$; $B_G^7(v,w,x,D(G)-1) = 0$ and $B_G^9(v,w,x,D(G)-1) = 0$.

It is well known that a bipartite graph does not contain any odd cycle. On the other hand, any of the Cases 1, 2, 3, 6 or 8 as shown in Table 1 mandates that u is of equidistance to either v and w, or w and x. The associated pair of shortest paths from u to any such a pair of vertices, together with the corresponding edge, induces a cycle of odd length. As a result, no vertex in any bipartite graph falls into any of the above five cases. We thus have the following observation.

Proposition 2.2 *Let G be bipartite, $p_2 = (v,w,x)$ be a path in G, then for $i \in [1, D(G)-1]$,*

$$B_G^{p_2}(i) = B_G^4(v,w,x,i) + B_G^5(v,w,x,i) + B_G^7(v,w,x,i) + B_G^9(v,w,x,i).$$

In the rest of this paper, we study the p_2-centered surface area for two important bipartite interconnection structures: the hypercube and the star graph.

3 The p_2-centered surface area of the hypercube

A hypercube of n dimensions, denoted as Q_n, contains 2^n vertices, often represented as n-bit vectors. The distance between any two vertices $v, w \in Q_n$, is given as follows:

$$d(v, w) = \sum_{l=1}^{n} (v_i \oplus w_i), \tag{2}$$

where $\oplus$ stands for the bitwise exclusive-or operation. Figure 1 shows Q_3.

It is relatively easy to see that, for $i \in [0, n]$, its vertex centered surface area with radius i is $\binom{n}{i}$, and, for any edge e in Q_n, the e-centered surface area of Q_n is $2\binom{n-1}{i}, i \in [0, n]$. For example, letting $v\,(e)$ be any vertex (edge) in Q_3, v- and e-centered surface area sequences for Q_3 are $(1,3,3,1)$, and $(2,4,2,0)$, respectively.

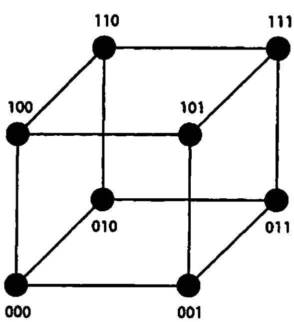

Figure 1: Q_3

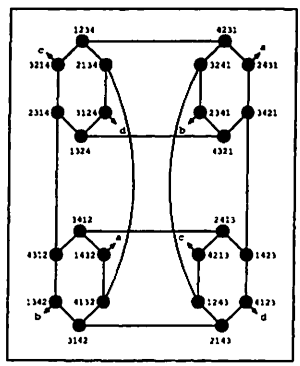

Figure 2: S_4

We now derive the surface area of Q_n, centered at $p_2 = (v, w, x)$, with radius $1 \le i \le n-1$. As Q_n is vertex symmetric[1], we choose $w = 0_n$. Moreover, without loss of generality, for $1 \le j < k \le n$, we choose $v = 0_j 1 0_{n-1-j}$, and $x = 0_k 1 0_{n-1-k}$.

1. We first consider those vertices u as counted in $B^4_{Q_n}(v, w, x, i)$, where $d(u, 0_n) = i$ and $d(u, 0_j 1 0_{n-1-j}) = d(u, 0_k 1 0_{n-1-k}) = i + 1$.

   If $u_{j+1} = 1$, since $u_{j+1} \oplus 0 = 1$ and $d(u, 0_n) = i$, $\sum_{l \ne j+1} (u_l \oplus 0) = i - 1$. But, since $d(u, 0_j 1 0_{n-1-j}) = i + 1$, $\sum_{l \ne j+1} (u_l \oplus 0) = i + 1$.

[1] A graph is vertex symmetric if for each pair of its vertices, a and b, there is an automorphism that maps a to b [1].

Thus, $u_{j+1} = 0$. By an analogous argument, $u_{k+1} = 0$. Finally, since $d(u, 0_n) = i$, by Eq. 2, there exist exactly i 1's in such a vertex u, where two bits are fixed to be 0. Thus, for $1 \leq i \leq n-2$, $B^4_{Q_n}(v, w, x, i) = \binom{n-2}{i}$.

2. For those vertices u as counted in $B^5_{Q_n}(v, w, x, i)$, where $d(u, 0_n) = i+1$ and $d(u, 0_j 1 0_{n-1-j}) = d(u, 0_k 1 0_{n-1-k}) = i$, by the same token, $u_{j+1} = u_{k+1} = 1$. Since $d(u, 0_j 1 0_{n-1-j}) = i$, for $1 \leq i \leq n-1$, $B^5_{Q_n}(v, w, x, i) = \binom{n-2}{i-1}$.

3. For those vertices u as counted in $B^7_{Q_n}(v, w, x, i)$, where $d(u, 0_j 1 0_{n-1-j}) = i, d(u, 0_n) = i+1$, and $d(u, 0_k 1 0_{n-1-k}) = i+2$, we can similarly derive that $u_{j+1} = 1$, but $u_{k+1} = 0$. Since $d(u, 0_j 1 0_{n-1-j}) = i$, for $1 \leq i \leq n-2$, $B^7_{Q_n}(v, w, x, i) = \binom{n-2}{i}$.

4. Finally, we consider those u as counted in $B^9_{Q_n}(v, w, x, i)$, where $d(u, 0_j 1 0_{n-1-j}) = i+2, d(u, 0_n) = i+1$, and $d(u, 0_k 1 0_{n-1-k}) = i$, we can similarly find out $u_{j+1} = 0$, but $u_{k+1} = 1$. Since $d(u, 0_k 1 0_{n-1-k}) = i$, we also have that, for $1 \leq i \leq n-2$, $B^9_{Q_n}(v, w, x, i) = \binom{n-2}{i}$.

By Proposition 2.2 and the above analysis, we have achieved the following general result, which also works for $i = 0$ and $i = n$, as $B^{p_2}_{Q_n}(0) = 3$ and $B^{p_2}_{Q_n}(n) = 0$, respectively.

Theorem 3.1 *Let p_2 be any length 2 path in Q_n, for $i \in [1, n-1]$,*

$$B^{p_2}_{Q_n}(i) = \binom{n-1}{i} + 2\binom{n-2}{i}.$$

For example, the p_2-centered surface area sequence of Q_3, centered at any such a path, is $(3, 4, 1, 0)$.

We also notice that for $n \geq 3$, $\min_{p_2 \in Q_n} B^{p_2}_{Q_n}(1) = 3n - 5$, confirming an observation made in [14, §5.1], discovered via structural analysis.

Finally, it is clear that the H-centered surface area sequence of a graph G necessarily constitutes a partition of all the vertices in G. To this regard, we have the following calculation:

$$\sum_{i=0}^{n} B^{p_2}_{Q_n}(i) \quad = \quad 3 + \sum_{i=1}^{n-1} \left[\binom{n-1}{i} + 2\binom{n-2}{i} \right] = 2^n,$$

which is indeed the total number of vertices in Q_n.

4 Star graph and its vertex structure

The symmetric star graph was proposed in [1] as an attractive alternative to the hypercube for interconnecting processors in a parallel computer, and compares quite favorably with the hypercube in several aspects. It has been widely studied and recent results include [5, 7, 8].

The vertex set of the n-dimensional star graph, denoted by $S_n, n \geq 3$, is simply the collection of all the permutations of $\langle n \rangle (= \{1, 2, \ldots, n\})$, where $e_n = 12 \cdots n$ is called its *identity vertex*. For any two permutations v and w, (v, w) is an edge in S_n if and only if, for some $j \in [2, n]$, w can be obtained from v by applying a transposition $(1, j)$, i.e., $w = v \circ (1, j)$. It is called star graph since, as a Cayley graph, its generators, $\{(1, j) | j \in [2, n]\}$, form a star. Thus, a better name for this graph might be star-generated graph. Figure 2 shows S_4, where 2134 is adjacent to 3124, since $3124 = 2134 \circ (1, 3)$.

It is well known that every permutation is a product of disjoint cycles of length ≥ 1; which is unique except for the order of these cycles. We thus refer to this unique factorization of $u \in S_n$ the *cycle structure of* u, denoted by $C(u)$, and will make no distinction between u and $C(u)$ in the rest of this paper. Furthermore, let $u \in S_n$, $C(u)$ be its cycle structure, and let $C \in C(u)$, we call C a *primary cycle*, if 1 belongs to C; otherwise, C is *normal*. We call a cycle *trivial* if it contains exactly one symbol, called a *fixed point* of $C(u)$; *non-trivial* otherwise. We often drop those fixed points from $C(u)$ when the context is clear.

For any $u \in S_n$, let $g(u)$ be the total number of non-trivial cycles contained in $C(u)$, containing a total of $b(u)$ symbols taken out of $\langle n \rangle$, Akers *et al* derived in [1] the following distance formula between u and e_n.

$$d(u, e_n) = b(u) + g(u) - \begin{cases} 0, & \text{if 1 is a fixed point in } C(u); \\ 2, & \text{otherwise.} \end{cases} \tag{3}$$

The diameter of S_n, is obtained by maximizing Eq. 3, and turns out to be $\left\lfloor \frac{3(n-1)}{2} \right\rfloor$.

For example, let $u = 635179284 \in S_9$, then $C(u) = (1, 6, 9, 4)(2, 3, 5, 7)$, where $(1, 6, 9, 4)$ is a primary cycle, and $(2, 3, 5, 7)$ a normal one. The trivial cycle (8) is dropped from $C(u)$. Since $b(u) = 8, g(u) = 2$, and 1 is not a fixed point in u, we have $d(u, e_9) = 8 + 2 - 2 = 8$. Indeed, one of the minimum routing paths from u to e_9, of eight steps, is the following:

$$635179284 \overset{(1,6)}{\rightarrow} 935176284 \overset{(1,9)}{\rightarrow} 435176289 \overset{(1,4)}{\rightarrow} 135476289 \overset{(1,2)}{\rightarrow}$$
$$315476289 \overset{(1,3)}{\rightarrow} 513476289 \overset{(1,5)}{\rightarrow} 713456289 \overset{(1,7)}{\rightarrow} 213456789 \overset{(1,2)}{\rightarrow} e_9,$$

where, with "$u \overset{(1,p)}{\rightarrow} v$", we apply transposition $(1, p)$ to a permutation u to obtain v, another permutation.

To identify the reference path, as S_n is vertex symmetric, we select e_n as vertex w. We may choose $v = e_n \circ (1,j)$, $j \in [2,n]$, in a total of $n-1$ ways, and then $x = e_n \circ (1,i), i \in [2,n] \setminus \{j\}$, in a total of $n-2$ ways. Considering the symmetry, there are a total of $(n-1)(n-2)/2$ ways of choosing $p_2 = (v, e_n, x)$. Out of this many choices, we choose $v = e_n \circ (1,3) = 3214 \cdots n$, and $x = e_n \circ (1,2) = 213 \cdots n$, based on a convenience consideration[2]. Such a selection leads to the definition of the following two automorphisms: For all $u \in S_n$, by $\varphi_1(u)$, we mean swapping symbols 1 and 3 in u; and by $\varphi_2(u)$, we mean swapping symbols 1 and 2 in u. Henceforth, we refer to $p_2 = (v, w, x) = (\varphi_1(e_n), e_n, \varphi_2(e_n))$ as the reference path for our enumeration.

Clearly, $\varphi_1(\varphi_1(e_n)) = \varphi_2(\varphi_2(e_n)) = e_n$, it thus follows, by the automorphic nature of φ_1 and φ_2, $d(u,v) = d(\varphi_1(u), \varphi_1(v)) = d(\varphi_1(u), e_n)$, and $d(u,x) = d(\varphi_2(u), \varphi_2(x)) = d(\varphi_2(u), e_n)$. As a result, we can use Eq. 3 to calculate the distance from any vertex u to v (resp. x) via the distance from its image under φ_1 (resp. φ_2) to e_n.

To derive $B^{p_2}_{S_n}(i)$, the surface area of S_n centered at p_2 with radius $i \in [1, D(S_n) - 1]$, we need to categorize, and enumerate, all the vertices in S_n. Let $u \in S_n$, the three symbols, 1, 2, and 3, can belong to three separate cycles, two separate cycles, or one cycle, in $C(u)$. When they all belong to the same cycle, since there are two distinct cyclic orderings of these three symbols, and for each of them, extra symbols may or may not occur after each of these symbols, there are sixteen different arrangements for this case. When they belong to two cycles, there are three ways of choosing two out of three symbols; and, depending on whether extra symbols occur after each of these three symbols, there are twenty-four different arrangements altogether for this case. Finally, when they belong to three separate cycles, there are eight arrangements, depending on whether additional symbols occur after each of these three symbols. After a thorough analysis of all these forty-eight arrangements, we arrive at the following categorizing result.

Theorem 4.1 *Let $v, w, x, u \in S_n$, $n \geq 3$, such that $v = \varphi_1(e_n), w = e_n$, $x = \varphi_2(e_n)$, and $d(u, p_2) = i$.*

1. *The vertex u falls into Case 4 of Table 1, i.e., $d(u, \varphi_1(e_n)) = i+1$, $d(u, e_n) = i$, and $d(u, \varphi_2(e_n)) = i+1$, if and only if $C(u)$ contains either*
   - *(a) $A = (1, a_2, \ldots, a_x)$, $B = (2), C = (3)$, where $(a_2, \ldots, a_x)$ may be empty; or*
   - *(b) $D = (1, a_2, \ldots, a_x, 2, b_2, \ldots, b_y)$, $C = (3)$, where $(a_2, \ldots, a_x)$ may be empty, but $(b_2, \ldots, b_y)$ is not; or*

[2] We will see later that the derivation of the formulas is independent of this choice.

(c) $D = (1, a_2, \ldots, a_x, 3, c_2, \ldots, c_z)$, $B = (2)$, *where* $(a_2, \ldots, a_x)$ *may be empty, but* $(c_2, \ldots, c_z)$ *is not; or*

(d) $D = (1, a_2, \ldots, a_x, 2, b_2, \ldots, b_y, 3, c_2, \ldots, c_z)$, *where* $(a_2, \ldots, a_x)$ *and* $(b_2, \ldots, b_y)$ *may be empty, but* $(c_2, \ldots, c_z)$ *is not; or*

(e) $D = (1, a_2, \ldots, a_x, 3, c_2, \ldots, c_z, 2, b_2, \ldots, b_y)$, *where* $(a_2, \ldots, a_x)$ *and* $(c_2, \ldots, c_z)$ *may be empty, but* $(b_2, \ldots, b_y)$ *is not.*

2. *The vertex* u *falls into Case 5 of Table 1, i.e.,* $d(u, \varphi_1(e_n)) = i$, $d(u, e_n) = i + 1$, *and* $d(u, \varphi_2(e_n)) = i$, *if and only if* $C(u)$ *contains either*

   (a) $A = (1, a_2, \ldots, a_x)$, $B = (2, b_2, \ldots, b_y)$, $C = (3, c_2, \ldots, c_z)$, *where* $(a_2, \ldots, a_x)$ *may be empty, but neither* $(b_2, \ldots, b_y)$ *nor* $(c_2, \ldots, c_z)$ *is; or*

   (b) $A = (1, a_2, \ldots, a_x)$, $D = (2, b_2, \ldots, b_y, 3, c_2, \ldots, c_z)$, *where* $(a_2, \ldots, a_x)$, $(b_2, \ldots, b_y)$ *and* $(c_2, \ldots, c_z)$ *may be empty; or*

   (c) $D = (1, a_2, \ldots, a_x, 3)$, $B = (2, b_2, \ldots, b_y)$, *where* $(a_2, \ldots, a_x)$ *may be empty, but* $(b_2, \ldots, b_y)$ *is not; or*

   (d) $D = (1, a_2, \ldots, a_x, 2)$, $C = (3, c_2, \ldots, c_z)$, *where* $(a_2, \ldots, a_x)$ *may be empty, but* $(c_2, \ldots, c_z)$ *is not.*

3. *The vertex* u *falls into Case 7 of Table 1, i.e.,* $d(u, \varphi_1(e_n)) = i$, $d(u, e_n) = i+1$, *and* $d(u, \varphi_2(e_n)) = i+2$, *if and only if* $C(u)$ *contains either*

   (a) $A = (1, a_2, \ldots, a_x)$, $B = (2)$, $C = (3, c_2, \ldots, c_z)$, *where* $(a_2, \ldots, a_x)$ *may be empty, but* $(c_2, \ldots, c_z)$ *is not; or*

   (b) $A = (1, a_2, \ldots, a_x, 3)$, $B = (2)$, *where* $(a_2, \ldots, a_x)$ *may be empty; or*

   (c) $D = (1, a_2, \ldots, a_x, 2, b_2, \ldots, b_y)$, $C = (3, c_2, \ldots, c_z)$, *where* $(a_2, \ldots, a_x)$ *may be empty, but neither* $(b_2, \ldots, b_y)$ *nor* $(c_2, \ldots, c_z)$ *is empty; or*

   (d) $D = (1, a_2, \ldots, a_x, 2, b_2, \ldots, b_y, 3)$, *where both* $(a_2, \ldots, a_x)$ *and* $(b_2, \ldots, b_y)$ *may be empty.*

4. *The vertex* u *falls into Case 9 of Table 1, i.e.,* $d(u, \varphi_1(e_n)) = i + 2$, $d(u, e_n) = i + 1$, *and* $d(u, \varphi_2(e_n)) = i$, *if and only if* $C(u)$ *contains either*

   (a) $A = (1, a_2, \ldots, a_x)$, $B = (2, b_2, \ldots, b_y)$, $C = (3)$, *where* $(a_2, \ldots, a_x)$ *may be empty, but* $(b_2, \ldots, b_y)$ *is not; or*

   (b) $D = (1, a_2, \ldots, a_x, 2)$, $C = (3)$, *where* $(a_2, \ldots, a_x)$ *may be empty; or* *(c)* $D = (1, a_2, \ldots, a_x, 3, c_2, \ldots, c_z)$, $B = (2, b_2, \ldots, b_y)$, *where* $(a_2, \ldots, a_x)$ *may be empty, but neither* $(b_2, \ldots, b_y)$ *nor* $(c_2, \ldots, c_z)$ *is; or*

(d) $D = (1, a_2, \ldots, a_x, 3, c_2, \ldots, c_z, 2)$, *where both* $(a_2, \ldots, a_x)$ *and* $(c_2, \ldots, c_z)$ *may be empty.*

Proof: This result does cover all the forty-eight possible vertex structures, but not just the seemingly seventeen of them, since some of these structures are equivalent to each other. For example, as Case 1a shows, when $C(u)$ contains $(1, a_2, a_3, \ldots, a_x)$, (2) and (3), it always falls into Case 4 of Table 1, whether $(a_2, \ldots, a_x) = \epsilon$ or not.

We will not give a complete proof of this result for all the seventeen cases, which is rather tedious, though not difficult. Basically, for each of the forty-eight cases into which u might fall, we apply Eq. 3 to find out $d(u, e_n), d(u, \varphi_1(u))$, and $d(u, \varphi_2(u))$, and then categorize u into one of the above four cases accordingly. We demonstrate this process with Case 1a: Let $C(u)$ contain $A = (1, a_2, \ldots, a_x)$, $B = (2)$, and $C = (3)$, where $(a_2, \ldots, a_x)$ may be empty. We discuss two cases:

- If $(a_2, \ldots, a_x) = \epsilon$, $A = (1)$ is a fixed point. Then, by Eq. 3, $d(u, e_n) = b(u) + g(u)$. Since $\varphi_1(u)$ contains a new primary cycle $(1, 3)$, containing two extra symbols, as compared with $u = (1)(2)(3)$, by Eq. 3, $d(\varphi_1(u), e_n) = (b(u)+2)+(g(u)+1)-2 = b(u)+g(u)+1 = d(u, e_n)+1$. Similarly, since $\varphi_2(u)$ contains a new cycle $(1, 2)$, we also have $d(\varphi_2(u), e_n) = d(u, e_n) + 1$. Thus, u falls into Case 4 of Table 1.

- Otherwise, by Eq. 3, $d(u, e_n) = b(u) + g(u) - 2$. For this case, we have that $\varphi_1(u)$ contains a new primary cycle $(1, a_2, \ldots, a_x, 3)$, and $\varphi_2(u)$ contains a new primary cycle $(1, a_2, \ldots, a_x, 2)$. In both cases, as the number of the cycles stays the same, but an additional symbol is added, $d(\varphi_1(u), e_n) = d(\varphi_2(u), e_n) = (b(u)+1)+g(u)-2 = d(u, e_n)+1$. Thus, u also falls into Case 4 of Table 1. □

We use S_3 as an example to demonstrate the results as reported in Theorem 4.1. Let $u = 123$, i.e., $C(u) = (1)(2)(3)$, falling into Case 4.a, then $\varphi_1(u) = 321$ and $C(\varphi_1(u)) = (1, 3)(2)$; and $\varphi_2(u) = 213$ and $C(\varphi_2(u)) = (1, 2)(3)$. As a result, $d(u, \varphi_1(e_3)) = d(\varphi_1(u), e_3) = 1$, and $d(u, \varphi_1(e_3)) = 1$. The whole situation regarding S_3 is summarized in Table 2, where Case label refers to the case index as given in the Theorem.

5 Derivation of the p_2-centered surface area

We note that, in Theorem 4.1, Cases 1b and 1c, Cases 1d and 1e, Cases 2c and 2d, and the corresponding structures in Cases 3 and 4 are symmetric to

Table 2: Vertex structures for S_3

u	$d(u,e_3)$	$d(u,\varphi_1(e_3))$	$d(u,\varphi_2(e_3))$	$d(u,p_2)$	Case
123	0	1	1	0	4.a
321	1	0	2	0	7.b
213	1	2	0	0	9.b
231	2	1	3	1	7.d
312	2	3	1	1	9.d
132	3	2	2	2	5.b

each other. Hence, we have ten cases of structures to enumerate, which we will carry out by constructing the associated cycle structures $C(u)$ containing $b(u)$ symbols organized in $g(u)$ non-trivial cycles, subject to constraints as imposed in various cases, as well as the distance formula. More specifically, we first construct those cycle(s) containing symbols 1, 2 and 3, then use the rest of the symbols to construct the remaining non-trivial cycles, each containing at least two symbols.

The general quantity of $\mathbf{d}(q,r)$, namely, *the number of ways of decomposing q distinct symbols into r non-trivial cycles,* is discussed in [11, §4.4]. Based on [11, Eqs. 4.9 and 4.18]: for $q \geq 2r \geq 1$,

$$\mathbf{d}(q,r) = \sum_{j=0}^{q}(-1)^{q+r-j}\binom{q}{j}s(q-j,r-j). \tag{4}$$

In the above, $s(\_,\_)$ stands for the signless Stirling numbers of the first kind, which can be represented as an explicit formula itself [5, Eqs. 5 and 6] in terms of a two-layer summation, when factorial is treated as a basic operation.

As mentioned earlier, when $(\varphi_1(e_n), e_n, \varphi_2(e_n))$ is used as the reference path, the three symbols 1, 2 and 3 can belong to one cycle, two cycles, or three cycles. When these three symbols belong to one cycle, D, we need to select $b(u)$ symbols, use some of them to construct D, and the rest for the other $g(u)-1$ non-trivial cycles. We have to deal with two different cases: Cases 1.d, which is equivalent to Case 1.e, and Case 3.d.

For Case 1.d, i.e., $C(u)$ contains $D = (1, a_2, \ldots, a_x, 2, b_2, \ldots, b_y, 3, c_2, \ldots, c_z)$, where $(a_2, \ldots, a_x)$ and $(b_2, \ldots, b_y)$ may be empty, but $(c_2, \ldots, c_z)$ is not, since D is a non-trivial primary cycle, $d(u,e) = b(u) + d(u) - 2$.

To construct $C(u)$, we select $b-3$ symbols out of $n-3$ symbols, excluding 1, 2 and 3, in $\binom{n-3}{b-3}$ ways. We first construct D, by selecting l_D symbols out of those $b-3$ symbols, $l_D \in [1, b-3]$, in $\binom{b-3}{l_D}$ ways. For each and every of those $l_D!$ permutations of this many symbols, out of l_D total positions:

to the left of the first symbol, and in between any two adjacent symbols, but not to the right of the last symbol, since $(c_2, \ldots, c_z) \neq \epsilon$, we select two positions to insert symbols 2 and 3, respectively, corresponding to the case that $(a_2, \ldots, a_x)$ may be empty, but neither $(b_2, \ldots, b_y)$ nor $(c_2, \ldots, c_z)$ is; and, out of the same l_D total positions, we select one position to insert both symbols 2 and 3, in this order, corresponding to the case that $(a_2, \ldots, a_x)$ may be empty, $(b_2, \ldots, b_y)$ is empty, but $(c_2, \ldots, c_z)$ is not. Thus, for each such a set of chosen l_D symbols, there are $l_D!\left(\binom{l_D}{2} + \binom{l_D}{1}\right)$ unique constructions of cycle D.

We finally use the remaining $b - 3 - l_D$ symbols to construct the other $g-1$ non-trivial cycles. To make sure that $\binom{n-3}{b-3} \geq 0$, $n \geq b = d-g+2$, thus $g \geq \max\{1, d-n+2\}$, since u contains at least one non-trivial cycle in this case. The fact that every non-trivial cycle contains at least two symbols leads to the upper bound of g, $b \geq 2g$, i.e., $g \leq \left\lfloor \frac{d+2}{3} \right\rfloor$. Finally, as Case 1.d falls into Case 4 of Table 1, $d(u, e_n) = i$. Therefore, for $i \in [1, D(S_n) - 1]$,

$$B_{1.d}(i) = B_{1.e}(i) = \sum_{g=\max\{1,i-n+2\}}^{\lfloor \frac{i+2}{3} \rfloor} \sum_{l_D=1}^{i-g-1} \binom{n-3}{i-g-1}\binom{i-g-1}{l_D} l_D! \left[\binom{l_D}{2} + \binom{l_D}{1}\right] \mathbf{d}(i-g-1-l_D, g-1). \quad (5)$$

By analogous arguments, we have also derived formulas for all the other cases as listed in Theorem 4.1 as follows. For $i \in [1, D(S_n) - 1]$,

$$\begin{aligned} B_{1,a}(i) &= \sum_{g=\max\{1,i-n+4\}}^{\lfloor \frac{i+2}{3} \rfloor} \sum_{l_A=1}^{i-g+1} \binom{n-3}{i-g+1}\binom{i-g+1}{l_A} l_A! \, \mathbf{d}(i-g+1-l_A, g-1) \\ &+ \sum_{g=\max\{1,i-n+3\}}^{\lfloor \frac{i}{3} \rfloor} \binom{n-3}{i-g} \mathbf{d}(i-g, g). \end{aligned} \quad (6)$$

$$B_{1.b}(i) = B_{1.c}(i) = \sum_{g=\max\{1,i-n+3\}}^{\lfloor \frac{i+2}{3} \rfloor} \sum_{l_D=1}^{i-g} \binom{n-3}{i-g}\binom{i-g}{l_D} l_D! l_D \mathbf{d}(i-g-l_D, g-1). \quad (7)$$

$B_{2,a}(i)$

$$= \sum_{g=\max\{2,i-n+2\}}^{\lfloor\frac{i+1}{3}\rfloor} \sum_{l_D=2}^{i-g-1} \binom{n-3}{i-g-1}\binom{i-g-1}{l_D} l_D!(l_D-1)$$

$$\mathbf{d}(i-g-1-l_D, g-2)$$

$$+ \sum_{g=\max\{3,i-n+3\}}^{\lfloor\frac{i+3}{3}\rfloor} \sum_{l_D=3}^{i-g} \binom{n-3}{i-g}\binom{i-g}{l_D} l_D! \binom{l_D}{2}$$

$$\mathbf{d}(i-g-l_D, g-3). \qquad (8)$$

$$B_{2,b}(i)$$

$$= \sum_{g=\max\{1,i-n+2\}}^{\lfloor\frac{i+1}{3}\rfloor} \sum_{l_D=0}^{i-g-1} \binom{n-3}{i-g-1}\binom{i-g-1}{l_D} l_D!(l_D+1)$$

$$\mathbf{d}(i-g-1-l_D, g-1)$$

$$+ \sum_{g=\max\{2,i-n+3\}}^{\lfloor\frac{i+3}{3}\rfloor} \sum_{l_D=0}^{i-g} \binom{n-3}{i-g}\binom{i-g}{l_D} l_D! \left[\binom{l_D}{2}+\binom{l_D}{1}\right]$$

$$\mathbf{d}(i-g-l_D, g-2). \qquad (9)$$

$$B_{2.c}(i) = B_{2.d}(i)$$

$$= \sum_{g=\max\{2,i-n+3\}}^{\lfloor\frac{i+3}{3}\rfloor} \sum_{l_D=1}^{i-g} \binom{n-3}{i-g}\binom{i-g}{l_D} l_D! l_D$$

$$\mathbf{d}(i-g-l_D, g-2). \qquad (10)$$

For $i \in [1, D(S_n)-2]$,

$$B_{3,a}(i) = \sum_{g=\max\{2,i-n+4\}}^{\lfloor\frac{i+3}{3}\rfloor} \sum_{l_D=2}^{i-g+1} \binom{n-3}{i-g+1}\binom{i-g+1}{l_D} l_D!(l_D-1)$$

$$\mathbf{d}(i-g+1-l_D, g-2)$$

$$+ \sum_{g=\max\{1,i-n+3\}}^{\lfloor\frac{i+1}{3}\rfloor} \sum_{l_C=1}^{i-g} \binom{n-3}{i-g}\binom{i-g}{l_C} l_C!$$

$$\mathbf{d}(i-g-l_C, g-1). \qquad (11)$$

$$B_{3.b}(i) = \sum_{g=\max\{1,i-n+4\}}^{\lfloor\frac{i+3}{3}\rfloor} \sum_{l_D=0}^{i-g+1} \binom{n-3}{i-g+1}\binom{i-g+1}{l_D} l_D! \\ \mathbf{d}(i-g+1-l_D, g-1). \tag{12}$$

$$B_{3.c}(i) = \sum_{g=\max\{2,i-n+3\}}^{\lfloor\frac{i+3}{3}\rfloor} \sum_{l_D=2}^{i-g} \binom{n-3}{i-g}\binom{i-g}{l_D}\binom{l_D}{2} l_D! \\ \mathbf{d}(i-g-l_D, g-2). \tag{13}$$

$$B_{3.d}(i) = \sum_{g=\max\{1,i-n+3\}}^{\lfloor\frac{i+3}{3}\rfloor} \sum_{l_D=0}^{i-g} \binom{n-3}{i-g}\binom{i-g}{l_D} l_D!(l_D+1) \\ \mathbf{d}(i-g-l_D, g-1). \tag{14}$$

Moreover, because of the aforementioned symmetry between Cases 3 and 4, for $i \in [1, D(S_n)-2]$,

$B_{4,a}(i) = B_{3,a}(i), B_{4,b}(i) = B_{3,b}(i), B_{4,c}(i) = B_{3,c}(i)$, and, $B_{4,d}(i) = B_{3,d}(i)$.

By Proposition 2.2 and the above analysis, we have achieved the following main result of this section.

Theorem 5.1 *Let $n \geq 3$, the surface area of the star graph, centered at $p_2(\varphi_1(e_n), e_n, \varphi_2(e_n))$ with radius $i \in [1, D(S_n)-1]$, is given as follows:*

$$\begin{aligned} B^{p_2}_{S_n}(i) &= B_{1.a}(i) + B_{2.a}(i) + B_{2,b}(i) + 2\left[B_{1,b}(i) + B_{1.d}(i) + B_{2,c}(i)\right] \\ &\quad +2\left[B_{3.a}(i) + B_{3,b}(i) + B_{3.c}(i) + B_{3,d}(i)\right], \end{aligned} \tag{15}$$

where explicit formulas for $B_{1.a}(i)$ through $B_{3,d}(i)$ are given in Eqs. 5 through 14.

Each and every of these expressions is given in terms of five layers of summation, including three layers for the $\mathbf{d}$ term, when factorial is considered as a basic operation. Obviously, the number of terms as included in all these expressions is polynomial in terms of n, thus computationally feasible. Indeed, it is straightforward to write a program to calculate $B^{p_2}_{S_n}$ for small values of n, as shown in Table 3, where we note the sum of all

Table 3: Sample data for $B^{p_2}_{S_n}(\varphi_1(e_n), e_n, \varphi_2(e_n), i), n \in [3, 8]$

	i									
n	0	1	2	3	4	5	6	7	8	9
3	3	2	1	0	0	0	0	0	0	0
4	3	5	9	7	0	0	0	0	0	0
5	3	8	23	45	38	3	0	0	0	0
6	3	11	43	132	249	241	41	0	0	0
7	3	14	69	286	840	1,648	1,737	428	15	0
8	3	17	101	525	2,090	6,220	12,577	14,013	4,429	345

the numbers in each row corresponding to n equals $n!$, the total number of vertices in S_n.

Although we choose a specific path $p_2(\varphi_1(e_n), e_n, \varphi_2(e_n))$ to work with in Section 4, it is clear that, if we select (v_1, e_n, x_1) as the reference path, where, for $k_1 \neq k_2$, $v_1 = e_n \circ (1, k_1)$, and $x_1 = e_n \circ (1, k_2)$, we would have the same structures as listed in Theorem 4.1, except the two symbols 2 and 3 will be replaced with k_1 and k_2, respectively. Thus, all such length 2 paths lead to the same surface area. We therefore have the following result.

Corollary 5.1 *Let $n \geq 3$, $v, w, x \in S_n$ such that $(v, w), (w, x)$ are edges of S_n. Then, let $p_2 = (v, w, x)$, for all $i \in [1, D(S_n) - 1]$,*

$$\begin{aligned} B^{p_2}_{S_n}(i) &= B_{1.a}(i) + B_{2.a}(i) + B_{2,b}(i) + 2\left[B_{1,b}(i) + B_{1.d}(i) + B_{2,c}(i)\right] \\ &\quad +2\left[B_{3.a}(i) + B_{3,b}(i) + B_{3.c}(i) + B_{3,d}(i)\right], \end{aligned} \tag{16}$$

where explicit formulas for $B_{1.a}(i)$ through $B_{3,d}(i)$ are given in Eqs. 5 through 14.

By Corollary 5.1 and Table 3, $\min_{p_2 \in S_n} B^{p_2}_{S_n}(1) = 3n-7, n \geq 3$, agreeing with the result as derived in [9], which is also obtained through a structural analysis.

6 Concluding remarks

We discussed the notion of length two path centered surface area for general graphs, particularly for bipartite graphs. We also derived a closed-from expression for the length two path centered surface area for the hypercube, and an explicit expression for the star graph, by enumeration and construction.

We believe that, when solving the subgraph centered surface area problem, this technique of identifying and enumerating vertices of equidistance from a subgraph can be applied to other Cayley graphs, particularly those defined on symmetric groups, when a distance formula is available.

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