A NOTE ON GRAPHIC SEQUENCES WITH NO REALIZATION CONTAINING AN INDUCED FOUR CYCLE

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ABSTRACT. The degree sequence of a finite graph G is its list $D = (d_1, \ldots, d_n)$ of vertex degrees in non-increasing order. The graph G is called a realization of D. In this paper, we characterize the graphic degree sequences D such that no realization of D contains an induced four cycle. Our characterization is stated in terms of the class of forcibly chordal graphs.

1. INTRODUCTION

Let G be a finite, simple graph and let $D(G) = (d_1, \ldots, d_n)$ be its sequence of vertex degrees listed in non-increasing order. The sequence $D(G)$ is known as the *degree sequence* of G, the graph G is said to *realize* $D(G)$, and G is said to be a *realization* of $D(G)$. We call a sequence $(d_1, \ldots, d_n)$ of nonnegative integers a *degree sequence* if it is realized by some graph. We adopt the convention that if D is used without comment, it denotes a degree sequence. Similarly for D_1, D_2, and so on.

Given D_1 and D_2, we define $D_1 \preceq D_2$ to mean there is G_1 realizing D_1 and G_2 realizing D_2 such that $G_1 \sqsubseteq G_2$, where $\sqsubseteq$ is the induced subgraph relation. The reader may check that $\preceq$ is a transitive relation on degree sequences. We say that G_2 excludes G_1 if $G_1 \not\sqsubseteq G_2$, that D_2 excludes D_1 if $D_1 \not\preceq D_2$, that D excludes G if D excludes $D(G)$, and that G excludes D if $D(G)$ excludes D.

A set X of vertices is called a *clique* if x and y are adjacent for each pair of distinct vertices x, y in X. We call a set X of vertices *independent* if no two distinct vertices x and y of X are adjacent. Given disjoint sets X and Y, we say that X is *complete to* Y if each x in X is adjacent to each y in Y. Similarly, we say that X is *anticomplete to* Y if no x in X is adjacent to any y in Y. A graph S is *split* if its vertex set has a partition (A, B) such

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that A is a clique and B is an independent set. Such partitions are called *split partitions.* Split partitions of a split graph are not in general unique. In writing a split partition (A, B), we understand that A is a clique and B is independent.

A graph is *chordal* if every induced cycle is a triangle. For a graph G and graph property $\mathcal{P}$, we say that G and $D(G)$ are *forcibly-$\mathcal{P}$* if every realization of $D(G)$ has property $\mathcal{P}$. We note that "forcibly $\mathcal{P}$-graphic" is the more common term. We let $G_1 \sqcup G_2$ denote the disjoint union of G_1 and G_2. We let $G[X]$ be the induced subgraph of G with vertex set X. We call the two edge matching $2K_2$. For other basic graph theoretic definitions and terminology, we refer the reader to [3].

In [2], Chudnovsky and Seymour prove Rao's Conjecture, which states that given infinitely many degree sequences $D_1, D_2, \ldots$, there are positive integers $i < j$ such that $D_i \preceq D_j$. To prove this they essentially give, for an arbitrary degree sequence D, an approximate structure theorem for those graphs excluding D. This general theorem is very powerful, allowing them to resolve a nearly thirty year old conjecture. It further suggests and leaves open a problem of independent interest: to give *exact* structure theorems for *specific* degree sequence exclusions. That is the focus of the current paper. Our main result is a structure theorem characterizing degree sequences excluding the square C_4.

We note that there is a large literature on determining when a degree sequence has a realization containing a particular subgraph. In particular, R. Luo characterizes the degree sequences with a realization containing a C_4 subgraph in [6]. However, degree sequence containment for the subgraph and induced subgraph relations behave quite differently. The results in [6] do not imply and are not implied by the results of this paper.

2. Technical Lemmas

We recall a folklore theorem, whose simple proof we omit.

Proposition 2.1. The following statements are equivalent for a graph G:

(1) G is a split graph.
(2) G excludes $2K_2$ and all cycles on at least four vertices.
(3) G excludes $2K_2$, C_4, and C_5.

In particular split graphs are chordal. We thus have a corollary.

Corollary 2.2. The following statements are equivalent for a degree sequence D:

(1) D is the degree sequence of a split graph.
(2) D excludes $2K_2$ and all cycles on at least 4 vertices.
(3) D excludes $2K_2$, C_4, and C_5.

By the well known characterization in [5] of split graphs as those graphs for which some Erdös-Gallai inequality [4] is equality, we see that every realization of a split graph is also split. Every split graph is thus forcibly split.

Let S be a split graph with split partition (A, B). Let H be an arbitrary graph. We define $(S, A, B) \circ H$ as the graph G with vertex set $V(S) \cup V(H)$ formed by joining H completely to A and anticompletely to B. This operation is defined by R. Tyshkevich in [7]. We use $\circ$ to state the following lemma, whose simple proof is left to the reader.

Lemma 2.3. Choose $n \geq 4$. Let S be a split graph with split partition (A, B) and let H be an arbitrary graph. If C_n is an induced subgraph of $(S, A, B) \circ H$, then C_n is an induced subgraph of S or H.

Lemma 2.4. Choose $k \geq 5$. Suppose D excludes C_{k-1}, but D does not exclude C_k. Let G be a realization of D containing a cycle C isomorphic to C_k, and let A and B be the sets of vertices of $G - C$ that are complete and anticomplete to C, respectively. Then $G = (G[A \cup B], A, B) \circ C$.

Proof. We have only to show that every vertex of $G - C$ is complete or anticomplete to C, that vertices complete to C are pairwise adjacent, and that vertices anticomplete to C are pairwise nonadjacent.

First we show that every vertex of $G - C$ is complete or anticomplete to C. Assume not. Then there is a vertex x outside of C adjacent to some vertex y of C and nonadjacent to some other vertex z of C. Choose a neighbor v of z in C distinct from y. Let $K = G[C \cup x]$. Define K' as the graph obtained from $K/\{v, z\}$ by subdividing the edge xy with a new vertex t. Simple checking shows that K and K' have the same degree sequence. But $K' - \{x, t\}$ is isomorphic to C_{k-1}. Therefore K' contains C_{k-1} as an induced subgraph. Thus K does not exclude $D(C_{k-1})$, and hence G does not exclude $D(C_{k-1})$ either. This implies that $D(C_{k-1}) \preceq D$, contrary to hypothesis. This contradiction shows that every vertex outside C is complete or anticomplete to C as claimed.

Next, assume there are nonadjacent vertices x and y, both complete to C. Write C in cyclic order as $c_1, c_2, \ldots, c_k$. Let $G' = G + c_1c_3 - c_3x + xy - yc_1$. One may check that $D(G) = D(G')$ and that $G'[c_1, c_3, c_4, \ldots, c_k]$ is a cycle in that cyclic order. Therefore G' contains an induced C_{k-1}. We thus see that D does not exclude C_{k-1}, contrary to hypothesis. This contradiction shows x and y must be adjacent. Since x and y are arbitrary elements of A, it follows that A is a clique as claimed.

Finally, let x and y be distinct vertices in B. It is enough to show x and y are not adjacent. Suppose they are adjacent. Then $G[C \cup \{x, y\}]$ is isomorphic to $C_k \sqcup P_2$, which has the same degree sequence as $C_{k-1} \sqcup P_3$. Therefore D does not exclude C_{k-1}, contrary to assumption. This completes the proof. □

3. The Main Results

We now state our main theorem, from which we derive our other main results as corollaries. A certain abuse of notation makes the statements of these results more concise, so we make the convention that $D = D(\text{SPLIT} \circ G)$ means that there is some split graph S with split partition (A, B) such that D is the degree sequence of $(S, A, B) \circ G$, and $D = D(\text{SPLIT})$ means D is the degree sequence of a split graph.

Theorem 3.1. Let $n \geq 4$. A degree sequence D excludes C_n if and only if either $D = D(\text{SPLIT} \circ C_{n+1})$, $D = D(\text{SPLIT} \circ C_{n+2})$, or D forcibly excludes each chordless cycle on at least n vertices.

Proof. First, if D excludes all chordless cycles on n or more vertices, then in particular D excludes C_n. If $D = D(\text{SPLIT} \circ C_{n+1})$, then by Lemma 2.3, if $D(C_n) \preceq D$ then $C_n \sqsubseteq C_{n+1}$ or $C_n \sqsubseteq S$ for some split graph S, a contradiction. Therefore D excludes C_n. Similarly if $D = D(\text{SPLIT} \circ C_{n+2})$ then D excludes C_n. One direction of the theorem is thus proved.

We now prove the converse. Let D exclude C_n. We must show D falls into one of the above three classes as claimed.

First, note D excludes C_{n+k} for all $k \geq 3$. To see this, assume not. Note that $D(C_{n+k}) = D(C_n \sqcup C_k)$, as C_k exists since $k \geq 3$ by assumption. Therefore $D(C_n \sqcup C_k) \preceq D$, so that D has a realization G such that $C_n \sqcup C_k \sqsubseteq G$. In particular $C_n \sqsubseteq G$, contrary to assumption that D excludes C_n. This contradiction proves our claim.

Next, we break into cases. The first case we consider is that D excludes C_{n+1} and C_{n+2}. D excludes C_n by hypothesis, and by the previous paragraph, D excludes C_{n+k} for k at least three. Therefore D excludes all cycles on at least n vertices. Therefore, as claimed, no realization has a chordless cycle on n or more vertices.

The other case is that D does not exclude both C_{n+1} and C_{n+2}. Then D has a realization G containing either C_{n+1} or C_{n+2} as an induced subgraph. Let $k = n+1$ if G contains an induced C_{n+1} and let $k = n+2$ otherwise. In either case, D excludes C_{k-1} but not C_k. Let $C_k = C$. It then follows by Lemma 2.4 that $G = (G[A \cup B], A, B) \circ C$, thus completing the proof. □

Applying Theorem 3.1 with $n = 4$, we obtain the following corollary.

Theorem 3.2. A degree sequence D excludes C_4 if and only if either D is forcibly chordal, $D = D(\text{SPLIT} \circ C_5)$, or $D = D(\text{SPLIT} \circ C_6)$.

Taking complements yields the following theorem as well.

Theorem 3.3. A degree sequence D excludes $2K_2$ if and only if either D is forcibly antichordal, $D = D(\text{SPLIT} \circ C_5)$, or $D = D(\text{SPLIT} \circ K_{3,3})$.

Proof. Take complements, use Theorem 3.2, and note antichordal graphs are the complements of chordal graphs by definition, the pentagon is self-complementary, and the complement of a hexagon has the same degree sequence as $K_{3,3}$. □

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