From Combinatorial Problems to Graph Colorings

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Abstract

Historically, a number of problems and puzzles have been introduced that initially appeared to have no connection to graph colorings but, upon further analysis, suggested graph colorings problems. In this paper, we discuss two combinatorial problems and several graph colorings problems that are inspired by these two problems. We survey recent results and open questions in this area of research as well as some relationships among these coloring parameters and well-known colorings and labelings.

Key Words: coloring, neighbor-distinguishing, vertex-distinguishing.

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1 Introduction

Graph coloring is one of the most popular research areas in graph theory. Among the most studied colorings are proper vertex colorings and proper edge colorings. A *proper vertex coloring* of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned distinct colors. The minimum number of colors required in a proper vertex coloring of G is the *chromatic number* $\chi(G)$ of G. A *proper edge coloring* of a graph G is an assignment of colors to the edges of G such that adjacent edges are assigned distinct colors and the minimum number of colors required in a proper edge coloring of G is the *chromatic index* $\chi'(G)$ of G. During the past 50 years, a number of problems and puzzles have been introduced that initially appeared to have no connection to graph colorings. However, upon further analysis, all of these suggested graph colorings problems. In this paper, we discuss two combinatorial problems and several graph colorings problems that are inspired by these two problems. We survey recent results and open questions in this area of research as well as relationships among these coloring parameters and some well-known colorings and labelings from the literature. We refer to [7] for graph theory notation and terminology not described in this paper.

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1.1 A Checkerboard Problem

Suppose that the squares of an $m \times n$ checkerboard (m rows and n columns), where $1 \leq m \leq n$ and $n \geq 2$, are alternately colored black and red. Figure 1(a) shows a 5×7 checkerboard where a shaded square represents a black square. Two squares are said to be *neighboring* if they belong to the same row or to the same column and there is no square between them. Thus every two neighboring squares are of different colors. A combinatorial problem was introduced by Gary Chartrand in 2010 and the following conjecture was stated [31].

The Checkerboard Conjecture *It is possible to place coins on some of the squares of an $m \times n$ checkerboard (at most one coin per square) such that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity.*

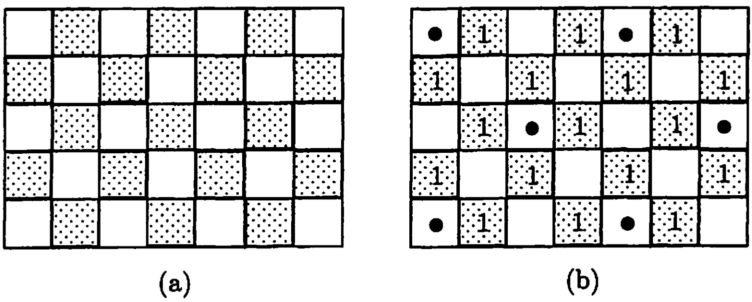


Figure 1: A 5×7 checkerboard and a coin placement on the checkerboard

Figure 1(b) shows a placement of 6 coins on the 5×7 checkerboard such that the number of coins on neighboring squares of every red square is even and the number of coins on neighboring squares of every black square is odd. Thus for every two squares of different colors, the numbers of coins on neighboring squares are of opposite parity. Consequently, the Checkerboard Conjecture is true for a 5×7 checkerboard. Observe that all 6 coins on the 5×7 checkerboard of Figure 1(b) are placed only on red squares. Thus the number of coins on neighboring squares of every red square is 0 and is therefore even, while the number of coins on neighboring squares of each black square is 1 and this is shown in Figure 1(b) as well.

In [31] it was shown that this problem could be placed in a graph theory setting. For example, let G be the graph whose vertices are the squares of the checkerboard and where two vertices of G are adjacent if the corresponding squares are neighboring. Then G is the grid (bipartite graph) $P_m \square P_n$ (or $P_m \times P_n$) which is the Cartesian product of the paths P_m and P_n. This suggests a function (coloring) c on $G = P_m \square P_n$, where $c : V(G) \to \mathbb{Z}_2$ such that

$$c(v) = \begin{cases} 0 & \text{if } v \text{ corresponds to a square with no coin} \\ 1 & \text{if } v \text{ corresponds to a square containing a coin.} \end{cases}$$

This induces another coloring $\sigma : V(G) \to \mathbb{Z}_2$ defined by

$$\sigma(v) = \sum_{u \in N(v)} c(u) \text{ in } \mathbb{Z}_2 \tag{1}$$

where $N(v)$ is the neighborhood of a vertex v (the set of vertices adjacent to v) and the addition is performed in $\mathbb{Z}_2$. If σ is a proper coloring, then the checkerboard problem has a solution on the checkerboard represented by G. With the aid of graph colorings described above, it was shown in [32] that the Checkerboard Conjecture is true for a checkerboard of any size.

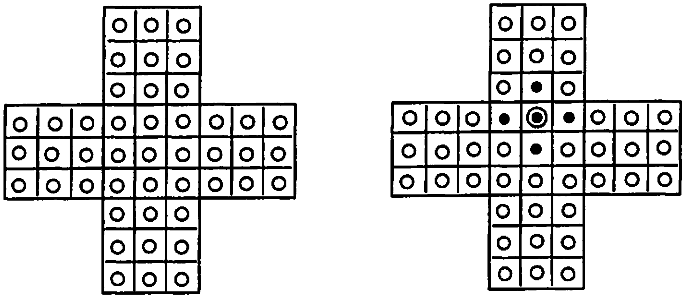
The Checkerboard Theorem *For every pair m, n of positive integers, it is possible to place coins on some of the squares of an $m \times n$ checkerboard (at most one coin per square) such that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity.*

If $\mathbb{Z}_2$ is replaced by $\mathbb{Z}_k$ for an integer $k \geq 2$, then this checkerboard problem gave rise to a new coloring in [31], which we will discuss in Section 2.1.

1.2 A Lights Out Problem

Another recreational problem concerns the electronic game of "Lights Out" consisting of a cube, each of whose six faces contains 9 squares in 3 rows and 3 columns. Thus there are 54 squares in all. Figure 2(a) shows the "front" of the cube as well as the faces on the top, bottom, left and right. The back of the cube is not shown. A button is placed on each square of a Lights Out cube containing a light which is either on or off. When a button is pushed, the light on that square changes from on to off or from off to on. Moreover, not only is the light on that square reversed when its button is pushed but the lights on its four neighboring squares (top, bottom, left, right) are reversed as well. The four neighboring squares of the middle square of a face lie on the same face as the middle square.

Only three neighboring squares of a "side square" (top middle, bottom middle, left middle, right middle) lie on the same face of such a square, with the remaining neighboring square lying on an adjacent face as the middle square. For example, if all 54 lights are on initially and the button on the top middle square on the front face is pushed, then this light goes off as well as the lights on its four neighboring squares (see Figure 2(b)). Only two neighboring squares of a "corner" square lie on the same face as that square; the other two neighboring squares lie on two other faces.



(a) (b)

Figure 2: Lights Out Game

One goal of the game "Lights Out" is to begin with such a cube where all lights are on and to push a set of buttons so that, at the end, all lights are out. Two observations are immediate: (1) No button need to be pushed more than once. (2) The order in which the buttons are pushed is immaterial.

This game has a setting in graph theory. Let each square be a vertex and join each vertex to the vertices corresponding to its neighboring squares. This results in a 4-regular graph G of order 54. The goal is to find a collection S of vertices of G, which correspond to the buttons to be pushed, such that every vertex of G is in the closed neighborhood of an odd number of vertices of S. This says that each vertex v, corresponding to a lit square, will have its light reversed an odd number of times, resulting in the light being turned out.

The Lights Out Game can in fact be played on any connected graph G on which there is a light at each vertex of G. The game has a solution for the graph G if all lights are on initially and if there exists a collection S of vertices which, when the button on each vertex of S is pushed, all lights of G will be out. This problem has another interpretation. A vertex v of a graph G *dominates* a vertex u if u belong to the closed neighborhood $N[v]$ of v (consisting of v and the vertices in the (open) neighborhood $N(v)$ of

v). The Lights Out Game has a solution for a given graph G if and only if G contains a set S of vertices such that every vertex of G is dominated by an odd number of vertices of S. In [41] Sutner showed that every graph has this property and so the Lights Out Game is solvable on every graph.

As discussed in [8], the Lights Out Game is also equivalent to beginning with a connected graph G where every vertex of G is initially assigned the color 1 in $\mathbb{Z}_2$ (corresponding to its light being on) and finding a set S of vertices of G and a coloring $c : V(G) \to \mathbb{Z}_2$ such that

$$c(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \notin S. \end{cases}$$

A new coloring $\sigma' : V(G) \to \mathbb{Z}_2$ induced by c is defined by

$$\sigma'(v) = 1 + \sum_{u \in N[v]} c(u) \ \text{ in } \mathbb{Z}_2. \tag{2}$$

The goal of the Lights Out Game is therefore to have $\sigma'(v) = 0$ for all $v \in V(G)$. The Lights Out Game suggests a new coloring problem introduced in [8], which we will discuss in Section 2.2.

2 Neighbor-Distinguishing Vertex Colorings

A coloring that provides a method of distinguishing every two adjacent vertices is said to be *neighbor-distinguishing.* Thus a proper vertex coloring of a graph is neighbor-distinguishing. A number of neighbor-distinguishing vertex colorings other than the standard proper colorings have been introduced in the literature (see [9, 10, 11, 12, 13], for example). In this section, we describe two recent neighbor-distinguishing vertex colorings, each of which is induced by a given coloring (proper or nonproper) and defined on sums of colors.

2.1 Modular Colorings

In 2010, a neighbor-distinguishing vertex coloring was introduced in [31] for the purpose of finding solutions to the checkboard problem as described in Section 1.1. For a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* $\sigma(v)$ of a vertex v of G is defined as

$$\sigma(v) = \sum_{u \in N(v)} c(u) \ \text{ in } \mathbb{Z}_k \tag{3}$$

where the addition is performed in $\mathbb{Z}_k$. Thus the vertex coloring c induces another vertex coloring $\sigma : V(G) \to \mathbb{Z}_k$ of G. If $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for every two adjacent vertices x and y of G, then the coloring c is called a *modular k-coloring* of G. The minimum k for which G has a modular k-coloring is called the *modular chromatic number* of G and is denoted by $\mathrm{mc}(G)$. Modular coloring in graphs have been studied in [17, 31, 32, 33]. To illustrate the concepts introduced above, Figure 3 shows a modular 3-coloring of a bipartite graph G (where the color of a vertex is placed within the vertex) together with the color sum $\sigma(v)$ for each vertex v of G (where the color sum of a vertex is placed next to the vertex). In fact, $\mathrm{mc}(G) = 3$ for this graph G.

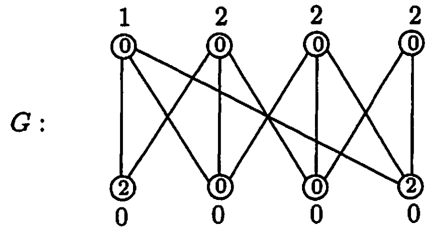


Figure 3: A bipartite graph G with $\mathrm{mc}(G) = 3$

The Checkerboard Theorem can consequently be stated in terms of graphs and modular colorings as follows.

Theorem 2.1 *For every two positive integers m and n with $mn \geq 2$,*

$$\mathrm{mc}(P_m \square P_n) = 2.$$

Modular colorings are closely related to another neighbor-distinguishing vertex colorings of a graph, called *sigma colorings*, which were introduced and studied in [12]. In the case of sigma colorings, a given coloring c is a function $c : V(G) \to \mathbb{N}$ and the color sum $\sigma(v)$ of a vertex v has the same formula as defined in (3) except that the addition is performed in $\mathbb{N}$. Among the results on modular colorings obtained in [31] are the following.

Theorem 2.2 *For every graph G,* $\mathrm{mc}(G) \geq \chi(G)$.

Theorem 2.3 *If G is a k-chromatic graph ($k \geq 2$) with maximum degree Δ, then*

$$\mathrm{mc}(G) \leq \Delta(\Delta + 1)^{k-2} + 1.$$

In particular, G is a bipartite graph, then $\mathrm{mc}(G) \leq \Delta(G) + 1$.

Theorem 2.4 *If T is a nontrivial tree, then* $\mathrm{mc}(T) = 2$ *or* $\mathrm{mc}(T) = 3$.

A nontrivial tree T is of type one if $\mathrm{mc}(T) = 2$ and is of type two if $\mathrm{mc}(T) = 3$. It is shown in [33] that all nontrivial trees of diameter at most 6 are of type one. A *caterpillar* is a tree of order 3 or more, the removal of whose end-vertices produces a path. A characterization of caterpillars that are of type two was established in [33]. An efficient algorithm has been established to compute the modular chromatic number of a given tree in [17]. Furthermore, modular chromatic numbers are determined for several classes of graphs in [31].

- If G is a complete multipartite graph, then $\mathrm{mc}(G) = \chi(G)$.
- For each integer $n \geq 3$, $\mathrm{mc}(C_n) = 2$ if $n \equiv 0 \pmod 4$ and $\mathrm{mc}(C_n) = 3$ otherwise.
- If G is a bipartite graph such that either (i) G contains a vertex that is adjacent to all vertices in a partite set or (ii) one of its partite sets consists only of odd vertices, then $\mathrm{mc}(G) = 2$.
- If G is a bipartite graph the degrees of whose vertices are of the same parity, then $\mathrm{mc}(G \,\square\, K_2) = 2$. In particular, $\mathrm{mc}(Q_n) = 2$ for each positive integer n.
- For every positive integer r, it follows that $r \leq \mathrm{mc}(K_r \,\square\, K_2) \leq r+1$ and $\mathrm{mc}(K_r \,\square\, K_2) = r$ if and only if $r \equiv 2 \pmod 4$. Thus for every integer $r \geq 3$ and $r \not\equiv 2 \pmod 4$, there exists an r-chromatic graph G with $\mathrm{mc}(G) = r+1$.
- For each integer $n \geq 3$, $\mathrm{mc}(W_n) = 4$ where $W_n = C_n \vee K_1$ (the join of C_n and K_1) is the wheel of order $n+1$. Also, for each integer $n \geq 2$, $\mathrm{mc}(P_n \vee K_2) = 4$. Thus there are infinitely many maximal planar graphs G with $\mathrm{mc}(G) = \chi(G)$.

There are many interesting questions in this topic (see [31]).

Problem 2.5 *Does there exist a planar graph whose modular chromatic number is* 5*?*

If the answer to this question is no (and we can verify this), then there is a new Four Color Theorem for which the classic Four Color Theorem is a corollary.

Problem 2.6 *Is there a constant* C *such that* $\mathrm{mc}(G) \leq C$ *for every bipartite graph* G*?*

Problem 2.7 *Is there a graph* G *such that* $\omega(G) < \chi(G) < \mathrm{mc}(G)$*?*

Problem 2.8 *Is there a graph* G *such that* $\mathrm{mc}(G) \geq \chi(G) + 2$*? Is there an upper bound for* $\mathrm{mc}(G)$ *in terms of* $\chi(G)$*?*

2.2 Closed Modular Colorings

Modular colorings and the Lights Out Game have suggested other coloring problems. For a positive integer k and a connected graph G, let $c : V(G) \to \mathbb{Z}_k$ be a vertex coloring where adjacent vertices may be assigned the same color. The *closed color sum* $\overline{\sigma}(v)$ of v as

$$\overline{\sigma}(v) = \sum_{u \in N[v]} c(u) \text{ in } \mathbb{Z}_k \tag{4}$$

where the addition is performed in $\mathbb{Z}_k$. Thus c induces another vertex coloring $\overline{\sigma} : V(G) \to \mathbb{Z}_k$ of G. If u and v are adjacent vertices of a graph G such that $N[u] = N[v]$, then $\overline{\sigma}(u) = \overline{\sigma}(v)$ for every vertex coloring c of G. Therefore, the coloring $\overline{\sigma}$ cannot be neighbor-distinguishing in general. For this reason, additional restrictions are needed.

Two vertices u and v in a connected graph G are *twins* if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, they are referred to as *true twins*; while if u and v are nonadjacent, they are *false twins*. If u and v are adjacent vertices of a graph G such that $N[u] = N[v]$, that is, if u and v are true twins, then $\overline{\sigma}(u) = \overline{\sigma}(v)$ for every vertex coloring c of G. Define a coloring $c : V(G) \to \mathbb{Z}_k$ to be a *closed modular k-coloring* if $\overline{\sigma}(u) \neq \overline{\sigma}(v)$ in $\mathbb{Z}_k$ for all pairs u, v of adjacent vertices for which $N[u] \neq N[v]$ in G (or u and v are not true twins in G). A vertex coloring c is a *closed modular coloring* of G if c is a closed modular k-coloring of G for some positive integer k. That is, in a closed modular coloring c of a graph, $\overline{\sigma}(u) = \overline{\sigma}(v)$ if u and v are true twins, $\overline{\sigma}(u) \neq \overline{\sigma}(v)$ if u and v are adjacent vertices that are not true twins and no condition is placed on $\overline{\sigma}(u)$ and $\overline{\sigma}(v)$ otherwise. The minimum k for which G has a closed modular k-coloring is called the *closed modular chromatic number* of G and is denoted by $\overline{\text{mc}}(G)$. To illustrate these concepts, Figure 4 shows a closed modular 3-coloring of a bipartite graph G (where the color of a vertex is placed within the vertex) together with the color sum $\overline{\sigma}(v)$ for each vertex v of G (where the color sum of a vertex is placed next to the vertex). In fact, $\overline{\text{mc}}(G) = 3$ for this graph G.

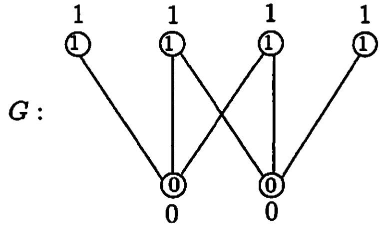


Figure 4: A bipartite graph G with $\overline{\text{mc}}(G) = 3$

These concepts were introduced and studied in [8] and studied further in [36, 37, 38, 39]. It was observed in [8] that the nontrivial complete graphs are the only nontrivial connected graphs G for which $\overline{\text{mc}}(G) = 1$. Next, we present some main results obtained in this topic.

Proposition 2.9 *If G is a nontrivial connected graph, then $\overline{\text{mc}}(G)$ exists. Furthermore, if G contains no true twins, then $\overline{\text{mc}}(G) \geq \chi(G)$.*

By Proposition 2.9, if G is a nontrivial connected graph that contains no true twins, then $\overline{\text{mc}}(G) \geq \chi(G)$. On the other hand, if G contains true twins, then it is possible that $\overline{\text{mc}}(G) < \chi(G)$. In fact, more can be said.

Theorem 2.10 *For each pair a, b of positive integers with $a \leq b$ and $b \geq 2$, there is a connected graph G such that $\overline{\text{mc}}(G) = a$ and $\chi(G) = b$.*

For an edge uv of a graph G, the graph G/uv obtained from G by *contracting the edge uv* has the vertex set $V(G)$ in which u and v are identified. If we denote the vertex $u = v$ in G/uv by w, then $V(G/uv) = (V(G) \cup \{w\}) - \{u, v\}$ and the edge set of G/uv is

$$\begin{aligned} E(G/uv) &= \{xy : xy \in E(G),\ x, y \in V(G) - \{u, v\}\} \cup \\ &\quad \{wx : ux \in E(G) \text{ or } vx \in E(G), x \in V(G) - \{u, v\}\}. \end{aligned}$$

The graph G/uv is referred to as an *elementary contraction* of G.

Theorem 2.11 *Let u and v be true twins of a nontrivial connected graph G. Then G has a closed modular k-coloring if and only if G/uv has a closed modular k-coloring.*

For a nontrivial connected graph G, define the *true twins closure* $TC(G)$ of G as the graph obtained from G by a sequence of elementary contractions of pairs of true twins in G until no such pair remains. In particular, if G contains no true twins, then $TC(G) = G$. Thus $TC(G)$ is a minor of G.The following then is a consequence of Theorem 2.11.

Corollary 2.12 *For a nontrivial connected graph G, $\overline{\text{mc}}(G) = \overline{\text{mc}}(TC(G))$.*

By Corollary 2.12, it suffices to consider nontrivial connected graphs containing no true twins. Closed modular chromatic numbers are determined for several classes of regular graphs. In particular, the following result on regular complete k-partite graphs has been established in [8].

Theorem 2.13 *For integers $r, k \geq 2$, let $G = K_{k(r)}$ be the regular complete k-partite graph, each partite set of which has r vertices.*

(a) *Then $\overline{\text{mc}}(G) = k$ if and only if $1 - r$ and k are relatively prime.*

(*b*) *If* $r \geq 2k+1$, *then* $\overline{\mathrm{mc}}(G) \leq 2k-1$. *Furthermore, if* $k \geq 3$ *and* $r = (2k-3)!+1$, *then* $\overline{\mathrm{mc}}(G) = 2k-1$.

(*c*) *If* r *is even, then* $\overline{\mathrm{mc}}(G) \leq 2k-2$.

(*d*) *If* r *and* $k \geq 4$ *are both even, then* $\overline{\mathrm{mc}}(G) \leq 2k-4$.

Exact values of $\overline{\mathrm{mc}}(G)$ are determined when G is a regular complete k-partite graph G for $2 \leq k \leq 5$. For example, for each integer $r \geq 2$,

$$\overline{\mathrm{mc}}(K_{4(r)}) = \begin{cases} 4 & \text{if } r \text{ is even} \\ 5 & \text{if } r \equiv 3,5,7,9 \pmod{10} \\ 7 & \text{if } r \equiv 1 \pmod{10}. \end{cases}$$

$$\overline{\mathrm{mc}}(K_{5(r)}) = \begin{cases} 5 & \text{if } r \not\equiv 1,6,11,16,21,26 \pmod{30} \\ 6 & \text{if } r \equiv 6,26 \pmod{30} \\ 7 & \text{if } r \equiv 1,11,16,21 \pmod{30} \text{ and } r \not\equiv 1 \pmod{7}. \\ 8 & \text{if } r \equiv 16 \pmod{30} \text{ and } r \equiv 1 \pmod{7}. \\ 9 & \text{if } r \equiv 1,11,21 \pmod{30} \text{ and } r \equiv 1 \pmod{7}. \end{cases}$$

By Theorem 2.13, if each partite set of G has at least $2k+1$ vertices, then $\overline{\mathrm{mc}}(G) \leq 2\chi(G)-1$ and this bound is sharp. Therefore, there is no positive integer constant c for which $\overline{\mathrm{mc}}(G) \leq \chi(G)+c$ for every graph G. In the case of trees, however, it was conjectured in [37] that $\overline{\mathrm{mc}}(T) \leq \chi(T)+1$ for every tree T of order at least 3.

Conjecture 2.14 *For every tree* T *of order at least* 3, $\overline{\mathrm{mc}}(T) \leq 3$.

It was shown in [37] that $\overline{\mathrm{mc}}(T) \leq 4$ for every tree T of order at least 3. In fact, more can be said. A closed modular k-coloring $c : V(G) \to \mathbb{Z}_k$ of a connected graph G of order 3 or more is a *nowhere-zero* coloring if $c(x) \neq 0$ for each vertex x of G. The following result has been established in [37].

Theorem 2.15 *Every tree of order at least* 3 *has a nowhere-zero closed modular* 4*-coloring.*

There is an infinite class of trees that do not have a nowhere-zero closed modular 3-coloring and so Theorem 2.15 cannot be improved. Conjecture 2.14 has been verified in [37, 38] for several classes of trees. For example, if T is a tree of order at least 4 each of whose vertices is odd, then $\overline{\mathrm{mc}}(T) = 3$ and if T is a caterpillar of order at least 3, then $\overline{\mathrm{mc}}(T) \leq 3$. A rooted tree T of order at least 3 is *even* if every vertex of T has an even number of children; while T is *odd* if every vertex of T has an odd number of children. Among the results for rooted trees obtained in [38] are the following.

Theorem 2.16 *Let T be a rooted tree of order at least 3.*

(a) *If T is an even rooted tree, then $\overline{\mathrm{mc}}(T) = 2$.*

(b) *If T is an odd rooted tree having no vertex with exactly one child, then $\overline{\mathrm{mc}}(T) \leq 3$.*

For each integer $p \in \{0, 1, 2, 3, 4, 5\}$, an odd rooted tree T of order at least 3 having root v is said to be of *type* p if $d(v, u) \equiv p \pmod 6$ for every leaf u in T.

Theorem 2.17 *For each integer $p \in \{0, 1, 2, 3, 4, 5\}$, let T be an odd rooted tree of order at least 3 that is of type p. Then $\overline{\mathrm{mc}}(T) = 2$ if and only if $p \neq 1$.*

For a nonempty subset $S \subseteq \{0, 2, 3, 4, 5\}$, an odd rooted tree T having root v is said to be of *type* S if for every leaf u in T, $d(v, u) \equiv p \pmod 6$ for some $p \in S$ and for each $p \in S$, there is at least one leaf u in T such that $d(v, u) \equiv p \pmod 6$. In particular, if $S = \{p\}$ where $p \in \{0, 2, 3, 4, 5\}$, then T is of type p.

Theorem 2.18 *Let S be a nonempty subset of $\{0, 2, 3, 4, 5\}$ such that S contains at most one of 2 and 5 and at most one of 0 and 3. If T is an odd rooted tree of order at least 3 that is of type S, then $\overline{\mathrm{mc}}(T) = 2$.*

For a nontrivial connected graph G, recall that $G \,\square\, K_2$ denotes the Cartesian product of G and K_2. The exact values of $\overline{\mathrm{mc}}(G \,\square\, K_2)$ where $G \in \{K_n, P_n, C_n\}$ have been determined in [39]. For each integer $n \geq 3$,

$$\begin{aligned}
\overline{\mathrm{mc}}(K_n \,\square\, K_2) &= n \\
\overline{\mathrm{mc}}(P_n \,\square\, K_2) &= \begin{cases} 2 & \text{if } n \equiv 0, 2, 4, 6, 7 \pmod 8 \\ 3 & \text{if } n \equiv 1, 3, 5 \pmod 8. \end{cases} \\
\overline{\mathrm{mc}}(C_n \,\square\, K_2) &= \begin{cases} 2 & \text{if } n \equiv 0 \pmod 8 \\ 3 & \text{otherwise} \end{cases}
\end{aligned}$$

For every connected graph G without true twins that we have encountered thus far, $\overline{\mathrm{mc}}(G) \leq 2\chi(G) - 1$. Thus, the following is the main open question on this topic.

Problem 2.19 *Let G be a connected graph G of order at least 3. Is it true that*

$$\overline{\mathrm{mc}}(G) \leq 2\chi(G) - 1?$$

3 Neighbor-Distinguishing Edge Colorings

Edge colorings (proper or nonproper) have also been introduced to distinguish every pair of adjacent vertices in a graph (see [1, 5, 22, 43] or [14, p. 385-391], for example). In this case, an edge coloring of a graph G induces a proper vertex coloring of G. Such an edge coloring is called a *neighbor-distinguishing edge coloring.* In this section, we describe two neighbor-distinguishing edge colorings, which are defined in terms of sums of colors and are closely related to modular vertex colorings discussed in Section 2.

3.1 Modular Chromatic Index

In [24] a neighbor-distinguishing edge coloring that is closely related to the modular vertex colorings was introduced. For a connected graph G of order at least 3, let $c : E(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be an edge coloring of G where adjacent edges may be colored the same. The *color sum* $s(v)$ of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the edges incident with v, that is,

$$s(v) = \sum_{e \in E_v} c(e) \ \text{ in } \mathbb{Z}_k, \tag{5}$$

where E_v denote the set of edges of G incident with a vertex v. An edge coloring c is a *modular k-edge coloring* of G if $s(x) \neq s(y)$ in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices of G. An edge coloring c is a *modular edge coloring* if c is a modular k-edge coloring for some integer $k \geq 2$. The *modular chromatic index* $\chi'_m(G)$ of G is the minimum k for which G has a modular k-edge coloring. Modular edge colorings have been studied in [23, 25, 26, 27]. To illustrate theses concepts, Figure 5 shows a modular 3-edge coloring of a tree T, where each edge is colored with an element in $\mathbb{Z}_3 = \{0, 1, 2\}$ and each vertex is labeled with its color sum. Since there is no modular 2-edge coloring of T, $\chi'_m(T) = 3$ for the tree T in Figure 5.

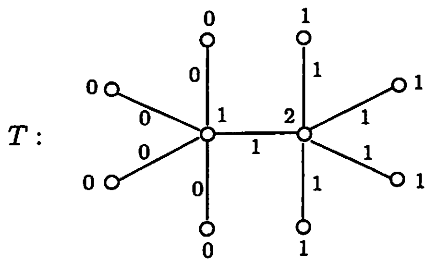


Figure 5: A modular 3-edge coloring of a graph

The classic theorem in the connection with the edge chromatic index of a graph due to Vizing [42], which states that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ for every graph G. A graph G is said to be of *Class* 1 if $\chi'(G) = \Delta(G)$ and of *Class* 2 if $\chi'(G) = \Delta(G)+1$. Determining which graphs belong to which class is a major problem of study in this area. In the case of modular edge coloring, there is a theorem [25] that has a similar style as Vizing's theorem.

Theorem 3.1 *If G is a connected graph of order at least* 3, *then*

$$\chi(G) \leq \chi'_m(G) \leq \chi(G)+1.$$

A characterization has been established for all connected graphs G of order at least 3 such that $\chi'_m(G) = \chi(G)+1$ in [25].

Theorem 3.2 *Let G be a connected graph of order at least* 3. *Then* $\chi'_m(G) = \chi(G)+1$ *if and only if* $\chi(G) \equiv 2 \pmod 4$ *and every proper* $\chi(G)$*-coloring of G results in color classes of odd size.*

For a positive integer k, a graph G is *k-colorable* if there is a proper coloring of G using k colors. Similarly, for an integer $k \geq 2$, a graph G is *modular k-edge colorable* if there is a modular k-edge coloring of G. It is clear that if G is a k-chromatic graph of order n, then a proper k-coloring of G can induce a proper k'-coloring of G for each integer k' with $k \leq k' \leq n$ by introducing a new color to a vertex of G. Therefore, it is easy to see every graph G of order n is k-colorable for all k with $\chi(G) \leq k \leq n$. In the case of modular edge colorings, the situation is quite different; that is, for positive integers k and k' where $k' > k$, a modular k-edge coloring of a graph G may not induce a modular k'-edge coloring of G by introducing a new color to the edge of G. Hence it is much more challenging to determine whether a connected graph G is modular k-edge colorable for an integer $k > \chi'_m(G)$ (see [27]).

Theorem 3.3 *If G is a connected graph of order at least* 3, *then G is modular k-edge colorable for each $k \geq \chi'_m(G)$.*

3.2 Sum Distinguishing Index

In 2004 a neighbor-distinguishing edge coloring $c : E(G) \to \{1, 2, \ldots, k\}$ of a graph G was introduced (see [14, p.385]) in which an induced vertex coloring $s : V(G) \to \mathbb{N}$ is defined by

$$s(v) = \sum_{e \in E_v} c(e) \quad \text{in } \mathbb{N} \tag{6}$$

for each $v \in V(G)$. If $s(x) \neq s(y)$ for every pair x, y of adjacent vertices of G, then c is called a *sum k-coloring*. The minimum k for which a graph G has a sum k-coloring is the *sum distinguishing index* and is denoted by $sd(G)$ of G. A sum $sd(G)$-coloring of G is a minimum sum coloring of G. For example, Figure 6 shows three graphs G_1, G_2, and G_3, where $sd(G) = i$ for $1 \leq i \leq 3$. A minimum sum coloring is given in each case.

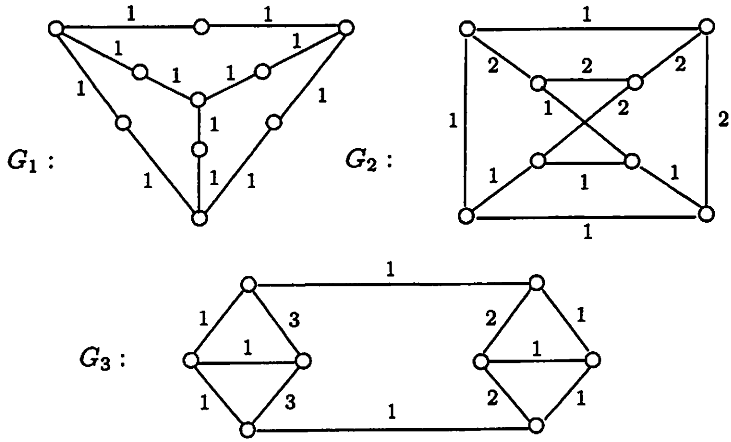


Figure 6: Minimum sum colorings of graphs

Karoński, Łuczak, and Thomason [35] proved the following.

Theorem 3.4 *If G is a 3-colorable graph of order 3 or more, then $sd(G) \leq 3$.*

Karoński, Łuczak, and Thomason conjectured that the requirement that G be 3-colorable is not necessary. This conjecture has developed a catchy name.

The 1-2-3 Conjecture *If G is a connected graph of order 3 or more, then $sd(G) \leq 3$.*

Consequently, if the 1-2-3 Conjecture is true, then for every connected graph G of order 3 or more, it is possible to assign each edge of G one of the colors 1, 2, 3 in such a way that if u and v are adjacent vertices of G, then the sums of the colors of the incident edges of u and v are different. Kalkowski, Karoński, and Pfender [34] proved that the sum distinguishing index cannot be too large.

Theorem 3.5 *If G is a connected graph of order 3 or more, then $sd(G) \leq 5$.*

4 Vertex- or Edge-Distinguishing Colorings

A vertex coloring (or labeling) of a graph G is *vertex-distinguishing* if distinct vertices of G are assigned distinct colors (or labels). There are numerous occasions when an edge coloring of a graph (not necessarily even proper) gives rise to a vertex-distinguishing coloring (see [14, 370-385] for example). An edge coloring (or labeling) of a graph G is *edge-distinguishing* if distinct edges of G are assigned distinct colors (or labels). There are occasions when a vertex coloring of a graph gives rise to an edge-distinguishing labeling (see [19, 40] or [14, p.359-370], for example).

4.1 Irregular Weighted Graphs

In 1986, one of best-known examples of vertex-distinguishing colorings was introduced by Chartrand et al. in [6]. At the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne, a weighting of a connected graph G was introduced for the purpose of producing a weighted graph whose degrees (obtained by adding the weights of the incident edges of each vertex) were distinct. Such a weighted graph was called *irregular.* This concept could be looked at in another manner, however. In particular, let $\mathbb{N}$ denote the set of positive integers and let E_v denote the set of edges of G incident with a vertex v. An edge coloring $c : E(G) \to \mathbb{N}$, where adjacent edges may be colored the same, is said to be *vertex-distinguishing* if the coloring $s : V(G) \to \mathbb{N}$ induced by c and defined by

$$s(v) = \sum_{e \in E_v} c(e) \quad \text{in } \mathbb{N} \tag{7}$$

has the property that $s(x) \neq s(y)$ for every two distinct vertices x and y of G. Note that the definition of the color sum $s(v)$ in (7) is the same formula as the one defined in (6). The main emphasis of this research however dealt with minimizing the largest color assigned to the edges of the graph to produce an irregular graph and such largest color is referred to as the *irregular strength* of the graph. Many research has been done in this area of research (see [2, 15, 16], for example). Furthermore, irregular Eulerian walks in graphs have been introduced and studied in [3, 4].

4.2 Graceful Graphs

The best known example of edge-distinguishing labeling is graceful labeling. In 1968, Rosa [40] introduced a vertex labeling that induces an edge-distinguishing labeling defined by subtracting labels. In particular, for a graph G of size m, a vertex labeling (an injective function) $f : V(G) \to$

$\{0, 1, \ldots, m\}$ was called a *β-valuation* by Rosa if the induced edge labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$ is bijective. In 1972 Golomb [21] called a β-valuation a *graceful labeling* and a graph possessing a graceful labeling a *graceful graph*. It is this terminology that became standard. Over the past few decades the subject of graph labelings has been growing in popularity. Gallian [18] has compiled a periodically updated survey of many kinds of labelings and numerous results, obtained from well over a thousand referenced research articles. A popular conjecture in graph theory, due to Anton Kotzig and Gerhard Ringel, is the following.

The Graceful Tree Conjecture *Every nontrivial tree is graceful.*

For a graph G of order n and size m, there is a smallest integer $k \geq m$ such that there exists a graceful labeling $f : V(G) \to \{0, 1, 2, \ldots, k\}$. This number k is called the *gracefulness* of G and is denoted by $grac(G)$. Thus $grac(G) = m$ if and only if G is graceful. Therefore, the gracefulness of a graph G is a measure of how close G is to being graceful. If G is a graph of order n and size m without isolated vertices, then $m \leq grac(G) \leq 2^{n-1}$.

4.3 Edge-Graceful Graphs

In 1985 Lo [30] introduced a dual type of graceful labeling – this one an edge labeling. Let G be a connected graph of order $n \geq 2$ and size m. For a vertex v of G, let $N(v)$ denote the neighborhood of v. An *edge-graceful labeling* of G is a bijective function $f : E(G) \to \{1, 2, \ldots, m\}$ that gives rise to a bijective function $f' : V(G) \to \{0, 1, 2, \ldots, n-1\}$ given by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. A graph that admits an edge-graceful labeling is called an *edge-graceful graph*. Figure 7 shows two edge-graceful graphs C_5 and $K_{1,4}$ together with an edge-graceful labeling for each of them. It is well known that C_n is graceful if and only if $n \equiv 0, 3 \pmod 4$ and so C_5 is not graceful.

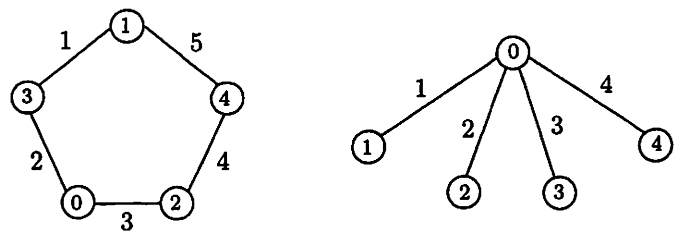


Figure 7: Two edge-graceful graphs

In the definition of an edge-graceful labeling of a connected graph G of order $n \geq 2$ and size m, the edge labeling f is required to be one-to-one.

Since, however, the induced vertex labels $f'(v)$ are obtained by addition in $\mathbb{Z}_n$, the function f is actually a function from $E(G)$ to $\mathbb{Z}_n$ and is in general not one-to-one. Dividing m by n, we obtain $m = nq + r$, where $q = \lfloor m/n \rfloor$ and $0 \le r \le n-1$. Hence in an edge-graceful labeling of G, $q+1$ edges are labeled i for each i with $1 \le i \le r$ and q edges are labeled i for each i with $r+1 \le i \le n$ (in $\mathbb{Z}_n$). Thus this edge labeling $f : E(G) \to \mathbb{Z}_n$ is a one-to-one function only when $m = n-1$ or $m = n$. This observation gives rise to another concept.

4.4 Modular Edge-Graceful Graphs

Let G be a connected graph of order $n \ge 3$ and let $f : E(G) \to \mathbb{Z}_n$, where f need not be one-to-one. Let $f' : V(G) \to \mathbb{Z}_n$ such that $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If f' is one-to-one, then f is called a *modular edge-graceful labeling* and G is a *modular edge-graceful graph*. Consequently, every edge-graceful graph is a modular edge-graceful graph. It turns out that this concept was introduced in 1991 by Jothi [20] under the terminology of *line-graceful graphs* (also see [18]). The graphs $G_1 = C_4$ and G_2 in Figure 8 are both modular edge-graceful. Modular edge-graceful labelings are shown in Figure 8 as well. In fact, the graph G_2 is not graceful while G_2 is not edge-graceful.

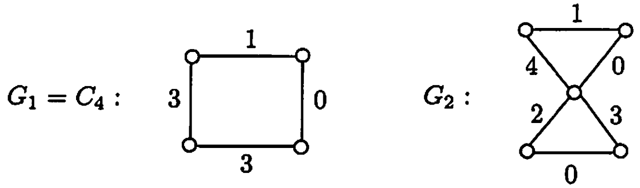


Figure 8: Two modular edge-graceful graphs

It was known that if G is a connected graph of order $n \ge 3$ for which $n \equiv 2 \pmod 4$, then G is not modular edge-graceful. Furthermore, it was conjectured that if T is a tree of order $n \ge 3$ for which $n \not\equiv 2 \pmod 4$, then T is modular edge-graceful (see [18]). This conjecture was verified and, in fact, the conjecture is not only true for trees but for all connected graphs (see [28]).

Theorem 4.1 *A connected graph of order $n \ge 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$.*

For every connected graph G of order n, there is a smallest integer $k \ge n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. The number k is defined in [28] as the *modular edge-gracefulness* $\text{meg}(G)$ of G. Thus $\text{meg}(G) \geq n$ and $\text{meg}(G) = n$ if and only if G is a modular edge-graceful graph of order n and if G is not modular edge-graceful, then $\text{meg}(G) \geq n+1$. As with the gracefulness of a graph, the modular edge-gracefulness of a graph G is a measure of how close G is to being modular edge-graceful. The number $\text{meg}(G)$ is determined for every connected graph G in [28].

Theorem 4.2 *If G is a nontrivial connected graph of order $n \geq 6$ that is not modular edge-graceful, then* $\text{meg}(G) = n + 1$.

If G is a modular edge-graceful spanning subgraph of a graph H, where G and H are connected, then a modular edge-graceful labeling of G can be extended to a modular edge-graceful labeling of H by assigning 0 to each edge of H that does not belong to G. Thus modular edge-graceful labelings of a graph that assign 0 to some edges of the graph play an important role in establishing Theorems 4.1 and 4.2. For this reason, we now investigate those modular edge-graceful labelings in which 0 is not permitted. This gives rise to a new concept along with additional challenging problems. More formally, for a connected graph G of order $n \geq 3$ let $f : E(G) \to \mathbb{Z}_n - \{0\}$, where f need not be one-to-one and let $f' : V(G) \to \mathbb{Z}_n$ be defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If f' is one-to-one, then f is called a *nowhere-zero modular edge-graceful labeling* and G is a *nowhere-zero modular edge-graceful graph.* A characterization of connected nowhere-zero modular edge-graceful graphs has been established [29].

Theorem 4.3 *A connected graph G of order $n \geq 3$ is nowhere-zero modular edge-graceful if and only if (i) $n \not\equiv 2 \pmod 4$, (ii) $G \neq K_3$ and (iii) G is not a star of even order.*

For every connected graph G of order n, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k - \{0\}$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. This number k is referred to as the *nowhere-zero modular edge-gracefulness* of G and is denoted by $\text{nzg}(G)$. Thus $\text{nzg}(G) = n$ if and only if G is nowhere-zero modular edge-graceful and so $\text{nzg}(G) \geq n+1$ if G is not nowhere-zero modular edge-graceful. For a connected graph G of order $n \geq 3$ with $n \not\equiv 2 \pmod 4$ that is not nowhere-zero modular edge-graceful, the exact value of $\text{nzg}(G)$ has been determined (see [29]).

Theorem 4.4 *If G is a connected graph of order $n \geq 3$ with $n \not\equiv 2$* $\pmod 4$ *that is not nowhere-zero modular edge-graceful, then* $\text{nzg}(G) \in$

$\{n+1, n+2\}$. *Furthermore,* $\text{nzg}(G) = n+1$ *if and only if* $G = K_3$ *and* $\text{nzg}(G) = n+2$ *if and only if* G *is a star of even order.*

By Theorem 4.1, if G is a connected graph of order $n \geq 6$ where $n \equiv 2 \pmod 4$, then G is not modular edge-graceful. Consequently, G is not nowhere-zero modular edge-graceful and so $\text{nzg}(G) \geq n+1$. For connected graphs of order $n \geq 3$ with $n \equiv 2 \pmod 4$, the following result is established in [29].

Theorem 4.5 *If* G *is a connected graph of order* $n \geq 6$ *such that* $n \equiv 2 \pmod 4$, *then* $\text{nzg}(G) \in \{n+1, n+2\}$ *and* $\text{nzg}(G) = n+2$ *if and only if* G *is a star.*

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