# On the Strong Metric Dimension of Permutation Graphs

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**Abstract**

A vertex $x$ in a graph $G$ *strongly resolves* a pair of vertices $v, w$ if there exists a shortest $x-w$ path containing $v$ or a shortest $x-v$ path containing $w$ in $G$. A set of vertices $S \subseteq V(G)$ is a *strong resolving set* of $G$ if every pair of distinct vertices of $G$ is strongly resolved by some vertex in $S$. The *strong metric dimension* $sdim(G)$ of a graph $G$ is the minimum cardinality over all strong resolving sets of $G$. Let $G_1$ and $G_2$ be disjoint copies of a graph $G$ and let $\sigma : V(G_1) \rightarrow V(G_2)$ be a permutation. Then a permutation graph $G_\sigma = (V, E)$ has the vertex set $V = V(G_1) \cup V(G_2)$ and the edge set $E = E(G_1) \cup E(G_2) \cup \{uv \mid v = \sigma(u)\}$. We show that $2 \leq sdim(G_\sigma) \leq 2n-2$, if $G$ is a connected graph of order $n \geq 3$; we also give an example showing that there is no function $f$ such that $f(sdim(G)) > sdim(G_\sigma)$ for all pairs $(G, \sigma)$. We prove that $sdim(G_{\sigma_0}) \leq 2sdim(G)$ for $\sigma_0$ the identity. Further, we characterize permutation graphs $G_\sigma$ satisfying $sdim(G_\sigma)$ equals $2n-2$ or $2n-3$ when $G$ is a complete $k$-partite graph, a cycle, or a path on $n$ vertices.

# 1 Introduction

Let $G = (V(G), E(G))$ be a finite, simple, undirected, and connected graph of order $|V(G)| = n \geq 2$. The *distance* between two vertices $v, w \in V(G)$, denoted by $d_G(v, w)$, is the length of the shortest path between $v$ and $w$ in $G$; we omit $G$ when ambiguity is not a concern. The diameter, $diam(G)$, of a graph $G$ is given by $\max\{d(u, v) \mid u, v \in V(G)\}$. For a vertex $v \in V(G)$, the *open neighborhood of* $v$ is the set $N_G(v) = \{u \mid uv \in E(G)\}$ and the *closed neighborhood of* $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. More generally, for $v \in V(G)$, let $N_G^k[v] = \{u \in V(G) \mid d_G(u, v) \leq k\}$; notice that $N_G^1[v] = N_G[v]$. The *degree* of a vertex $v \in V(G)$ is the the number of edges incident to the vertex $v$; an *end-vertex* (or *leaf*) is a vertex of degree

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one. We denote by $K_n$, $C_n$, and $P_n$ the complete graph, the cycle, and the path, respectively, on $n$ vertices. For other terminologies in graph theory, refer to [5].

A vertex $x \in V(G)$ *resolves* a pair of vertices $v, w \in V(G)$ if $d(v, x) \neq d(w, x)$. A set of vertices $S \subseteq V(G)$ *resolves* $G$ if every pair of distinct vertices of $G$ is resolved by some vertex in $S$; then $S$ is called a *resolving set* of $G$. For an ordered set $S = \{u_1, u_2, \ldots, u_k\} \subseteq V(G)$ of distinct vertices, the (metric) *representation of* $v \in V(G)$ *with respect to* $S$ is the $k$-vector $r_G(v|S) = (d(v, u_1), d(v, u_2), \ldots, d(v, u_k))$. The *metric dimension* of $G$, denoted by $dim(G)$, is the minimum cardinality over all resolving sets of $G$. Slater [21, 22] introduced the concept of a resolving set for a connected graph under the term *locating set*. He referred to a minimum resolving set as a *reference set*, and the cardinality of a minimum resolving set as the *location number* of a graph. Independently, Harary and Melter [12] studied these concepts under the term *metric dimension*. Metric dimension as a graph parameter has numerous applications, among them are robot navigation [15], sonar [21], combinatorial optimization [20], and pharmaceutical chemistry [3]. In [10], it is noted that determining the metric dimension of a graph is an NP-hard problem. Metric dimension has been extensively studied; for surveys, see [1, 6]. For more articles on the metric dimension of graphs, see [2, 7, 8, 9, 11, 13, 14, 18, 19].

A vertex $x \in V(G)$ *strongly resolves* a pair of vertices $v, w \in V(G)$ if there exists a shortest $x - w$ path containing $v$ or a shortest $x - v$ path containing $w$. A set of vertices $S \subseteq V(G)$ *strongly resolves* $G$ if every pair of distinct vertices of $G$ is strongly resolved by some vertex in $S$; then $S$ is called a *strong resolving set* of $G$. The *strong metric dimension* of $G$, denoted by $sdim(G)$, is the minimum cardinality over all strong resolving sets of $G$. Sebö and Tannier [20] introduced strong metric dimension; they observed that if $S$ is a strong resolving set, then the vectors $\{r_G(v|S) \mid v \in V(G)\}$ uniquely determine the graph $G$, i.e., if $H$ is a graph with $V(H) = V(G)$ such that a strong resolving set $S$ of $H$ satisfies $r_H(v|S) = r_G(v|S)$ for all $v \in V(H) = V(G)$, then $H = G$. In [20], it is also noted that if $S$ is a resolving set, then the vectors $\{r_G(v|S) \mid v \in V(G)\}$ may not uniquely determine $G$. In [17], Oellermann and Peters-Fransen showed that determining the strong metric dimension of a graph is an NP-hard problem. For more articles on the strong metric dimension of graphs, see [16, 23].

Chartrand and Harary [4] introduced a "permutation graph", which is also called a "generalized prism".

**Definition 1.1.** [4] Let $G_1$ and $G_2$ be disjoint copies of a graph $G$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation. A *permutation graph* $G_\sigma = (V, E)$ consists of the vertex set $V = V(G_1) \cup V(G_2)$ and the edge set $E = E(G_1) \cup E(G_2) \cup \{uv \mid v = \sigma(u)\}$.

In this paper, we study the strong metric dimension of permutation graphs. We show that $2 \le sdim(G_\sigma) \le 2n - 2$, if $G$ is a connected graph of order $n \ge 3$; we also give an example showing that there is no function $f$ such that $f(sdim(G)) > sdim(G_\sigma)$ for all pairs $(G, \sigma)$. We prove that $sdim(G_{\sigma_0}) \le 2sdim(G)$ for $\sigma_0$ the identity. Further, we characterize permutation graphs $G_\sigma$ satisfying $sdim(G_\sigma)$ equals $2n - 2$ or $2n - 3$ when $G$ is a complete $k$-partite graph, a cycle, or a path on $n$ vertices.

# 2 Preliminaries on the strong metric dimension of graphs

We first recall the following observations.

**Observation 2.1.** *(a) [20] For any graph $G$, $dim(G) \le sdim(G)$.*

*(b) [20] If $T$ is a tree, then $sdim(T) = L(T) - 1$, where $L(T)$ denotes the number of leaves of $T$.*

*(c) [17] If $C_n$ is the cycle of order $n \ge 3$, then $sdim(C_n) = \lceil \frac{n}{2} \rceil$.*

*(d) [17] If $K_n$ is the complete graph of order $n \ge 2$, then $sdim(K_n) = n - 1$.*

We say that $u \in V(G)$ is *maximally distant* from $v \in V(G)$ if for every $w \in N_G(u)$, $d_G(w, v) \le d_G(u, v)$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then we say that $u$ and $v$ are *mutually maximally distant*. It was shown in [17] that if two vertices $x$ and $y$ are mutually maximally distant in $G$, then any strong resolving set of $G$ must contain either $x$ or $y$.

**Theorem 2.2.** *[23] If $G$ is a connected graph of order $n \ge 2$ and diameter $d$, then*

$$f(n, d) \le sdim(G) \le n - d,$$

*where $f(n, d)$ is the least positive integer $k$ for which $k + d^k \ge n$.*

Next, we recall another upper bound of $sdim(G)$ that is obtained in [16]. Two vertices $u, v \in V(G)$ are called *true twins* if $N_G[u] = N_G[v]$. We say that $X \subseteq G$ is a *twin-free clique* in $G$ if $X$ is a clique containing no true twins. The *twin-free clique number* of $G$, denoted by $\overline{\omega}(G)$, is the maximum cardinality among all twin-free cliques in $G$.

**Theorem 2.3.** *[16] Let G be a connected graph of order $n \geq 2$. Then $sdim(G) \leq n - \overline{\omega}(G)$, where the equality holds when $diam(G) = 2$.*

Next, we recall characterizations of graphs (of order $n \geq 2$) with strong metric dimension 1, $n-1$, or $n-2$.

**Theorem 2.4.** *[23] Let G be a connected graph of order $n \geq 2$. Then*

*(a) $sdim(G) = 1$ if and only if $G = P_n$,*

*(b) $sdim(G) = n - 1$ if and only if $G = K_n$,*

*(c) for $n \geq 4$, $sdim(G) = n-2$ if and only if $diam(G) = 2$ and $\overline{\omega}(G) = 2$.*

# 3 $sdim(G)$ versus $sdim(G_\sigma)$

In this section, we show that $2 \leq sdim(G_\sigma) \leq 2n - 2$, if $G$ is a connected graph of order $n \geq 3$; we also give an example showing that there is no function $f$ such that $f(sdim(G)) > sdim(G_\sigma)$ for all pairs $(G, \sigma)$. We prove that $sdim(G_{\sigma_0}) \leq 2sdim(G)$ for $\sigma_0$ the identity.

First, we obtain general bounds for the strong metric dimension of permutation graphs. If $G$ is a connected graph of order 2, then $G \cong P_2$ and $sdim(G_\sigma) = 2$ for any permutation $\sigma$. So, we consider a connected graph $G$ of order $n \geq 3$ for the rest of the paper.

**Proposition 3.1.** *Let G be a connected graph of order $n \geq 3$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation. Then $2 \leq sdim(G_\sigma) \leq 2n - 2$.*

*Proof.* Since $G_\sigma$ contains a cycle, the lower bound follows from Theorem 2.4(a); for the sharpness of the lower bound, take $G = P_n$ and $\sigma = id$, the identity (see Lemma 6.1). Since $G_\sigma \not\cong K_{2n}$, the upper bound follows from Theorem 2.4(b); for an example of $G_\sigma$ achieving the upper bound, take $G = C_5$ and $G_\sigma \cong \mathcal{P}$, the Petersen graph (see Theorem 5.1). □

Next, we give an example showing that there is no function $f$ such that $f(sdim(G)) > sdim(G_\sigma)$ for all pairs $(G, \sigma)$.

**Remark 3.2.** *There's no function $f$ such that $f(sdim(G)) > sdim(G_\sigma)$ for all pairs $(G, \sigma)$. Let $G = P_{2k}$, $V(G_1) = \{u_i \mid 1 \leq i \leq 2k\}$, and $V(G_2) = \{v_i \mid 1 \leq i \leq 2k\}$, where $k \geq 2$. Let $\sigma : V(G_1) \to V(G_2)$ be defined by $\sigma(u_{2i-1}) = v_{2i}$ and $\sigma(u_{2i}) = v_{2i-1}$, where $1 \leq i \leq k$ (see Figure 1). Then $sdim(G) = 1$ by Theorem 2.4(a), and $sdim(G_\sigma) \geq 2k - 2$ since, for each $j$ $(1 \leq j \leq k-1)$, $u_{2j}$ and $v_{2j+1}$ are mutually maximally distant and $v_{2j}$ and $u_{2j+1}$ are mutually maximally distant in $G_\sigma$.*

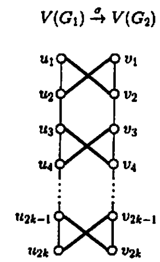



Figure 1: An example showing that there's no function $f$ such that $f(sdim(G)) > sdim(G_\sigma)$ for all pairs $(G, \sigma)$

**Question.** Is there an example showing that there's no function $g$ such that $sdim(G) < g(sdim(G_\sigma))$ for all pairs $(G, \sigma)$?

Next, we prove the following theorem, where $G\Box K_2$ can be viewed as the permutation graph $G_{\sigma_0}$ with $\sigma_0 = id$, the identity, on a connected graph $G$.

**Theorem 3.3.** *For a connected graph $G$, $sdim(G\Box K_2) \leq 2sdim(G)$, where $A\Box B$ denotes the Cartesian product of two graphs $A$ and $B$.*

*Proof.* Let $G_1$ and $G_2$ be the two copies of $G$ in $G\Box K_2$. Let $S$ be a minimum strong resolving set for $G$, and let $S_1 = \{w_1, w_2, \ldots, w_k\}$ and $S_2 = \{w'_1, w'_2, \ldots, w'_k\}$ be the minimum strong resolving set of $G_1$ and $G_2$, respectively, corresponding to $S$. We will show that $S_1 \cup S_2$ is a strong resolving set for $G\Box K_2$. Let $x, y \in V(G\Box K_2) - (S_1 \cup S_2)$. We consider two cases.

*Case 1: Either $\{x, y\} \subseteq V(G_1)$ or $\{x, y\} \subseteq V(G_2)$, say the former.* Notice that $d_{G\Box K_2}(x, w_i) = d_{G_1}(x, w_i)$ and $d_{G\Box K_2}(y, w_i) = d_{G_1}(y, w_i)$ for $1 \leq i \leq k$. So, $x$ and $y$ are strongly resolved by a vertex in $S_1 \subseteq S_1 \cup S_2$.

*Case 2: Either $x \in V(G_1)$ and $y \in V(G_2)$, or $x \in V(G_2)$ and $y \in V(G_1)$, say the former.* Notice that $d_{G\Box K_2}(x, w'_j) = d_{G\Box K_2}(x, w_j) + 1 = d_{G_1}(x, w_j) + 1$ and $d_{G\Box K_2}(y, w_j) = d_{G\Box K_2}(y, w'_j) + 1 = d_{G_2}(y, w'_j) + 1$ for $1 \leq j \leq k$. If $d_{G_1}(x, w_j) \leq d_{G_2}(y, w'_j)$, then there exists a shortest $y - w_j$ path containing $x$ in $G\Box K_2$; if $d_{G_1}(x, w_j) > d_{G_2}(y, w'_j)$, then there exists a shortest $x - w'_j$ path containing $y$ in $G\Box K_2$. So, $x$ and $y$ are strongly resolved by a vertex in $S_1 \cup S_2$.

For the sharpness of the bound, take $G = C_{2m}$, an even cycle. By Observation 2.1(c), $sdim(G) = m$. We will show that $sdim(G\Box K_2) = 2m$. Let $V(G_1) = \{u_i \mid 0 \leq i \leq 2m - 1\}$ and $E(G_1) = \{u_i u_{i+1} \mid 0 \leq i \leq$

$2m-1 \pmod{2m}\}$; similarly, let $V(G_2) = \{v_i \mid 0 \le i \le 2m-1\}$ and $E(G_2) = \{v_iv_{i+1} \mid 0 \le i \le 2m-1 \pmod{2m}\}$. Notice that, for each $i \in \{0, 1, \ldots, 2m-1\}$, $u_i$ and $v_{i+m} \pmod{2m}$ are mutually maximally distant in $G\square K_2$; thus $sdim(G\square K_2) \ge 2m$. Since $V(G_1)$ forms a strong resolving set for $G\square K_2$, $sdim(G\square K_2) \le 2m$. Thus, $sdim(G\square K_2) = 2m$. □

# 4 The strong metric dimension of permutation graphs on complete $k$-partite graphs

In this section, we characterize permutation graphs $G_\sigma$ such that $sdim(G_\sigma)$ equals $|V(G_\sigma)| - 2$ or $|V(G_\sigma)| - 3$ when $G$ is a complete $k$-partite graph. For $k \ge 2$, let $G = K_{a_1,a_2,\ldots,a_k}$ be a complete $k$-partite graph of order $n = \sum_{i=1}^{k} a_i \ge 3$. Throughout this section, let $V(G_1)$ be partitioned into $k$-partite sets $V_1, V_2, ..., V_k$, and let $V(G_2)$ be partitioned into $k$-partite sets $V_1', V_2', ..., V_k'$, where $|V_i| = |V_i'| = a_i$ $(1 \le i \le k)$; further, for each $i$ $(1 \le i \le k)$, let $V_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,a_i}\}$ and let $V_i' = \{u_{i,1}', u_{i,2}', \ldots, u_{i,a_i}'\}$.

**Proposition 4.1.** *Let $G = K_n$ be the complete graph of order $n \ge 3$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation. Then $sdim(G_\sigma) = n$.*

*Proof.* Since $diam(G_\sigma) = 2$ and $\overline{\omega}(G_\sigma) = n$, $sdim(G_\sigma) = n$ by Theorem 2.3. □

**Proposition 4.2.** *For $k \ge 2$, let $G = K_{a_1,a_2,\ldots,a_k}$ be a complete $k$-partite graph of order $n = \sum_{i=1}^{k} a_i \ge 3$. Let $s$ be the number of partite sets of $G$ consisting of one element; if each $a_i \ge 2$ $(1 \le i \le k)$, let $s = 0$. Then*

$$sdim(G) = \begin{cases} n-k & \text{if } s = 0 \\ n+s-k-1 & \text{if } s \neq 0. \end{cases}$$

*Proof.* For $k \ge 2$, let $G = K_{a_1,a_2,\ldots,a_k}$ be a complete $k$-partite graph of order $n = \sum_{i=1}^{k} a_i \ge 3$. If $s = k$, then $sdim(G) = sdim(K_n) = n-1$ by Observation 2.1(d). So, suppose that $G \ncong K_n$ (i.e., $s \neq k$); notice that $diam(G) = 2$. If $0 \le s \le 1$, then $\overline{\omega}(G) = k$; thus $sdim(G) = n-k$ by Theorem 2.3. If $2 \le s < k$, then $\overline{\omega}(G) = k+1-s$; thus $sdim(G) = n+s-k-1$ by Theorem 2.3. □

Next, we give bounds for the strong metric dimension of permutation graphs on complete $k$-partite graphs.

**Lemma 4.3.** *For $k \ge 2$, let $G = K_{a_1,a_2,\ldots,a_k}$ be a complete $k$-partite graph of order $n = \sum_{i=1}^{k} a_i \ge 3$. Then $2 \le sdim(G_\sigma) \le 2n-k$.*

*Proof.* The lower bound follows from Proposition 3.1. The upper bound follows from Theorem 2.3, since $\overline{\omega}(G_\sigma) = k$. □

Next, we characterize permutation graphs $G_\sigma$ such that $sdim(G_\sigma)$ equals $|V(G_\sigma)| - 2$ or $|V(G_\sigma)| - 3$ when $G$ is a complete $k$-partite graph.

**Theorem 4.4.** *For $k \geq 2$, let $G = K_{a_1,a_2,\ldots,a_k}$ be a complete $k$-partite graph of order $n = \sum_{i=1}^{k} a_i \geq 3$. Then $sdim(G_\sigma) = 2n - 2$ if and only if $G_\sigma$ is isomorphic to one of the permutation graphs in Figure 2.*

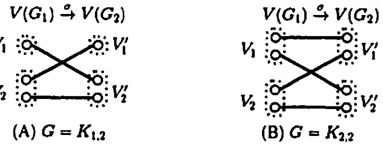



Figure 2: Permutation graphs $G_\sigma$ on complete $k$-partite graphs with $sdim(G_\sigma) = 2|V(G)| - 2$

*Proof.* For $k \geq 2$, let $G = K_{a_1,a_2,\ldots,a_k}$ be a complete $k$-partite graph of order $n = \sum_{i=1}^{k} a_i \geq 3$.

($\Longleftarrow$) Suppose that $G_\sigma$ is isomorphic to (A) or (B) of Figure 2. Since $diam(G_\sigma) = 2$ and $\overline{\omega}(G_\sigma) = 2$, $sdim(G_\sigma) = 2n - 2$ by Theorem 2.3.

($\Longrightarrow$) Suppose that $sdim(G_\sigma) = 2n - 2$. We consider two cases.
*Case 1: $|\sigma(V_i) \cap V_j'| \geq 2$ for some $i, j$, where $1 \leq i, j \leq k$.* Assume that $\{\sigma(u_{i,1}), \sigma(u_{i,2})\} \subseteq V_j'$ by relabeling if necessary, where $1 \leq i, j \leq k$. Since $d_{G_\sigma}(u_{i,1}, \sigma(u_{i,2})) = 3$, $diam(G_\sigma) \geq 3$. Thus, $sdim(G_\sigma) \leq 2n - 3$ by Theorem 2.2.
*Case 2: For each $i, j$ $(1 \leq i, j \leq k)$, $|\sigma(V_i) \cap V_j'| \leq 1$.* By Lemma 4.3, $k = 2$. Further, notice that $a_1 \leq 2$ and $a_2 \leq 2$; otherwise, two vertices of one partite set in $G_1$ must be mapped to the same partite set in $G_2$. So, $G = K_{1,2}$ (see (A) of Figure 2) or $G = K_{2,2}$ (see (B) of Figure 2). □

**Theorem 4.5.** *For $k \geq 2$, let $G = K_{a_1,a_2,\ldots,a_k}$ be a complete $k$-partite graph of order $n = \sum_{i=1}^{k} a_i \geq 3$. Then $sdim(G_\sigma) = 2n - 3$ if and only if $G_\sigma$ is isomorphic to one of the permutation graphs in Figure 3 or $G_\sigma$ is isomorphic to (C) of Figure 5.*

*Proof.* For $k \geq 2$, let $G = K_{a_1,a_2,\ldots,a_k}$ be a complete $k$-partite graph of order $n = \sum_{i=1}^{k} a_i \geq 3$; further, assume that $a_k \geq a_{k-1} \geq \ldots \geq a_2 \geq a_1$.

($\Longleftarrow$) First, suppose that $G_\sigma$ is isomorphic to one of the permutation graphs in Figure 3. Then $diam(G_\sigma) = 2$ and $\overline{\omega}(G_\sigma) = 3$; thus $sdim(G_\sigma) = 2n - 3$ by Theorem 2.3.

Next, suppose that $G_\sigma$ is isomorphic to (C) of Figure 5; we will show that $sdim(G_\sigma) = 2n - 3$. Since $diam(G_\sigma) = 3$, $sdim(G_\sigma) \leq 2n - 3$ by Theorem 2.2. On the other hand, note that (i) any two vertices in $\{u_{2,1}, u_{2,2}, u'_{2,1}, u'_{2,3}\}$ are mutually maximally distant in $G_\sigma$; (ii) any two vertices in $\{u_{2,1}, u_{2,3}, u'_{2,1}, u'_{2,2}\}$ are mutually maximally distant in $G_\sigma$; (iii) $u_{1,1}$ and $u'_{1,1}$ are mutually maximally distant in $G_\sigma$. So, for any minimum strong resolving set $S$ of $G_\sigma$, $|S| \geq 5 = 2n - 3$; thus $sdim(G_\sigma) \geq 2n - 3$. Therefore, $sdim(G_\sigma) = 5 = 2n - 3$.

($\Longrightarrow$) Suppose that $sdim(G_\sigma) = 2n - 3$. By Lemma 4.3, $k = 2$ or $k = 3$; otherwise, $k \geq 4$ and hence $sdim(G_\sigma) \leq 2n - 4$. Noting that $2 \leq diam(G_\sigma) \leq 3$, we consider two cases.

*Case 1:* $diam(G_\sigma) = 2$. That is, for each $i, j$ $(1 \leq i, j \leq k)$, $|\sigma(V_i) \cap V'_j| \leq 1$; so, $a_\ell \leq k$ for each $\ell$ $(1 \leq \ell \leq k)$. By Theorem 2.3, $\overline{\omega}(G_\sigma) = 3$, and hence $k = 3$. So, $(a_1, a_2, a_3)$ must take one of the following values: $(1,1,1)$, $(1,1,2)$, $(1,1,3)$, $(1,2,2)$, $(1,2,3)$, $(1,3,3)$, $(2,2,2)$, $(2,2,3)$, $(2,3,3)$, or $(3,3,3)$. If $(a_1, a_2, a_3) = (1,3,3)$, then two vertices in one partite set must be mapped to the same partite set, contradicting the assumption that $diam(G_\sigma) = 2$. One can readily check that there are 11 non-isomorphic permutation graphs $G_\sigma$ such that $diam(G_\sigma) = 2$ and $\overline{\omega}(G_\sigma) = 3$ (see Figure 3).

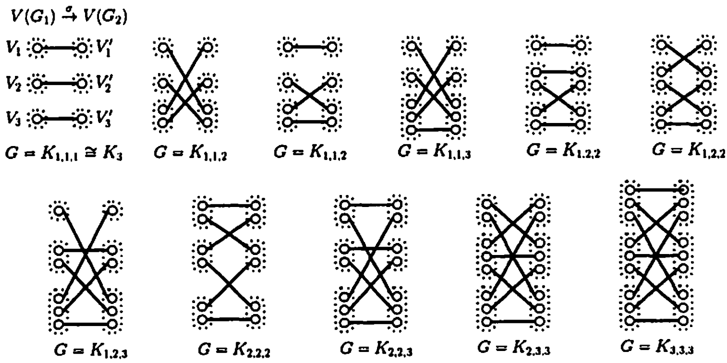



Figure 3: Permutation graphs $G_\sigma$ on complete $k$-partite graphs with $diam(G_\sigma) = 2$ and $sdim(G_\sigma) = 2|V(G)| - 3$

*Case 2:* $diam(G_\sigma) = 3$. That is, $|\sigma(V_i) \cap V'_j| \geq 2$ for some $i, j$ $(1 \leq i, j \leq k)$. By Lemma 4.3, $k = 2$ or $k = 3$. First, we consider $k = 3$. Assume that $\sigma(u_{i,1}) = u'_{j,1}$ and $\sigma(u_{i,2}) = u'_{j,2}$ by relabeling if necessary, where $1 \le i, j \le 3$. Since $V(G_\sigma) - \{u'_{x,1}, u'_{y,1}, u'_{j,1}, u'_{j,2}\}$, where $x, y, j \in \{1, 2, 3\}$ are all distinct, forms a strong resolving set for $G_\sigma$, $sdim(G_\sigma) \le 2n - 4$. Next, we consider $k = 2$. If $G_\sigma$ contains one of the five configurations in Figure 4 as a subgraph, then $V(G_\sigma) - \{\sigma(x_1), \sigma(x_2), \sigma(x_3), \sigma(x_4)\}$ forms a strong resolving set for $G_\sigma$; thus, $sdim(G_\sigma) \le 2n - 4$. Since $a_2 \ge 4$ implies

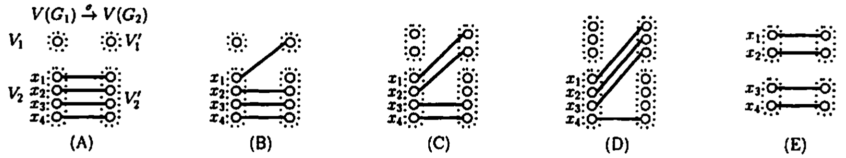



Figure 4: Subgraphs of $G_\sigma$ on complete bi-partite graphs such that $diam(G_\sigma) = 3$ and $sdim(G_\sigma) \le 2|V(G)| - 4$

$sdim(G_\sigma) \le 2n - 4$, $a_2 \le 3$. So, $(a_1, a_2)$ must take one of the following values: $(1, 2)$, $(1, 3)$, $(2, 2)$, $(2, 3)$, or $(3, 3)$. Among them, there are four non-isomorphic permutation graphs $G_\sigma$ such that $diam(G_\sigma) = 3$ and $G_\sigma$ does not contain (E) of Figure 4 as a subgraph. If $G_\sigma$ is isomorphic to (A), (B), or (D) of Figure 5, then the solid vertices form a strong resolving set for $G_\sigma$, and thus $sdim(G_\sigma) \le 2n - 4$. If $G_\sigma$ is isomorphic to (C) of Figure 5, then $sdim(G_\sigma) = 5 = 2n - 3$ as shown earlier. □

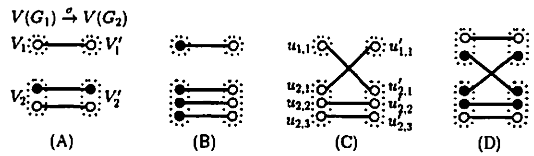



Figure 5: Permutation graphs $G_\sigma$ on complete bi-partite graphs such that $diam(G_\sigma) = 3$ and $G_\sigma$ does not contain any graph in Figure 4 as a subgraph

# 5 The strong metric dimension of permutation graphs on cycles

In this section, we characterize permutation graphs $G_\sigma$ such that $sdim(G_\sigma)$ equals $|V(G_\sigma)| - 2$ or $|V(G_\sigma)| - 3$ when $G$ is a cycle $C_n$ on $n \ge 3$ vertices. Throughout this section, let $V(G_1) = \{u_i \mid 1 \le i \le n\}$ and let $E(G_1) =$

$\{u_iu_{i+1} \mid 1 \le i \le n-1\} \cup \{u_1u_n\}$; similarly, let $V(G_2) = \{v_i \mid 1 \le i \le n\}$ and let $E(G_2) = \{v_iv_{i+1} \mid 1 \le i \le n-1\} \cup \{v_1v_n\}$.

**Theorem 5.1.** *Let $G = C_n$ be the cycle of order $n \ge 3$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation. Then $sdim(G_\sigma) = 2n-2$ if and only if*

*(i) $n = 4$ and $G_\sigma \not\cong C_4 \Box K_2$, or*

*(ii) $n = 5$ and $G_\sigma \cong \mathcal{P}$, the Petersen graph.*

*Proof.* Let $G = C_n$ be the cycle of order $n \ge 3$.

($\Longleftarrow$) Suppose that $G_\sigma$ is isomorphic to (B) of Figure 6 or (D) of Figure 7 (the Petersen graph $\mathcal{P}$). In each case, $diam(G_\sigma) = 2$ and $\overline{\omega}(G_\sigma) = 2$; thus $sdim(G_\sigma) = 2n-2$ by Theorem 2.3.

($\Longrightarrow$) Suppose that $sdim(G_\sigma) = 2n-2$. By Theorem 2.4 (c), $diam(G_\sigma) = 2$ and $\overline{\omega}(G_\sigma) = 2$. We may assume that $\sigma(u_1) = v_1$ by relabeling if necessary. If $n \ge 6$, then $d_{G_\sigma}(u_1, u_4) = 3$ and hence $diam(G_\sigma) \ge 3$; thus $n \le 5$. If $n = 3$, then $\overline{\omega}(G_\sigma) = 3$ for any permutation $\sigma$. If $n = 4$, $G_\sigma$ is isomorphic to (A) or (B) of Figure 6 (see [4]): if $G_\sigma$ is isomorphic to (A) of Figure 6, $diam(G_\sigma) = 3$; if $G_\sigma$ is isomorphic to (B) of Figure 6, then $sdim(G_\sigma) = 2n-2$ as shown above. If $n = 5$, one can easily check that $diam(G_\sigma) = 2$ implies that $G_\sigma \cong \mathcal{P}$ (the Petersen graph), and $sdim(\mathcal{P}) = 2n-2$ as shown above. $\square$

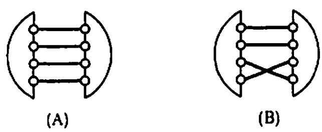



Figure 6: Two non-isomorphic permutation graphs $G_\sigma$ for $G = C_4$

**Theorem 5.2.** *Let $G = C_n$ be the cycle of order $n \ge 3$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation. Then $sdim(G_\sigma) = 2n-3$ if and only if*

*(i) $n = 3$ (for any permutation $\sigma$), or*

*(ii) $n = 5$ and $G_\sigma$ is isomorphic to (C) of Figure 7.*

*Proof.* Let $G = C_n$ be the cycle of order $n \ge 3$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation.

($\Longleftarrow$) If $n = 3$, then $G_\sigma \cong C_3 \Box K_2$ (for any permutation $\sigma$); thus $sdim(G_\sigma) = 3 = 2n-3$ by Theorem 2.3, since $diam(G_\sigma) = 2$ and $\overline{\omega}(G_\sigma) =$

3. If $G_\sigma$ is isomorphic to (C) of Figure 7, we will show that $sdim(G_\sigma) = 2n-3$. Note that (i) any two vertices in $\{u_1, u_4, v_2, v_4\}$ are mutually maximally distant in $G_\sigma$; (ii) any two vertices in $\{u_3, v_1, v_3\}$ are mutually maximally distant in $G_\sigma$; (iii) any two vertices in $\{u_2, v_3, v_5\}$ are mutually maximally distant in $G_\sigma$; (iv) any two vertices in $\{u_2, u_5, v_5\}$ are mutually maximally distant in $G_\sigma$. Let $S$ be a minimum strong resolving set for $G_\sigma$. If $v_3 \notin S$, then $S_0 = \{u_2, u_3, v_1, v_5\} \subseteq S$ by (ii) and (iii), and $|S - S_0| \geq 3$ by (i). If $v_3 \in S$, then $|S| \geq 7$ by (i), (ii), and (iv). In each case, $|S| \geq 7 = 2n-3$, and thus $sdim(G_\sigma) \geq 7 = 2n-3$. Since $diam(G_\sigma) = 3$, $sdim(G_\sigma) \leq 2n-3$ by Theorem 2.2. Thus, $sdim(G_\sigma) = 2n-3$.

($\Longrightarrow$) Suppose that $sdim(G_\sigma) = 2n-3$. Let $\sigma(u_1) = v_1$ by relabeling if necessary. By Theorem 2.2, $diam(G_\sigma) \leq 3$; thus $n \leq 9$: notice that $N_{G_\sigma}[u_1] = \{u_1, u_2, u_n, v_1\}$, $N^2_{G_\sigma}[u_1] = N_{G_\sigma}[u_1] \cup \{u_3, u_{n-1}, \sigma(u_2), \sigma(u_n), v_2, v_n\}$, and $N^3_{G_\sigma}[u_1] \cap V(G_1) = [N^2_{G_\sigma}[u_1] \cap V(G_1)] \cup \{u_4, u_{n-2}, \sigma^{-1}(v_2), \sigma^{-1}(v_n)\}$. We consider six cases.

*Case 1: $n = 3$ or $n = 4$.* If $n = 3$, $sdim(G_\sigma) = 3 = 2n-3$ as shown above. If $n = 4$, see Figure 6 for two non-isomorphic $G_\sigma$: (i) if $G_\sigma$ is isomorphic to (A) of Figure 6, then $sdim(G_\sigma) \leq 2n-4$ since $V(G_1)$ forms a strong resolving set for $G_\sigma$; (ii) if $G_\sigma$ is isomorphic to (B) of Figure 6, then $sdim(G_\sigma) = 2n-2$ (see Theorem 5.1).

*Case 2: $n = 5$.* There are four non-isomorphic permutation graphs $G_\sigma$ (see Figure 7): (i) if $G_\sigma$ is isomorphic to (A) or (B) of Figure 7, then $V(G_\sigma) - \{u_1, u_2, v_1, v_2\}$ forms a strong resolving set for $G_\sigma$, and thus $sdim(G_\sigma) \leq 2n-4$; (ii) if $G_\sigma$ is isomorphic to (C) of Figure 7, $sdim(G_\sigma) = 2n-3$ as shown above; (iii) if $G_\sigma$ is isomorphic to (D) of Figure 7, then $sdim(G_\sigma) = 8 = 2n-2$ (see Theorem 5.1).

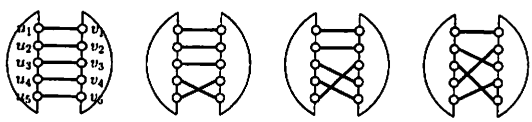



Figure 7: Four non-isomorphic permutation graphs $G_\sigma$ for $G = C_5$

*Case 3: $n = 6$.* Since $diam(G_\sigma) \leq 3$, $G_\sigma \not\cong C_6 \square K_2$. We consider four subcases; in each case, we will show that $sdim(G_\sigma) \leq 2n-4$.

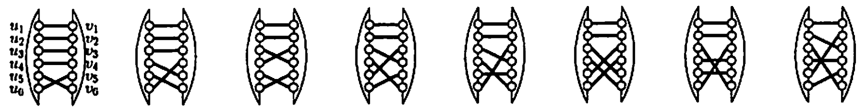
*Subcase 3.1:* $P_4 \square K_2 \subseteq G_\sigma \not\subseteq P_6 \square K_2$. There is a unique $G_\sigma$ up to isomorphism (see (A) of Figure 8). If $G_\sigma$ is isomorphic to (A) of Figure 8, $V(G_\sigma) - \{u_2, u_3, u_4, v_3\}$ forms a strong resolving set for $G_\sigma$.

*Subcase 3.2:* $P_3 \square K_2 \subseteq G_\sigma \not\subseteq P_4 \square K_2$. Assume that $\sigma(u_i) = v_i$ for

$i = 1, 2, 3$, by relabeling if necessary; then $\sigma(u_4) \in \{v_5, v_6\}$. If $\sigma(u_4) = v_5$, see (B) of Figure 8. If $\sigma(u_4) = v_6$, $d_{G_\sigma}(u_2, v_5) \le 3$ implies that $\sigma(u_6) = v_5$; then $G_\sigma$ is isomorphic to (B) of Figure 8. If $G_\sigma$ is isomorphic to (B) of Figure 8, $V(G_\sigma) - \{u_3, v_2, v_3, v_4\}$ forms a strong resolving set for $G_\sigma$.

*Subcase 3.3:* $P_2 \square K_2 \subseteq G_\sigma \not\subseteq P_3 \square K_2$. Assume that $\sigma(u_i) = v_i$ for $i = 1, 2$, by relabeling if necessary. One can readily check that there are five non-isomorphic $G_\sigma$ with $diam(G_\sigma) = 3$ (see (C), (D), (E), (F), and (G) of Figure 8). If $G_\sigma$ is isomorphic to (C), (D), or (F) of Figure 8, $V(G_\sigma) - \{u_2, v_1, v_2, v_3\}$ forms a strong resolving set for $G_\sigma$; if $G_\sigma$ is isomorphic to (E) or (G) of Figure 8, $V(G_\sigma) - \{u_6, v_2, v_3, v_4\}$ forms a strong resolving set for $G_\sigma$.

*Subcase 3.4:* $G_\sigma \not\supseteq P_2 \square K_2$. One can easily check that there exists a unique $G_\sigma$ with $diam(G_\sigma) = 3$, up to isomorphism (see (H) of Figure 8). If $G_\sigma$ is isomorphic to (H) of Figure 8, $V(G_\sigma) - \{u_5, v_1, v_2, v_3\}$ forms a strong resolving set for $G_\sigma$.




(A) (B) (C) (D) (E) (F) (G) (H)

Figure 8: Permutation graphs $G_\sigma$ for $G = C_6$ with $diam(G_\sigma) = 3$

*Case 4:* $n = 7$. Since $diam(G_\sigma) \le 3$, $G_\sigma$ does not contain $P_4 \square K_2$ as a subgraph: if $\sigma(u_i) = v_i$ for each $i \in \{1, 2, 3, 4\}$, then $d_{G_\sigma}(u_2, v_5) = 4$ or $d_{G_\sigma}(u_2, v_6) = 4$. We consider three subcases. In each case, we will show that $sdim(G_\sigma) \le 2n - 4$.

*Subcase 4.1:* $P_3 \square K_2 \subseteq G_\sigma \not\subseteq P_4 \square K_2$. Assume that $\sigma(u_i) = v_i$ for $i = 1, 2, 3$, by relabeling if necessary. Since $diam(G_\sigma) \le 3$, $d_{G_\sigma}(u_2, v_5) \le 3$ and $d_{G_\sigma}(u_2, v_6) \le 3$ imply that $\{\sigma(u_4), \sigma(u_7)\} = \{v_5, v_6\}$. There are two non-isomorphic $G_\sigma$ with $diam(G_\sigma) = 3$ (see (A) and (B) of Figure 9). In each case, $V(G_\sigma) - \{u_2, v_1, v_2, v_3\}$ forms a strong resolving set for $G_\sigma$.

*Subcase 4.2:* $P_2 \square K_2 \subseteq G_\sigma \not\subseteq P_3 \square K_2$. Assume that $\sigma(u_i) = v_i$ for $i = 1, 2$, by relabeling if necessary. One can readily check that there are 9 non-isomorphic $G_\sigma$ with $diam(G_\sigma) = 3$ (see (C), (D), (E), (F), (G), (H), (I), (J), and (K) of Figure 9). If $G_\sigma$ is isomorphic to (C), (E), or (F) of Figure 9, $V(G_\sigma) - \{u_3, v_3, v_4, v_5\}$ forms a strong resolving set for $G_\sigma$; if $G_\sigma$ is isomorphic to (D) of Figure 9, $V(G_\sigma) - \{u_3, v_4, v_5, v_6\}$ forms a strong resolving set for $G_\sigma$; if $G_\sigma$ is isomorphic to (G) of Figure 9, $V(G_\sigma) - \{u_6, v_2, v_3, v_4\}$ forms a strong resolving set for $G_\sigma$; if $G_\sigma$ is isomorphic to (H) of Figure 9, $V(G_\sigma) - \{u_4, v_3, v_4, v_5\}$ forms a strong resolving set for $G_\sigma$; if $G_\sigma$ is isomorphic to (I) of Figure 9, $V(G_\sigma) - \{u_6, v_5, v_6, v_7\}$ forms a strong resolving set for $G_\sigma$; if $G_\sigma$ is isomorphic to (J) of Figure 9, $V(G_\sigma) - \{u_5, v_2, v_3, v_4\}$ forms a strong resolving set for $G_\sigma$; if $G_\sigma$ is isomorphic to (K) of Figure 9, $V(G_\sigma) - \{u_7, v_3, v_4, v_5\}$ forms a strong resolving set for $G_\sigma$.

*Subcase 4.3:* $G_\sigma \not\supseteq P_2 \Box K_2$. Noting that $\sigma(u_1) = v_1$, one can readily check that there are three non-isomorphic $G_\sigma$ with $diam(G_\sigma) = 3$ (see (L), (M), and (N) of Figure 9). In each case, $V(G_\sigma) - \{u_1, u_2, v_1, v_3\}$ forms a strong resolving set for $G_\sigma$.

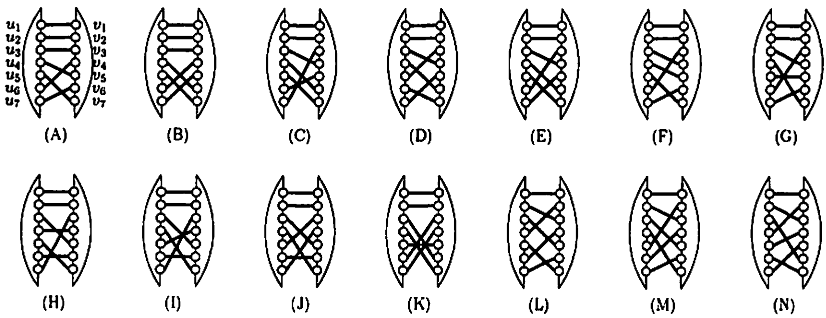
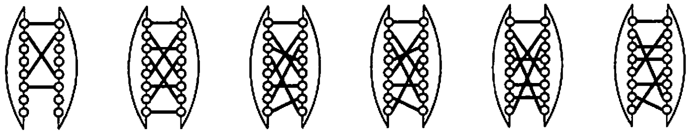



Figure 9: Permutation graphs $G_\sigma$ for $G = C_7$ with $diam(G_\sigma) = 3$

*Case 5:* $n = 8$. Since $diam(G_\sigma) \le 3$, $d_{G_\sigma}(u_1, u_5) \le 3$ implies that $\sigma(u_5) \in \{v_2, v_8\}$: assume that $\sigma(u_5) = v_2$ by relabeling if necessary. Similarly, $d_{G_\sigma}(v_2, v_6) \le 3$ implies that $\sigma^{-1}(v_6) \in \{u_4, u_6\}$: we may assume that $\sigma^{-1}(v_6) = u_6$ by relabeling if necessary. Further, $d_{G_\sigma}(v_1, v_5) \le 3$ implies that $\sigma^{-1}(v_5) \in \{u_2, u_8\}$.

First, we consider $\sigma^{-1}(v_5) = u_8$; we will show that $diam(G_\sigma) \ge 4$. Note that (i) $d_{G_\sigma}(u_6, u_2) \le 3$ implies $\sigma(u_2) = v_7$; (ii) $d_{G_\sigma}(u_8, u_4) \le 3$ implies $\sigma(u_4) = v_4$; (iii) $d_{G_\sigma}(v_7, v_3) \le 3$ implies $\sigma(u_3) = v_3$ (see (B) of Figure 10). If $G_\sigma$ is isomorphic to (B) of Figure 10, then $d_{G_\sigma}(u_3, u_7) = 4$; thus $diam(G_\sigma) \ge 4$.

Next, suppose that $\sigma^{-1}(v_5) = u_2$ (see (A) of Figure 10). Notice that $\sigma(u_3) \in \{v_3, v_4, v_7, v_8\}$. If $\sigma(u_3) = v_3$, then $d_{G_\sigma}(u_3, u_7) \le 3$ implies $\sigma(u_7) = v_4$, and $d_{G_\sigma}(v_3, v_7) \le 3$ implies $\sigma(u_4) = v_7$ (see $(A_1)$ of Figure 10). If $\sigma(u_3) = v_4$, then $d_{G_\sigma}(u_3, u_7) \le 3$ implies $\sigma(u_7) = v_3$, and $d_{G_\sigma}(v_3, v_7) \le 3$ implies $\sigma(u_8) = v_7$ (see $(A_2)$ of Figure 10). If $\sigma(u_3) = v_7$, then $d_{G_\sigma}(u_3, u_7) \le 3$ implies $\sigma(u_7) = v_8$, and $d_{G_\sigma}(v_7, v_3) \le 3$ implies $\sigma(u_4) = v_3$ (see $(A_3)$ of Figure 10). If $\sigma(u_3) = v_8$, then $d_{G_\sigma}(u_3, u_7) \le 3$ implies $\sigma(u_7) = v_7$, and $d_{G_\sigma}(v_7, v_3) \le 3$ implies $\sigma(u_8) = v_3$ (see $(A_4)$ of Figure 10). One can easily check that $(A_1)$, $(A_2)$, $(A_3)$, and $(A_4)$ of Figure 10 are isomorphic. Since $V(G_\sigma) - \{u_2, u_3, v_3, v_5\}$ forms a strong resolving set for $G_\sigma$ in $(A_1)$ of Figure 10, $sdim(G_\sigma) \le 2n - 4$.



(A) $(A_1)$ $(A_2)$ $(A_3)$ $(A_4)$ (B)

Figure 10: Permutation graphs $G_\sigma$ for $G = C_8$

*Case 6:* $n = 9$. Since $diam(G_\sigma) \leq 3$, $d_{G_\sigma}(u_1, u_5) \leq 3$ and $d_{G_\sigma}(u_1, u_6) \leq 3$ imply that $\{\sigma(u_5), \sigma(u_6)\} = \{v_2, v_9\}$: assume that $\sigma(u_5) = v_2$ and $\sigma(u_6) = v_9$ by relabeling if necessary. Similarly, $d_{G_\sigma}(v_1, v_5) \leq 3$ and $d_{G_\sigma}(v_1, v_6) \leq 3$ imply that $\{\sigma^{-1}(v_5), \sigma^{-1}(v_6)\} = \{u_2, u_9\}$ (see Figure 11). In each case, $d_{G_\sigma}(u_2, u_6) = 4$, and thus $diam(G_\sigma) \geq 4$. □

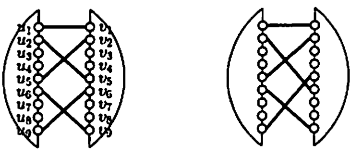



Figure 11: Subgraphs of permutation graphs $G_\sigma$ for $G = C_9$

# 6 The strong metric dimension of permutation graphs on paths

In this section, we characterize permutation graphs $G_\sigma$ such that $sdim(G_\sigma)$ equals $|V(G_\sigma)| - 2$ or $|V(G_\sigma)| - 3$ when $G$ is a path $P_n$ on $n \geq 3$ vertices. Throughout this section, let $V(G_1) = \{u_i \mid 1 \leq i \leq n\}$ and let $E(G_1) = \{u_iu_{i+1} \mid 1 \leq i \leq n-1\}$; similarly, let $V(G_2) = \{v_i \mid 1 \leq i \leq n\}$ and let $E(G_2) = \{v_iv_{i+1} \mid 1 \leq i \leq n-1\}$.

**Lemma 6.1.** *Let $G = P_n$ be the path of order $n \geq 3$, and let $id : V(G_1) \to V(G_2)$ be the identity. Then $sdim(G_{id}) = 2$.*

*Proof.* Since $G_{id} \not\cong P_{2n}$, $sdim(G_{id}) \geq 2$ by Theorem 2.4(a). On the other hand, $sdim(G_{id}) \leq 2$ by Theorem 3.3 and the fact that $sdim(P_n) = 1$. Thus $sdim(G_{id}) = 2$. □

**Theorem 6.2.** *Let $G = P_n$ be the path of order $n \geq 3$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation. Then $sdim(G_\sigma) = 2n-2$ if and only if $n = 3$ and $G_\sigma \not\cong P_3 \Box K_2$.*

*Proof.* Let $G = P_n$ be the path of order $n \geq 3$. If $n = 3$, then there are two non-isomorphic permutation graphs (see Figure 12): if $G_\sigma$ is isomorphic to (A) of Figure 12, then $sdim(G_\sigma) = 2 < 2n-2$ by Lemma 6.1; if $G_\sigma$ is isomorphic to (B) of Figure 12, then $diam(G_\sigma) = 2$ and $\overline{\omega}(G_\sigma) = 2$, and thus $sdim(G_\sigma) = 4 = 2n-2$ by Theorem 2.3. If $n \geq 4$, then $diam(G_\sigma) \geq 3$ since $d_{G_\sigma}(u_1, u_4) = 3$; thus $sdim(G_\sigma) \leq 2n-3$ by Theorem 2.2. □

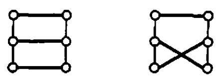



Figure 12: Two non-isomorphic permutation graphs $G_\sigma$ for $G = P_3$

As an immediate consequence of Theorem 6.2, we have the following

**Corollary 6.3.** *Let $G = P_n$ be the path of order $n \geq 4$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation. Then $2 \leq sdim(G_\sigma) \leq 2n-3$.*

Next, we characterize permutation graphs $G_\sigma$ such that $sdim(G_\sigma) = |V(G_\sigma)| - 3$ when $G$ is a path.

**Theorem 6.4.** *Let $G = P_n$ be the path of order $n \geq 4$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation. Then $sdim(G_\sigma) = 2n-3$ if and only if*

*(i) $G_\sigma$ is isomorphic to (A) of Figure 13, or*

*(ii) $G_\sigma$ is isomorphic to one of the permutation graphs in Figure 14.*

*Proof.* Let $G = P_n$ be the path of order $n \geq 4$, and let $\sigma : V(G_1) \to V(G_2)$ be a permutation.

($\Longleftarrow$) Let $S$ be a minimum strong resolving set for $G_\sigma$.

First, suppose that $G_\sigma$ is isomorphic to (A) of Figure 13; we will show that $sdim(G_\sigma) = 2n-3$. Note that (i) the following pairs are mutually maximally distant in $G_\sigma$: $\{u_1, u_4\}$, $\{u_1, v_4\}$, $\{v_1, u_4\}$, and $\{v_1, v_4\}$; (ii) $u_2$ and $v_2$ are mutually maximally distant in $G_\sigma$; (iii) $u_3$ and $v_3$ are mutually maximally distant in $G_\sigma$; (iv) the following pairs are mutually maximally distant in $G_\sigma$: $\{u_3, v_4\}$ and $\{v_3, u_4\}$. By (i), $|S \cap \{u_1, u_4, v_1, v_4\}| \geq 2$. If $|S \cap \{u_1, u_4, v_1, v_4\}| \geq 3$, then $|S| \geq 5 = 2n-3$ by (ii) and (iii). If $|S \cap \{u_1, u_4, v_1, v_4\}| = 2$, say $S_0 = \{u_1, v_1\} \subseteq S$, by (i) and relabeling if necessary, then $|S-S_0| \geq 3$ by (ii) and (iv). Thus, $sdim(G_\sigma) \geq 5 = 2n-3$. Since $sdim(G_\sigma) \leq 2n-3$ by Corollary 6.3, we have $sdim(G_\sigma) = 5 = 2n-3$.

Next, suppose that $G_\sigma$ is isomorphic to one of the permutation graphs in Figure 14; we will show that $sdim(G_\sigma) = 2n-3$ in each case. Note that (i) since any two vertices in $\{u_1, u_4, v_1, v_4\}$ are mutually maximally distant in $G_\sigma$, $|S \cap \{u_1, u_4, v_1, v_4\}| \geq 3$; (ii) since $u_2$ and $v_3$ are mutually maximally distant in $G_\sigma$, $|S \cap \{u_2, v_3\}| \geq 1$; (iii) since $u_3$ and $v_2$ are mutually maximally distant in $G_\sigma$, $|S \cap \{u_3, v_2\}| \geq 1$. So, $|S| \geq 5 = 2n-3$, and hence $sdim(G_\sigma) \geq 2n-3$. Since $sdim(G_\sigma) \leq 2n-3$ by Corollary 6.3, we have $sdim(G_\sigma) = 2n-3$.

($\Longrightarrow$) Suppose that $sdim(G_\sigma) = 2n-3$. Then $diam(G_\sigma) \leq 3$ by Theorem 2.2. We consider two cases.

*Case 1:* $\{\sigma(u_1), \sigma(u_n)\} \cap \{v_1, v_n\} \neq \emptyset$. We may assume that $\sigma(u_1) = v_1$, by relabeling if necessary. Notice that $N_{G_\sigma}[u_1] = \{u_1, u_2, v_1\}$, $N^2_{G_\sigma}[u_1] = N_{G_\sigma}[u_1] \cup \{u_3, v_2, \sigma(u_2)\}$, and $N^3_{G_\sigma}[u_1] \cap V(G_1) = [N^2_{G_\sigma}[u_1] \cap V(G_1)] \cup \{u_4, \sigma^{-1}(v_2)\}$. Since $diam(G_\sigma) \leq 3$, $n \leq 5$.

First, we consider $n = 4$. Noting that $sdim(P_4 \Box K_2) = 2$ by Lemma 6.1, there are four non-isomorphic permutation graphs to consider (see (A), (B), (C), and (D) of Figure 13). In each case, $diam(G_\sigma) = 3$. If $G_\sigma$ is isomorphic to (A) of Figure 13, then $sdim(G_\sigma) = 5 = 2n-3$ as shown above. If $G_\sigma$ is isomorphic to (B), (C), or (D) of Figure 13, then $sdim(G_\sigma) \leq 2n-4$: (i) if $G_\sigma$ is isomorphic to (B) of Figure 13, $V(G_\sigma) - \{u_1, u_2, v_1, v_2\}$ forms a strong resolving set for $G_\sigma$; (ii) if $G_\sigma$ is isomorphic to (C) of Figure 13, $V(G_\sigma) - \{u_1, u_2, u_3, v_1\}$ forms a strong resolving set for $G_\sigma$; (iii) if $G_\sigma$ is isomorphic to (D) of Figure 13, $V(G_\sigma) - \{u_2, u_3, u_4, v_3\}$ forms a strong resolving set for $G_\sigma$.

Next, we consider $n = 5$. Since $diam(G_\sigma) \leq 3$, $d_{G_\sigma}(u_1, u_5) \leq 3$ implies that $\sigma^{-1}(v_2) = u_5$; similarly, $d_{G_\sigma}(v_1, v_5) \leq 3$ implies that $\sigma(u_2) = v_5$. If $\sigma(u_3) = v_4$ and $\sigma(u_4) = v_3$, then $d_{G_\sigma}(u_5, v_5) = 4$, and thus $diam(G_\sigma) \geq 4$. So, $\sigma(u_3) = v_3$ and $\sigma(u_4) = v_4$ (see (E) of Figure 13); here, notice that $V(G_\sigma) - \{u_4, u_5, v_2, v_4\}$ forms a strong resolving set for $G_\sigma$, and thus $sdim(G_\sigma) \leq 2n-4$.

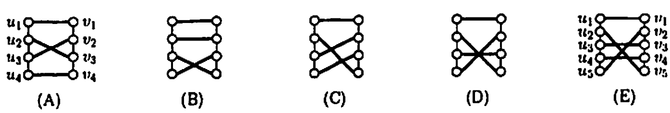



Figure 13: Permutation graphs $G_\sigma$ for $G \in \{P_4, P_5\}$ such that $diam(G_\sigma) = 3$ and $\sigma(u_1) = v_1$

*Case 2:* $\{\sigma(u_1), \sigma(u_n)\} \cap \{v_1, v_n\} = \emptyset$. Then $N_{G_\sigma}[u_1] = \{u_1, u_2, \sigma(u_1)\}$, $N^2_{G_\sigma}[u_1] = N_{G_\sigma}[u_1] \cup \{u_3, \sigma(u_2)\} \cup N_{G_2}(\sigma(u_1))$, and $N^3_{G_\sigma}[u_1] \cap V(G_1) = [N^2_{G_\sigma}[u_1] \cap V(G_1)] \cup \{u_4, \sigma^{-1}(v_x), \sigma^{-1}(v_y)\}$, where $\{v_x, v_y\} = N_{G_2}(\sigma(u_1))$. Since $diam(G_\sigma) \leq 3$, $n \leq 6$. We consider three subcases.

*Subcase 2.1:* $n = 4$. Assume that $\sigma(u_1) = v_2$ and $\sigma(u_4) = v_3$, by relabeling if necessary. There are two permutation graphs to consider (see Figure 14): in each case, $sdim(G_\sigma) = 2n - 3$ as shown above.

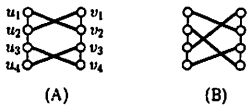



Figure 14: Permutation graphs $G_\sigma$ for $G = P_4$ such that $\sigma(u_1) = v_2$ and $\sigma(u_4) = v_3$

*Subcase 2.2:* $n = 5$. We may assume that $\sigma(u_1) \in \{v_2, v_3\}$, by relabeling if necessary. If $\sigma(u_1) = v_2$, then $d_{G_\sigma}(u_1, u_5) \leq 3$ implies $u_5 \in \{\sigma^{-1}(v_1), \sigma^{-1}(v_3)\}$; since $u_5 \neq \sigma^{-1}(v_1)$, $u_5 = \sigma^{-1}(v_3)$. If $\sigma(u_1) = v_3$, then $u_5 \in \{\sigma^{-1}(v_2), \sigma^{-1}(v_4)\}$, say $u_5 = \sigma^{-1}(v_4)$, by relabeling if necessary; here, notice that this configuration is isomorphic to $G_\sigma$ satisfying $\sigma(u_1) = v_2$ and $u_5 = \sigma^{-1}(v_3)$. So, let $\sigma(u_1) = v_2$ and $\sigma(u_5) = v_3$. If $\sigma(u_2) = v_1$, $d_{G_\sigma}(v_1, v_5) \leq 3$ implies that $v_5 = \sigma(u_3)$ (see (A) of Figure 15); if $\sigma(u_2) = v_5$, $d_{G_\sigma}(v_5, v_1) \leq 3$ implies that $v_1 = \sigma(u_3)$ (see (D) of Figure 15); if $\sigma(u_2) = v_4$, see (B) and (C) of Figure 15. In each case, we will show that $sdim(G_\sigma) \leq n - 4$: (i) if $G_\sigma$ is isomorphic to (A) of Figure 15, $V(G_\sigma) - \{u_4, u_5, v_3, v_4\}$ forms a strong resolving set for $G_\sigma$; (ii) if $G_\sigma$ is isomorphic to (B) of Figure 15, $V(G_\sigma) - \{u_1, u_2, v_3, v_4\}$ forms a strong resolving set for $G_\sigma$; (iii) if $G_\sigma$ is isomorphic to (C) of Figure 15, $V(G_\sigma) - \{u_5, v_2, v_3, v_4\}$ forms a strong resolving set for $G_\sigma$; (iv) if $G_\sigma$ is isomorphic to (D) of Figure 15, $V(G_\sigma) - \{u_2, u_3, v_1, v_2\}$ forms a strong resolving set for $G_\sigma$.

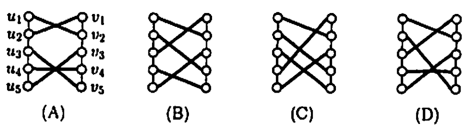



Figure 15: Permutation graphs $G_\sigma$ for $G = P_5$ such that $diam(G_\sigma) = 3$ and $\{\sigma(u_1), \sigma(u_5)\} \cap \{v_1, v_5\} = \emptyset$

*Subcase 2.3:* $n = 6$. We may assume that $\sigma(u_1) \in \{v_2, v_3\}$, by relabeling if necessary. If $\sigma(u_1) = v_2$, then $d_{G_\sigma}(u_1, u_5) \leq 3$ and $d_{G_\sigma}(u_1, u_6) \leq 3$ imply that $\{\sigma^{-1}(v_1), \sigma^{-1}(v_3)\} = \{u_5, u_6\}$; since $u_6 \neq \sigma^{-1}(v_1)$, $\sigma^{-1}(v_1) = u_5$ and $\sigma^{-1}(v_3) = u_6$ (see (A) of Figure 16). If $G_\sigma$ contains (A) of Figure 16 as a subgraph, then $d_{G_\sigma}(v_1, v_5) = 4$ or $d_{G_\sigma}(v_1, v_6) = 4$; thus $diam(G_\sigma) \geq 4$. If $\sigma(u_1) = v_3$, then $d_{G_\sigma}(u_1, u_5) \leq 3$ and $d_{G_\sigma}(u_1, u_6) \leq 3$ imply that $\{\sigma^{-1}(v_2), \sigma^{-1}(v_4)\} = \{u_5, u_6\}$: if $\sigma^{-1}(v_2) = u_6$ and $\sigma^{-1}(v_4) = u_5$ (see (B) of Figure 16), then $d_{G_\sigma}(v_2, v_6) = 4$, and hence $diam(G_\sigma) \geq 4$; thus, $\sigma^{-1}(v_2) = u_5$ and $\sigma^{-1}(v_4) = u_6$. If $\sigma^{-1}(v_6) \neq u_4$ (see (C) of Figure 16), then $d_{G_\sigma}(v_2, v_6) = 4$, and thus $diam(G_\sigma) \geq 4$. So, $\sigma(u_4) = v_6$ and $\{\sigma(u_2), \sigma(u_3)\} = \{v_1, v_5\}$. If $\sigma(u_2) = v_1$ and $\sigma(u_3) = v_5$, then $d_{G_\sigma}(v_1, v_6) = 4$, and hence $diam(G_\sigma) \geq 4$; thus $\sigma(u_2) = v_5$ and $\sigma(u_3) = v_1$ (see (D) of Figure 16). If $G_\sigma$ is isomorphic to (D) of Figure 16, then $diam(G_\sigma) = 3$ and $V(G_\sigma) - \{u_5, v_2, v_3, v_4\}$ forms a strong resolving set for $G_\sigma$; thus $sdim(G_\sigma) \leq 2n - 4$. □

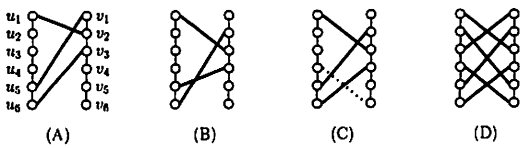



Figure 16: Permutation graphs $G_\sigma$ for $G = P_6$ such that $\{\sigma(u_1), \sigma(u_6)\} \cap \{v_1, v_6\} = \emptyset$

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