A Note on Vertex-Covering Walks

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Abstract

For a nontrivial connected graph G of order n and a cyclic ordering $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of $V(G)$, let $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1} for $1 \leq i \leq n$. The Hamiltonian number $h(G)$ and upper Hamiltonian number $h^+(G)$ of G are defined as $h(G) = \min\{d(s)\}$ and $h^+(G) = \max\{d(s)\}$, respectively, where the minimum and maximum are taken over all cyclic orderings s of $V(G)$. All connected graphs G with $h^+(G) = h(G)$ and $h^+(G) = h(G) + 1$ have been characterized in [6, 13]. In this note, we first present a new and much improved proof of the characterization of all graphs whose Hamiltonian and upper Hamiltonian numbers differ by 1 and then determine all pairs of integers that can be realized as the order and upper Hamiltonian number of some tree.

Keywords: Hamiltonian number, upper Hamiltonian number.

AMS subject classification: 05C12, 05C45.

1 Introduction

For a nontrivial connected graph G of order n and a cyclic ordering $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of $V(G)$, the number $d(s)$ is defined in [6] as $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and

v_{i+1}. Therefore, $d(s) \geq n$ for each cyclic ordering s of $V(G)$. The *Hamiltonian number* $h(G)$ of G is defined in [6] by $h(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings s of $V(G)$. Therefore, $h(G) = n$ if and only if G is Hamiltonian. In [7, 8] Goodman and Hedetniemi introduced the concept of a *Hamiltonian walk* in a connected graph G, defined as a closed spanning walk of minimum length in G. During the 10-year period 1973-1983, this concept received considerable attention. For example, Hamiltonian walks were also studied by Asano, Nishizeki, and Watanabe [1, 2], Bermond [3], Nebeský [14], and Vacek [17]. It was shown in [6] that the Hamiltonian number of a connected graph G is, in fact, the length of a Hamiltonian walk in G. This concept was studied further by many (see [5, 11, 15], for example).

For a connected graph G, the *upper Hamiltonian number* $h^+(G)$ of G is defined in [6] as $h^+(G) = \max\{d(s)\}$, where the maximum is taken over all cyclic orderings s of $V(G)$. Obviously, $h^+(G) \geq h(G)$ for every connected graph G. Upper Hamiltonian numbers of graphs were studied in [6, 9, 10, 11, 12, 15]. Not surprisingly, $h^+(G)$ can be considerably larger than $h(G)$. In contrast, there are only two types of graphs G for which $h^+(G) = h(G)$. Also, there are only four types of graphs G for which $h^+(G) = h(G) + 1$. These two facts were established in [6, 13].

Theorem 1.1 [6] *Let G be a nontrivial connected graph. Then $h(G) = h^+(G)$ if and only if G is a complete graph or a star.*

The complement of a graph G is denoted by $\overline{G}$. The join and union of two graphs G and H are denoted by $G \vee H$ and $G + H$, respectively.

Theorem 1.2 [13] *Let G be a nontrivial connected graph of order n. Then $h^+(G) - h(G) = 1$ if and only if $n \geq 4$ and $G = K_1 \vee H$, where $H \in \{K_{1,1,\ldots,1,2}, \overline{K_{1,1,\ldots,1,2}}, K_{1,n-2}, \overline{K_{1,n-2}}\}$.*

The upper Hamiltonian numbers of trees have been studied in [6, 10, 11, 12, 15]. In particular, upper and lower bounds were established for $h^+(T)$ of a tree T in terms of its order, as stated next.

Theorem 1.3 [6] *Let T be a nontrivial tree of order n. Then $2(n-1) = h(T) \leq h^+(T) \leq \lfloor n^2/2 \rfloor$. Furthermore,* (i) *$h^+(T) = 2(n-1)$ if and only if T is a star and* (ii) *$h^+(T) = \lfloor n^2/2 \rfloor$ if and only if T is a path.*

The proof of Theorem 1.2 presented in [6] was an extensive case-by-case analysis and was very lengthy. In Section 2, we present a new and much improved proof for Theorem 1.2. In Section 3, we determine all pairs of integers that can be realized as the order and upper Hamiltonian number of some tree. All graphs under consideration are nontrivial connected graphs. We refer to the book [4] for graph-theoretical notation and terminology not described in this paper.

2 A New Proof of a Charaterization

As mentioned earlier, we present a new proof for Theorem 1.2 in this section. In order to do this, the following lemma will be useful.

Lemma 2.1 *For a graph G, let $G_1 = K_1 \vee G$ and $G_2 = K_1 \vee \overline{G}$. Then $h^+(G_1) - h(G_1) = h^+(G_2) - h(G_2)$.*

Proof. For a graph G of order $n-1$ (≥ 1), construct each of G_1 and G_2 by adding a new vertex v and joining it to every vertex of G and $\overline{G}$, respectively. Let $V = V(G_1) = V(G_2)$. Since $G_1 = G_2 = K_2$ if $n = 2$, assume that $n \geq 3$. For every two distinct vertices $u, v \in V(G)$, we have $d_{G_1}(u,v) + d_{G_2}(u,v) = 3$. Therefore, $d_{G_1}(s) + d_{G_2}(s) = 3n - 2$ for every cyclic ordering s of V. Let s_1 and s_2 be cyclic orderings of $V(G_1) = V(G_2)$ such that $d_{G_1}(s_1) = h(G_1)$ and $d_{G_2}(s_2) = h^+(G_2)$. Then $3n-2 = d_{G_1}(s_1)+d_{G_2}(s_1) \leq h(G_1)+h^+(G_2) \leq d_{G_1}(s_2)+d_{G_2}(s_2) = 3n-2$, implying that $h(G_1) + h^+(G_2) = 3n - 2$. We can similarly show that $h(G_2) + h^+(G_1) = 3n - 2$. Thus, $h^+(G_1) - h(G_1) = h^+(G_2) - h(G_2)$. ∎

Theorem 2.2 *Let G be a nontrivial connected graph of order n. Then $h^+(G) - h(G) = 1$ if and only if $n \geq 4$ and $G = K_1 \vee H$, where $H \in \{K_{1,1,\ldots,1,2}, \overline{K_{1,1,\ldots,1,2}}, K_{1,n-2}, \overline{K_{1,n-2}}\}$.*

Proof. For $n \geq 4$, let $H_1 = K_{1,1,\ldots,1,2}$, $H_2 = \overline{K_{1,n-2}}$, $H_3 = K_{1,n-2}$, and $H_4 = \overline{K_{1,1,\ldots,1,2}}$. Then it is straightforward to verify that $h(K_1 \vee H_1) = n = h^+(K_1 \vee H_1) - 1$, $h(K_1 \vee H_2) = n+1 = h^+(K_1 \vee H_2) - 1$, $h(K_1 \vee H_3) = 2n - 4 = h^+(K_1 \vee H_3) - 1$, and $h(K_1 \vee H_4) = 2n - 3 = h^+(K_1 \vee H_4) - 1$.

For the converse, suppose that G is a connected graph of order n and $h^+(G) - h(G) = 1$. By Theorem 1.1, G is neither complete nor a star; thus we may assume that $n \geq 4$.

We first show that G cannot contain a 4-path (v_1, v_2, v_3, v_4) where $v_1v_3, v_2v_4 \notin E(G)$. If there is such a path, then let $s_1 : v_1, v_2, v_3, v_4, v_1$ and $s_2 : v_1, v_3, v_2, v_4, v_1$. If $n = 4$, then both s_1 and s_2 are cyclic orderings of $V(G)$ and $h(G) \leq d(s_1) = d(v_1, v_4) + 3$ and $h^+(G) \geq d(s_2) = d(v_1, v_4) + 5$. Therefore, $h^+(G) - h(G) \geq 2$. Similarly, if $n \geq 5$, then let s be a linear ordering of $V(G) - \{v_1, v_2, v_3, v_4\}$. For $i = 1, 2$, let s_i' be the cyclic ordering of $V(G)$ obtained by inserting s between v_4 and v_1 in s_i. Then again

$h^+(G) - h(G) \geq d(s_2') - d(s_1') = 2$. If $h^+(G) - h(G) = 1$, therefore, then G contains neither P_4 nor C_4 as an induced subgraph, which also implies that $\mathrm{diam}(G) = 2$ and $\Delta(G) = n-1$. Thus, $G = K_1 \vee H$ for some graph H of order $n-1$. By Theorem 1.1, note that H is neither complete nor empty. For $n = 4$, therefore, $H \in \{K_{1,2}, \overline{K_{1,2}}\}$.

Now assume that $n \geq 5$. We next show that none of $2K_2$, P_4, and C_4 is an induced subgraph in H. We have already seen that neither P_4 nor C_4 can be an induced subgraph in G, that is, neither is contained in H as an induced subgraph. Also, $2K_2 = \overline{C}_4$ cannot be an induced subgraph in H by Lemma 2.1. For $n = 5$, therefore, $H \in \{K_{1,1,2}, \overline{K_{1,1,2}}, K_{1,3}, \overline{K_{1,3}}\}$ or $H \in \{H_0, \overline{H}_0\}$, where $H_0 = K_1 + P_3$. One can quickly verify that $h(K_1 + H_0) = 6 = h^+(K_1 + H_0) - 2$ and so $h^+(K_1 + H_0) - h(K_1 + H_0) = h^+(K_1 + \overline{H}_0) - h(K_1 + \overline{H}_0) = 2$ by Lemma 2.1.

Finally, assume that $n \geq 6$. We next show that $\deg_H v \in \{0, 1, n-3, n-2\}$ for every $v \in V(H)$. Assume, to the contrary, that v_1 is a vertex in H with $2 \leq \deg_H v_1 \leq n-4$. Then let v_2, v_3, v_4, v_5 be vertices in H such that v_2 and v_3 are adjacent to v_1 while v_4 and v_5 are not. Let v_0 be the vertex in G that is adjacent to every vertex in H. Then by considering two orderings $s_1 : v_2, v_1, v_3, v_4, v_0, v_5, v_2$ and $s_2 : v_2, v_0, v_3, v_4, v_1, v_5, v_2$ (and by inserting some fixed linear ordering of $V(G) - \{v_0, v_1, \ldots, v_5\}$ between v_5 and v_2 in each of s_1 and s_2 in case $n \geq 7$), we see that $h^+(G) - h(G) \geq 2$. This verifies the claim. Furthermore, $\Delta(H) \in \{1, n-3, n-2\}$ since H is nonempty. If $\Delta(H) = 1$, then $H = \overline{K_{1,1,\ldots,1,2}}$ since $2K_2$ cannot be an induced subgraph in H. Thus, we now consider the following two cases. Let $V(H) = \{v_1, v_2, \ldots, v_{n-1}\}$ and $\deg_H v_1 = \Delta(H)$.

Case 1. $\Delta(H) = n-3$. Then suppose that $v_1v_2 \notin E(H)$. If $\deg_H v_2 \geq 1$, say $v_2v_3 \in E(H)$, then we may assume that $v_3v_4 \notin E(H)$ since $\deg_H v_3 \leq n-3$. However, this implies that the subgraph induced by $\{v_1, v_2, v_3, v_4\}$ is either C_4 or P_4, which cannot occur. Hence, $\deg_H v_2 = 0$. If $H \neq \overline{K_{1,n-2}}$, then $H = K_1 + K_{1,n-3}$ since $\deg_H v \in \{0, 1, n-3, n-2\}$ for every $v \in V(H)$. To see that this cannot occur, observe that $\overline{H}$ is traceable and $K_1 \vee \overline{H}$ is Hamiltonian while $d_{K_1 \vee \overline{H}}(s) \geq n+2$ for any cyclic ordering of $V(K_1 \vee \overline{H})$ whose first three terms are v_3, v_1, v_4. Thus, $h^+(K_1 \vee H) - h(K_1 \vee H) = h^+(K_1 \vee \overline{H}) - h(K_1 \vee \overline{H}) \geq 2$ by Lemma 2.1. Therefore, $H = \overline{K_{1,n-2}}$ is the only possibility in this case.

Case 2. $\Delta(H) = n-2$. Then $\delta(H) \in \{1, n-3\}$ since H is not complete. If there are two or more vertices having degree $n-2$ in H, then $\delta(H) = n-3$. Furthermore, $H = K_{1,1,\ldots,1,2}$ since C_4 cannot occur as an induced subgraph in H. On the other hand, if v_1 is the only vertex whose degree in H equals $n-2$, then the number of end-vertices in H is either 1 or $n-2$. If the former occurs, then $\overline{H} = K_1 + K_{1,n-3}$. However, this is impossible by Case 1 and Lemma 2.1. Therefore, $H = K_{1,n-2}$. ∎

3 A New Result on Upper Hamiltonian Numbers of Trees

For each edge e of a tree T, the *component number* $\text{cn}(e)$ of e is defined in [6] as the minimum order of a component of $T - e$. In 2008 a formula for the upper Hamiltonian number of a tree T was established in [12] in terms of component numbers of the edges of T.

Theorem 3.1 [12] *If T is a nontrivial tree, then $h^+(T) = 2\sum_{e\in E(T)} \text{cn}(e)$.*

The upper Hamiltonian number of a nontrivial tree was studied further in [10], where this number was expressed in terms of a distance parameter. In order to present this result, we introduce some additional definitions. For a connected graph G, the *Hamiltonian spectrum* $\mathcal{H}(G)$ of G is defined in [9] as $\mathcal{H}(G) = \{d(s) : s \text{ is a cyclic ordering of } V(G)\}$. For a vertex v in a connected graph G, the *total distance* $\text{td}(v)$ of v is the sum of distances from v to all other vertices. The minimum total distance over all vertices of G is the *median number* of G and is denoted by $\text{med}(G)$.

Theorem 3.2 [10] *For a nontrivial tree T of order n, $\mathcal{H}(T) = \{2k : k = n-1, n, n+1, \ldots, \text{med}(T)\}$.*

The following is a consequence of Theorem 3.2.

Corollary 3.3 *For every nontrivial tree T, $h^+(T) = 2\,\text{med}(T)$.*

According to Theorems 1.3, 3.1 and Corollary 3.3, the upper Hamiltonian number of a nontrivial tree of order n is an even integer between $2(n-1)$ and $\lfloor n^2/2 \rfloor$. In fact, for each integer $n \geq 2$, every even integer between $2(n-1)$ and $\lfloor n^2/2 \rfloor$ is the upper Hamiltonian number of some tree of order n. In order to show this, we first present some preliminary results. A vertex of a connected graph G whose total distance equals $\text{med}(G)$ is a *median vertex* of G. The subgraph of G induced by its median vertices of G is the *median* of G. The following two lemmas will be useful to us, the first of which is an easy observation and the second of which was established by Truszczyński [16].

Lemma 3.4 *No end-vertex of a tree T of order at least 3 is a median vertex of T.*

Lemma 3.5 *The median of every connected graph G lies in a single block of G.*

It therefore follows by Lemma 3.5 that the median of a tree is isomorphic to either K_1 or K_2. We are prepared to present the main result of this section.

Theorem 3.6 *For each pair n, k of integers satisfying $1 \leq n-1 \leq k \leq \lfloor n^2/4 \rfloor$, there exists a tree T of order n such that $h^+(T) = 2k$.*

Proof. By Theorem 1.3, the result holds when $k \in \{n-1, \lfloor n^2/4 \rfloor\}$. Thus, let $n \geq 5$ be a fixed integer and suppose that k is an integer such that $n+1 \leq k \leq \lfloor n^2/4 \rfloor$ and there exists a tree T_k of order n with $h^+(T_k) = 2k$. We show that there exists a tree T of order n with $h^+(T) = 2(k-1)$.

Let x be a median vertex of T_k and select a vertex y furthest from x. Thus, y is an end-vertex in T_k while x is not by Lemma 3.4. Also, $\text{td}_{T_k}(x) = \text{med}(T_k) = k$ by Corollary 3.3. Now consider the $y-x$ geodesic $P = (y = v_0, v_1, v_2, \ldots, v_{e(x)} = x)$, where $e(x)$ is the eccentricity of x. Note that $e(x) \geq 2$ since T_k is not a star. Let T be the tree obtained from T_k by deleting the edge v_0v_1 and adding the edge v_0v_2. Then y is an end-vertex in T while v_1 may or may not. We claim that $\text{med}(T) = k-1$. For each vertex $v \in V(T) - \{y\}$, observe that

$$\text{td}_T(v) = \begin{cases} \text{td}_{T_k}(v) + 1 & \text{if } v \in V(T') \\ \text{td}_{T_k}(v) - 1 & \text{otherwise,} \end{cases}$$

where T' is the component of $T - v_1v_2$ containing v_1. Since $\text{td}_T(y) > \text{med}(T)$ again by Lemma 3.4, it follows that $\text{med}(T) = \text{td}_T(x) = k-1$ and so $h^+(T) = 2(k-1)$ by Corollary 3.3. ■

References

[1] T. Asano, T. Nishizeki, and T. Watanabe, An upper bound on the length of a Hamiltonian walk of a maximal planar graph. *J. Graph Theory* 4 (1980) 315–336.

[2] T. Asano, T. Nishizeki, and T. Watanabe, An approximation algorithm for the Hamiltonian walk problems on maximal planar graphs. *Discrete Appl. Math.* 5 (1983) 211–222.

[3] J.C. Bermond, On Hamiltonian walks. *Congr. Numer.* 15 (1976) 41–51.

[4] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs: 5th Edition*, Chapman & Hall/CRC, Boca Raton, FL (2010).

[5] G. Chartrand, T. Thomas, V. Saenpholphat, and P. Zhang, On the Hamiltonian number of a graph. *Congr. Numer.* **165** (2003) 51–64.

[6] G. Chartrand, T. Thomas, V. Saenpholphat, and P. Zhang, A new look at Hamiltonian walks. *Bull. Inst. Combin. Appl.* **42** (2004) 37–52.

[7] S.E. Goodman and S.T. Hedetniemi, On Hamiltonian walks in graphs. *Congr. Numer.* (1973) 335–342.

[8] S.E. Goodman and S.T. Hedetniemi, On Hamiltonian walks in graphs. *SIAM J. Comput.* **3** (1974) 214–221.

[9] D. Král, L.D. Tong and X. Zhu, Upper Hamiltonian numbers and Hamiltonian spectra of graphs. *Australas. J. Combin.* **35** (2006) 329–340.

[10] D. Liu, Hamiltonian Spectrum for Trees. *Ars Combin.* **99** (2011) 415–419.

[11] F. Okamoto, V. Saenpholphat, and P. Zhang, Measures of traceability in graphs. *Math. Bohem.* **131** (2006) 63–83.

[12] F. Okamoto, V. Saenpholphat, and P. Zhang, The upper traceable number of a graph. *Czech. Math. J.* **58** (2008) 271–287.

[13] F. Okamoto and P. Zhang, A characterization of graphs whose Hamiltonian and upper Hamiltonian numbers differ by 1. *Congr. Numer.* **180** (2006) 129–144.

[14] L. Nebeský, A generalization of Hamiltonian cycles for trees. *Czech. Math. J.* **26** (1976) 596–603.

[15] V. Saenpholphat and P. Zhang, Graphs with prescribed order and Hamiltonian number. *Congr. Numer.* **175** (2005) 161–173.

[16] M. Truszczyński, Centers and centroids of unicyclic graphs. *Math. Slovaca* **35** (1985) 223–228.

[17] P. Vacek, On open Hamiltonian walks in graphs. *Arch Math. (Brno)* **27A** (1991) 105–111.