New Construction Techniques for $H_2(8t, 3)$s

Dinesh G. Sarvate *
Department of Mathematics
College of Charleston
Charleston, SC 29424
SarvateD@cofc.edu

Li Zhang †
Department of Mathematics
and Computer Science
The Citadel
Charleston, SC 29409
li.zhang@citadel.edu

Abstract

An H_2 graph is a multigraph on three vertices with a double edge between a pair of distinct vertices and a single edge between the other two pairs. The problem of decomposition of a complete multigraph $3K_{8t}$ into H_2 graphs has been completely solved. In this paper, we describe some new procedures for such decompositions and ask a question: Can these procedures be adapted or extended to find a unified proof of the existence of $H_2(8t, \lambda)$'s?

1 Introduction

A graph can be decomposed into a collection of subgraphs such that every edge of the graph is contained in one of the subgraphs. Decomposing a graph into simple graphs has been well studied in the literature. For a well written survey on the decomposition of a complete graph into simple graphs with small number of points and edges, see [1].

A *multigraph* is a graph where more than one edge between a pair of points is allowed. The decomposition of copies of a complete graph into proper multigraphs has not received much attention yet, see [3, 4, 7, 8]. A complete multigraph λK_v ($\lambda \geq 1$) is a graph on v points with λ edges between every pair of distinct points. In this paper we address different techniques used in the decomposition of a $3K_{8t}$ ($t \geq 1$) into H_2 graphs (defined in section 1.1). A well studied combinatorial design (BIBD, which can also be used to find graph decompositions) is defined below. On the other hand, a BIBD(v, k, λ) can be considered as a decomposition of λK_v into complete graphs K_k's.

*Thanks to the Department of Mathematics, College of Charleston for continued support.
†Thanks to The Citadel Foundation for their support.

Definition 1 Given a finite set V of v points and integers k and $\lambda \geq 1$, a *balanced incomplete block design* (BIBD), denoted as BIBD(v, k, λ), is a pair (V, B) where B is a collection of subsets (also called blocks) of V such that every block contains exactly $k < v$ points and every pair of distinct points is contained in exactly λ blocks.

Definition 2 A *2-factor* of a graph G is a spanning subgraph of G which is regular of degree 2. A *2-factorization* of a graph G is an edge disjoint decomposition of G into 2-factors.

It is also known that a K_{2n+1} $(n \geq 1)$ has n 2-factors.

Lemma 1 [2] (Agrawal's Lemma) In every binary equi-replicate design of constant block size k (hence $bk = vr$ and $b = mv$), the treatments in each block can be rearranged such that in the k by b array, formed with ordered blocks as columns, every treatment occurs in each row exactly m times.

1.1 H_2 Graphs

Definition 3 An H_2 graph is a multigraph on three points with a double edge between a pair of distinct points and single edges between the other two pairs of distinct points.

If the set of points of an H_2 is $V = \{a, b, c\}$ and the double edge is between a and b, then we denote the H_2 graph by $\langle a, b, c\rangle_{H_2}$ (see Figure 1). An $H_2(v, \lambda)$ is a decomposition of λK_v into H_2 graphs. In particular, an $H_2(8t, 3)$ is a decomposition of a $3K_{8t}$ graph into $3t(8t-1)$ H_2 graphs.

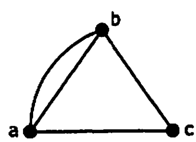

Figure 1: An H_2 Graph

1.2 Difference Sets for $H_2(8t, 3)$ Decompositions

One of the powerful techniques to construct combinatorial designs is based on *difference sets* and *difference families*; for example, see Stinson [9] for details and how to develop the difference sets to get the blocks of a design including the use of ("dummy") elements ∞'s (refer Example 1 below).

Definition 4 Suppose $(G, +)$ is a finite group of order v in which the identity element is denoted "0". Let k and λ be positive integers such that $2 \leq k < v$. A (v, k, λ) difference set in $(G, +)$ is a subset $D \subseteq G$ that satisfies the following properties: 1. $|D| = k$, 2. the multiset $[x-y : x, y \in D, x \neq y]$ contains every element in $G \backslash \{0\}$ exactly λ times. A difference family is a collection $[D_1, \ldots, D_l]$ of k-subsets of G such that the multiset of the differences from all sets in the collection $[D_1, \ldots, D_l]$ together cover all nonzero elements of G as differences exactly λ times.

In many cases, G is $(Z_v, +)$, the integers modulo v. For example, a $(7,3,1)$-difference set in $(Z_7, +)$ is $D = \{3, 0, 2\}$. Note $0-3 = 4, 2-3 = 6, 3-0 = 3, 2-0 = 2, 3-2 = 1$ and $0-2 = 5$, hence we get every element of $Z_7 \backslash \{0\}$ exactly once as a difference of two distinct elements in D.

To connect the difference set concept to an $H_2(8t, 3)$, we define the difference set $D = \langle a, b, c \rangle$ for H_2 graphs as the difference set such that it gives $|a-b|$ twice (corresponding to a double edge between a and b), $|a-c|$ once (corresponding to a single edge between a and c) and $|b-c|$ once (corresponding to a single edge between b and c). For example, the difference set $\langle 3, 0, 2 \rangle$ gives the difference 3 twice, the difference 1 once and the difference 2 once. A graphical illustration is given in Figure 2.

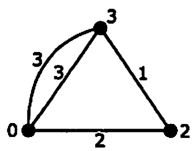

Figure 2: An H_2 graph $\langle 3, 0, 2 \rangle_{H_2}$ corresponding to a difference set $\langle 3, 0, 2 \rangle$

We label the points of $3K_{8t}$ with the points in $V = \{\infty, 0, 1, 2, \ldots, 8t-2\} = Z_{8t-1} \bigcup \{\infty\}$, where Z_{8t-1} is the set of the integers modulo $8t-1$. The aim here is to construct a difference family $[D_1, \ldots, D_{3t}]$ where all differences in $\{1, 2, \ldots, \frac{8t-2}{2}\}$ appear exactly 3 times except one difference d which occurs twice in $3t-1$ difference sets, and then it occurs once in the difference set $\langle d, \infty, 0 \rangle$. We expand the difference sets in the difference family modulo $8t-1$ to obtain an $H_2(8t, 3)$ where each difference set (or base block) is expanded to obtain $8t-2$ additional blocks (i.e., H_2 graphs). The total number of blocks after the expansion is $3t(8t-1)$, each of which corresponds to an H_2 graph in the decomposition, and each edge between a pair of distinct points appears 3 times in these H_2 graphs as required.

Example 1 *For an $H_2(8, 3)$, we have $t = 1$, so we need 3 difference sets in a difference family where each difference in $\{1, 2, 3\}$ appears exactly 3 times. One such difference family is $[\langle 3, 0, 2 \rangle, \langle 2, 0, 3 \rangle, \langle 1, \infty, 0 \rangle]$. Next, we expand the difference sets cyclically modulo 7 to obtain an $H_2(8, 3)$.*

Hurd and Sarvate [4] show that the necessary condition for existence of an $H_2(v, 3)$ is $v(v-1) \equiv 0 \pmod 8$, i.e., $v = 8t$ or $8t+1$, for all $t \geq 1$. They proofed that this necessary condition is sufficient for the existence of an $H_2(v, 3)$, except possibly for the cases $v = 8t$ and $24 \leq v \leq 1680$. Sarvate and Zhang resolve all these cases in the affirmative in [8]. Although an $H_2(8t, 3)$ exists for all $t \geq 1$ [8], the different techniques used for obtaining decompositions can be very interesting and intriguing. Our focus here is to present some new techniques believing that they may be useful for other combinatorial design problems but mainly to ask if they can be generalized to obtain another unified existence proof for $H_2(v, \lambda)$.

2 New Techniques and Tools

Procedure SPLIT($\{b_1, b_2, b_3\}, a$): Given a triangle $\{b_1, b_2, b_3\}$ and a new point a, construct three H_2 graphs $\langle a, b_1, b_2\rangle_{H_2}$, $\langle a, b_2, b_3\rangle_{H_2}$ and $\langle a, b_3, b_1\rangle_{H_2}$.

A graphical illustration of SPLIT($\{b_1, b_2, b_3\}, a$) is shown in Figure 3.

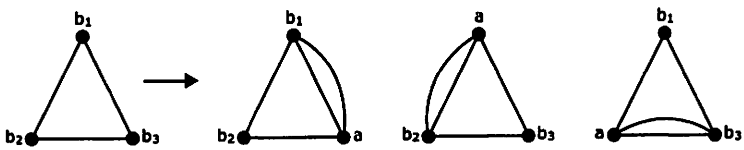

Figure 3: SPLIT($\{b_1, b_2, b_3\}, a$) results in three H_2 graphs $\langle a, b_1, b_2\rangle_{H_2}$, $\langle a, b_2, b_3\rangle_{H_2}$ and $\langle a, b_3, b_1\rangle_{H_2}$.

Clearly, SPLIT($\{b_1, b_2, b_3\}, a$) results in three H_2 graphs where each of the three pairs ($\{a, b_1\}$, $\{a, b_2\}$, $\{a, b_3\}$) involving the new point a appears three times and the three pairs ($\{b_1, b_2\}$, $\{b_2, b_3\}$, $\{b_1, b_3\}$) of the original triangle appear once.

Procedure COPY($\{b_1, b_2, b_3\}, a$): Given a triangle $\{b_1, b_2, b_3\}$ and a new point a, construct three H_2 graphs $\langle b_1, b_2, b_3\rangle_{H_2}$, $\langle b_1, b_3, b_2\rangle_{H_2}$ and $\langle a, b_2, b_3\rangle_{H_2}$.

A graphical illustration of COPY($\{b_1, b_2, b_3\}, a$) is shown in figure 4.

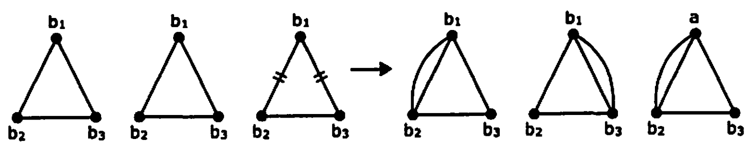

Figure 4: COPY($\{b_1, b_2, b_3\}, a$) results in three H_2 graphs $\langle b_1, b_2, b_3\rangle_{H_2}$, $\langle b_1, b_3, b_2\rangle_{H_2}$ and $\langle a, b_2, b_3\rangle_{H_2}$.

Clearly, COPY($\{b_1, b_2, b_3\}, a$) results in three H_2 graphs, where each edge (or pair) of the original triangle appears three times, and the edge between a and b_2

appears twice, and the edge between a and b_3 appears once and no edge is created between a and b_1.

2.1 Techniques Utilizing STS and Difference Sets

A BIBD$(v, 3, 1)$ is also called a Steiner triple system (STS), and it is *cyclic* if it has an automorphism that is a permutation consisting of a single cycle of length v. The necessary condition for the existence of a STS(v) is that $v \equiv 1, 3 (\text{mod } 6)$ ([5]). Also, there exists a cyclic STS(v) for all $v \equiv 1, 3 (\text{mod } 6)$ except $v = 9$ ([6]). Difference sets can be used to construct a cyclic STS(v). The number of difference sets in a difference family for a STS(v) is s if $v = 6s + 1$ or $s + 1$ if $v = 6s + 3$ where one of the $(s + 1)$ difference sets is a short difference set. This short difference set when developed gives a parallel class. This concept will be needed in the proofs of upcoming results.

For a STS(v) on $V = \{0, 1, 2, \ldots, 14\}$ there should be 3 difference sets in a difference family, e.g., $\{0, 1, 4\}$, $\{0, 2, 8\}$ and $\{0, 5, 10\}$. Note that the first two difference sets account for the differences $1, 2, 3, 4, 6$ and 7. Each of the first two can be be expanded cyclically modulo 15 to obtain 15 triangles/blocks. The third difference set $\{0, 5, 10\}$ is a short difference set and when expanded cyclically modulo 15, we get 5 triangles, $\{0, 5, 10\}$, $\{1, 6, 11\}$, $\{2, 7, 12\}$, $\{3, 8, 13\}$ and $\{4, 9, 14\}$ (note if we continue, the next one would be $\{5, 10, 0\}$, same as $\{0, 5, 10\}$). These 5 triangles form a *parallel class* (or *resolution class*) since they partition the point set V. The 35 triangles thus obtained give a solution to STS(15). Note that the short difference set is $\{0, 2s + 1, 4s + 2\}$, (if $V = \{0, \ldots, 6s + 2\}$) and gives the difference $2s + 1$ and generates $2s + 1$ triangles(blocks) for the design when expanded cyclically modulo $6s + 3$.

Theorem 1 *An $H_2(8t, 3)$ exists for $t \equiv 1 (mod\ 7)$.*

Proof: Let $s \equiv 1 (\text{mod } 8) = 8z + 1$ $(z \geq 0)$, then an $H_2(s, 3)$ exists [4]. Let the set of points of an $H_2(s, 3)$ be $U = \{\infty_1, \ldots, \infty_s\}$. Also, let $v = 6s + 1$, then cyclic STS(v) exists [6]. Let the set of v points of STS(v) be $V = \{0, 1, 2, \ldots, 6s\}$ and $D_i = \{a_i, b_i, c_i\}, i = 1, \ldots, s$ be the i^{th} difference set for the STS(v).

For $i = 1, \ldots, s$, we perform the procedure COPY(D_i, ∞_i). Each procedure results in three H_2 graphs, to be used as the difference sets for the H_2 graph decomposition. The three resulting difference sets are $\langle a_i, b_i, c_i \rangle$, $\langle a_i, c_i, b_i \rangle$ and $\langle \infty_i, b_i, c_i \rangle$. Notice that these three difference sets give the difference $|a_i - b_i|$ three times, the difference $|a_i - c_i|$ three times and the difference $|b_i - c_i|$ three times. Since the s difference sets for STS(v) give each difference between 1 and $3s$ exactly once, after s procedures are performed, we will have $3s$ difference sets which give each difference between 1 and $3s$ three times. Therefore, these $3s$ difference sets can be used to form a difference family to generate H_2 graphs by expanding each difference set modulo $6s + 1$ cyclically.

Note that when we expand the difference set $\langle\infty_i, b_i, c_i\rangle$ ($i = 1, \ldots, s$) cyclically, ∞_i remains the same in each expansion. Furthermore, each point in V appears once in the second position in some H_2 block and once in the third position in another H_2 block after the cyclical expansion on the difference set $\langle\infty_i, b_i, c_i\rangle$. This implies that there are three edges between ∞_i and every point in V in the H_2 blocks generated. The H_2 graphs generated by the $3s$ difference sets in the difference family together with the H_2 graphs from the $H_2(s, 3)$ on the points in U form an $H_2(v+s, 3) = H_2(7s+1, 3) = H_2(56z+8, 3) = H_2(8(7z+1), 3)$, i.e., an $H_2(8t, 3)$ exists for $t \equiv 1 (\text{mod } 7)$. □

Theorem 2 *An $H_2(8t, 3)$ exists for $t \equiv 6(mod\ 7)$.*

Proof: Let $s = 8z + 6$. Since $s + 3 \equiv 1(\text{mod } 8)$, an $H_2(s+3, 3)$ exists. Let the set of points of an $H_2(s+3, 3)$ be $U = \{\infty_1, \ldots, \infty_{s+1}, \infty_{s+2}, \infty_{s+3}\}$. Also, let $v = 6s+3$ (note $s = 8z+6 \geq 6, v = 6s+3 \geq 39$), then cyclic STS($v$) exists [6] and there are $s+1$ different sets. Let the set of v points of STS(v) be $V = \{0, 1, 2, \ldots, 6s+2\}$ and $D_i = \{a_i, b_i, c_i\}, i = 1, \ldots, s+1$ be the i^{th} difference set for the STS(v). The last difference set D_{s+1} is a short difference set $\{0, 2s+1, 4s+2\}$.

For $i = 1, \ldots, s$, we perform the procedure COPY(D_i, ∞_i). Each procedure results in three difference sets $\langle a_i, b_i, c_i\rangle$, $\langle a_i, c_i, b_i\rangle$ and $\langle\infty_i, b_i, c_i\rangle$. Except for difference $2s+1$, each difference between 1 and $3s+1$ appears three times in the $3s$ difference sets obtained. Expand each of the $3s$ difference sets cyclically modulo $6s+3$ to obtain H_2 graphs. For the short difference set D_{s+1}, first expand it cyclically modulo $6s+3$ to obtain a total of $2s+1$ triangles/blocks which form a parallel class $\{B_1, \ldots, B_{2s+1}\}$. For $k = s+1, s+2, s+3$ and $j = 1, \ldots, 2s+1$, perform SPLIT(B_j, ∞_k), and as a result, we will have $9(2s+1)$ H_2 graphs where there are three edges between ∞_k ($k = s+1, s+2, s+3$) and every point in V and three edges between each pair in the parallel class. Combine all the H_2 graphs obtained and the H_2 graphs from the $H_2(s+3, 3)$, we have an $H_2(v+s+3, 3) = H_2(7s+6, 3) = H_2(56z+48, 3) = H_2(8(7z+6), 3)$. □

Let us call the procedure performed in the proofs of Theorem 1 and Theorem 2 **DSET**(v, A, n) where $v = 6s+1$ or $6s+3$ and A is the collection of n new points ∞'s. If $v = 6s+1$, then $n = s$. If $v = 6s+3$, then $n = s+3$. The H_2 graphs resulted in DSET(v, A, n) contain three edges between each pair of distinct points from V and three edges between every pair of points where one point is from V and the other point is from A.

Theorem 3 *An $H_2(8t, 3)$ exists for $t \equiv 2(mod\ 15)$.*

Proof: Let $s \equiv 1(\text{mod } 8) = 8z+1$ ($z \geq 0$). Since $9s \equiv 1(\text{mod } 8)$, an $H_2(9s, 3)$ exists. Let the set of points of an $H_2(9s, 3)$ be $U = \{\infty_1^1, \ldots, \infty_1^9, \ldots, \infty_s^1, \ldots, \infty_s^9\}$. Also, let $v = 6s+1$, then cyclic STS(v) exists [6]. Let the set of v points of STS(v) be $V = \{0, 1, 2, \ldots, 6s\}$ and $D_i = \{a_i, b_i, c_i\}, i = 1, \ldots, s$ be the i^{th} difference set for the STS(v).

For $i = 1, \ldots, s$, we obtain 9 H_2 difference sets $\langle \infty_i^1, a_i, b_i \rangle$, $\langle \infty_i^2, a_i, b_i \rangle$, $\langle \infty_i^3, a_i, b_i \rangle$, $\langle \infty_i^4, a_i, c_i \rangle$, $\langle \infty_i^5, a_i, c_i \rangle$, $\langle \infty_i^6, a_i, c_i \rangle$, $\langle \infty_i^7, b_i, c_i \rangle$, $\langle \infty_i^8, b_i, c_i \rangle$ and $\langle \infty_i^9, b_i, c_i \rangle$. Notice that each difference appears three times. These $9s$ difference sets form a difference family. Expand each difference set in the difference family cyclically modulo $6s+1$ to obtain H_2 graphs. Combining the resulting H_2 graphs and the H_2 graphs from the $H_2(9s, 3)$ on U, we have an $H_2(v + 9s, 3) = H_2(15s + 1, 3) = H_2(120z + 16, 3) = H_2(8(15z + 2), 3)$, i.e., an $H_2(8t, 3)$ exists for $t \equiv 2 (\text{mod } 15)$. □

Theorem 4 *An $H_2(8t, 3)$ exists for $t \equiv 12(mod\ 15)$.*

Proof: Let $s = 8z + 6$. Since $s \equiv 6(\text{mod } 8)$, $9s + 3 \equiv 1(\text{mod } 8)$, an $H_2(9s + 3, 3)$ exists. Let the set of points of an $H_2(9s+3, 3)$ be $U = \{\infty_1^1, \ldots, \infty_1^9, \ldots, \infty_s^1, \ldots, \infty_s^9, \infty_{s+1}, \infty_{s+2}, \infty_{s+3}\}$. Also, let $v = 6s + 3$, then cyclic STS(v) exists [6] and there are $s + 1$ different sets. Let the set of v points of STS(v) be $V = \{0, 1, 2, \ldots, 6s + 2\}$ and $D_i = \{a_i, b_i, c_i\}, i = 1, \ldots, s + 1$ be the i^{th} difference set for the STS(v). The last difference set D_{s+1} is a short difference set $\{0, 2s + 1, 4s + 2\}$.

Similar to the proof of Theorem 3, for $i = 1, \ldots, s$, we obtain 9 H_2 difference sets using the 9 ∞'s: $\infty_i^1 \cdots \infty_i^9$. Expand each of the 9 H_2 difference sets cyclically modulo $6s+3$ to obtain H_2 graphs. For the short difference set D_{s+1}, similar to the proof of Theorem 2, we first expand it cyclically modulo $6s + 3$ to obtain a total of $2s + 1$ blocks/triangles which form a parallel class $\{B_1, \ldots, B_{2s+1}\}$. For $k = s + 1, s + 2, s + 3$ and $j = 1, \ldots, 2s + 1$, perform SPLIT(B_j, ∞_k), and as a result, we will have $9(2s + 1)$ H_2 graphs using 3 ∞'s. Combine all the H_2 graphs obtained and the H_2 graphs from the $H_2(9s + 3, 3)$ on U, we have an $H_2(v+9s+3, 3) = H_2(15s+6, 3) = H_2(120z+96, 3) = H_2(8(15z+12)$. □

Let us call the procedure performed in the proofs of Theorem 3 and Theorem 4 **NINE-DSET**(v, A, n) where $v = 6s + 1$ or $6s + 3$ and A is the collection of n new points ∞'s. If $v = 6s + 1$, then $n = 9s$. If $v = 6s + 3$, then $n = 9s + 3$. The H_2 graphs resulted in NINE-DSET(v, A, n) contain three edges between each pair of distinct points from V and three edges between every pair of points where one point is from V and the other point is from A.

Theorem 5 *An $H_2(8t, 3)$ exists for $t \equiv 7(mod\ 11)$.*

Proof: Let $s \equiv 5(\text{mod } 8) = 8z + 5$ $(z \geq 0)$. Since $5s \equiv 1(\text{mod } 8)$, an $H_2(5s, 3)$ exists. Let the set of points of an $H_2(5s, 3)$ be $U = \{\infty_1^1, \ldots, \infty_1^5, \ldots, \infty_s^1, \ldots, \infty_s^5\}$. Also, let $v = 6s + 1$, then cyclic STS(v) exists [6]. Let the set of v points of STS(v) be $V = \{0, 1, 2, \ldots, 6s\}$ and $D_i = \{a_i, b_i, c_i\}, i = 1, \ldots, s$ be the i^{th} difference set for the STS(v).

For $i = 1, \ldots, s$, we obtain 6 H_2 difference sets $\langle a_i, b_i, c_i \rangle$, $\langle \infty_i^1, a_i, c_i \rangle$, $\langle \infty_i^2, b_i, c_i \rangle$, $\langle \infty_i^3, a_i, b_i \rangle$, $\langle \infty_i^4, a_i, c_i \rangle$ and $\langle \infty_i^5, b_i, c_i \rangle$. Notice that each difference appears three times. These $6s$ difference sets obtained using $5s$ ∞'s form a difference family. Expand each difference set in the difference family cyclically modulo $6s+1$ to obtain H_2 graphs. Combining the resulting H_2 graphs and the H_2 graphs from the $H_2(5s,3)$ on U, we have an $H_2(v+5s,3) = H_2(11s+1,3) = H_2(88z+56,3) = H_2(8(11z+7),3)$, i.e., an $H_2(8t,3)$ exists for $t \equiv 7 (\bmod 11)$. □

Theorem 6 *An $H_2(8t,3)$ exists for $t \equiv 9 (mod\ 11)$.*

Proof: Let $s \equiv 6 (\bmod 8) = 8z+6$ $(z \geq 0)$. Since $5s+3 \equiv 1 (\bmod 8)$, an $H_2(5s+3,3)$ exists. Let the set of points of an $H_2(5s+3,3)$ be $U = \{\infty_1^1, \ldots, \infty_1^5, \ldots, \infty_s^1, \ldots, \infty_s^5, \infty_{s+1}, \infty_{s+2}, \infty_{s+3}\}$. Also, let $v = 6s+3$, then cyclic STS(v) exists [6] and there are $s+1$ different sets. Let the set of v points of STS(v) be $V = \{0,1,2,\ldots,6s+2\}$ and $D_i = \{a_i, b_i, c_i\}, i = 1,\ldots,s+1$ be the i^{th} difference set for the STS(v). The last difference set D_{s+1} is a short difference set $\{0, 2s+1, 4s+2\}$.

Similar to the proof of Theorem 5, for $i = 1,\ldots,s$, we obtain 6 H_2 difference sets using $5s$ ∞'s. Expand each of the $6s$ H_2 difference sets cyclically modulo $6s+3$ to obtain H_2 graphs. For the short difference set D_{s+1}, similar to the proof of Theorem 2, we first expand it cyclically modulo $6s+3$ to obtain a total of $2s+1$ triangles/blocks which form a parallel class $\{B_1,\ldots,B_{2s+1}\}$. For $k = s+1, s+2, s+3$ and $j = 1,\ldots,2s+1$, perform SPLIT(B_j, ∞_k), this will result in $9(2s+1)$ H_2 graphs using 3 ∞'s. Combine all the H_2 graphs obtained and the H_2 graphs from the $H_2(5s+3,3)$ on U, we have an $H_2(v+5s+3,3) = H_2(11s+6,3) = H_2(88z+72,3) = H_2(8(11z+9),3)$, i.e., an $H_2(8t,3)$ exists for $t \equiv 9 (\bmod 11)$. □

Let us call the procedure performed in the proofs of Theorem 5 and Theorem 6 **FIVE-DSET**(v, A, n) where $v = 6s+1$ or $6s+3$ and A is the collection of n new points ∞'s. If $v = 6s+1$, then $n = 5s$. If $v = 6s+3$, then $n = 5s+3$. The H_2 graphs resulted in FIVE-DSET(v, A, n) contain three edges between each pair of distinct points from V and three edges between every pair of points where one point is from V and the other point is from A.

2.2 Techniques Utilizing 2-Factorization

Theorem 7 *An $H_2(8t,3)$ exists for $t \equiv 2 (mod\ 5)$.*

Proof: Let $v = 2n+1$ $(n \geq 1)$, then a K_v has n 2-factors $T_1,\ldots,T_n$. For $i = 1,\ldots,n$, let $T_i = \{(a_1^i, b_1^i), \ldots, (a_v^i, b_v^i)\}$ where each pair represents an edge between the two points. For $i = 1,\ldots,n$ and $j = 1,\ldots,v$ and $w = 1,2,3$, construct a $\langle \infty_i^w, a_j^i, b_j^i \rangle_{H_2}$. That is, three new points (∞'s) are applied to each of the v edges in T_i to create H_2 graphs. As a result, $3n$ new points are used to create H_2 graphs which contain three edges between any pair of two distinct points from V and three edges between a point from V and a new point (note that for any new point used in a 2-factor to create H_2 graphs, every point in V appears exactly once in the second position and once in the third position, so three edges between every point in V and the new point are created in these H_2 graphs constructed). If $n \equiv 3 (\text{mod } 8)$, then $3n \equiv 1 (\text{mod } 8)$, so an $H_2(3n, 3)$ exists. Obtain an $H_2(3n, 3)$ on the $3n$ new points. Combine all the H_2 graphs obtained, we have an $H_2(v + 3n, 3) = H_2(5n + 1, 3) = H_2(5(8z + 3) + 1, 3) = H_2(8(5z + 2), 3)$, i.e., an $H_2(8t, 3)$ exists for $t \equiv 2 (\text{mod } 5)$. □

Let us call the procedure used in the proof of Theorem 7 **FACTOR**(T, A, n') where $v = 2n + 1$, T is the collection of n 2-factors, A is the set of $n' = 3n$ new points.

3 Summary

In this paper, we discussed various procedures that can be used for $H_2(8t, 3)$ decompositions. It is interesting to see that so many well-known combinatorial designs such as STS, difference sets and 2-factorization can be utilized for the problem under consideration. We hope that the procedures developed in this note will be used to give a more unified proof for the existence of $H_2(v, \lambda)$ in general and the existence of $H_2(8t, 3)$ in particular.

References

[1] P. Adams, D. Bryant, and M. Buchanan. A survey on the existence of G-Designs. *Journal of Combinatorial Designs*, 5:373–410, 2008.

[2] H. Agrawal. Some generalizations of distinct representives with applications to statistical designs. *Ann. Math. Statis.*, 37:25–528, 1966.

[3] H. Chan. Stanton graph decompositions. *Bulletin of the ICA*, 64:21–29, 2012.

[4] S. P. Hurd and D. G. Sarvate. Graph decompositions of $K(v, \lambda)$ into modified triangles using Langford and Skolem sequences. Accepted, Ars Combinatoria.

[5] T. P. Kirkman. On a problem in combinations. *Cambridge Dublin Math. J.*, 2:191–204, 1847.

[6] R. Peltesohn. Eine lôsung der beiden heffterschen differenzenprobleme. *Compositio Math.*, 6:251–257, 1939.

[7] N. Punim. Loop designs. *J. Combin. Math. Combin. Comput.*, 78:261–272, 2011.

[8] D. G. Sarvate and L. Zhang. Decomposition of a $3K_{8t}$ into H_2 graphs. Accepted, Ars Combinatoria.

[9] D. R. Stinson. *Combinatorial Designs: Constructions and Analysis*. Springer, New York, 2004.