On friendly index sets of prisms and Möbius ladders

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Abstract Let G be a graph with vertex set V(G) and edge set E(G), and let A be an abelian group. A labeling f: V(G) → A induces an edge labeling f^*: E(G) → A defined by $f^*(xy) = f(x) + f(y)$, for each edge xy ∈ E(G). For i ∈ A, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$, and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. Let $c(f) = \{|e_f(i) - e_f(j)| : (i, j) \in A \times A\}$. A labeling f of a graph G is said to be A-friendly if $|v_f(i) - v_f(j)| \leq 1$ for all $(i, j) \in A \times A$. If c(f) is a (0, 1)-matrix for an A-friendly labeling f, then f is said to be A-cordial. When $A = Z_2$, the friendly index set of the graph G, FI(G), is defined as $\{|e_f(0) - e_f(1)|$: the vertex labeling f is Z_2-friendly$\}$. In this paper the friendly index sets for two classes of cubic graphs, prisms and Möbius ladders, are completely determined.

Key words: vertex labeling, friendly labeling, cordiality, friendly index set, cubic graphs.

AMS 2000 MSC: 05C78, 05C25

1. Introduction

Let G be a graph with vertex set V(G) and edge set E(G). Let A be an abelian group. A labeling f: V(G) → A induces an edge labeling f^*: E(G) → A defined by $f^*(xy) = f(x) + f(y)$, for each edge xy ∈ E(G). For i ∈ A, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$, and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. Let $c(f) = \{|e_f(i) - e_f(j)| : (i, j) \in A \times A\}$. A labeling f of a graph G is said to be A-friendly if $|v_f(i) - v_f(j)| \leq 1$ for all $(i, j) \in A \times A$. If c(f) is a (0, 1)-matrix for an A-friendly labeling f, then f is said to be A-cordial.

The notion of A-cordial labelings was first introduced by Hovey [10], who generalized the concept of cordial graphs of Cahit [2]. Several constructions of cordial graphs were proposed in [1, 3, 6, 7, 8, 10, 11, 12, 13, 14, 16, 21, 22, 23, 24]. For more details of known results and open problems on cordial graphs, see [4, 6].

In this paper, we will exclusively focus on $A = Z_2$, and drop the reference to the group. In [6] the friendly index set FI(G) of the graph G was introduced. The set FI(G) is defined as $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}$. When the context is clear, we will drop the subscript f. Note that if 0 or 1 is in FI(G), then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality.

Cairnie and Edwards [5] have determined the computational complexity of cordial labeling and Z_k-cordial labeling. They proved that to decide whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. Thus in general it is difficult to determine the friendly index sets of graphs.

In [17] the friendly index sets of complete bipartite graphs and cycles, are determined. In [9, 15, 17, 18, 19, 20] the friendly index sets of other classes of graphs are determined.

The following result was established in [17].

Theorem 1.1. For any graph with q edges, the friendly index set $FI(G) \subseteq \{0, 2, 4, \ldots, q\}$ if q is even and $FI(G) \subseteq \{1, 3, \ldots, q\}$ if q is odd.

In [15], the friendly index sets of 2-regular graphs are studied. In 1989, the first author, Ho and Shee [8] completely characterized cordial generalized Petersen graphs. It is natural to extend our study of friendly index sets to cubic graphs.

The smallest cubic graph is K_4, and $FI(K_4) = \{2\}$.

Example 1. There are two non-isomorphic cubic graphs of order 6, $K_{3,3}$ and the prism $C_3 \times P_2$. They have different friendly index sets: $FI(K_{3,3}) = \{1, 9\}$ and $FI(C_3 \times P_2) = \{1, 3, 5\}$.

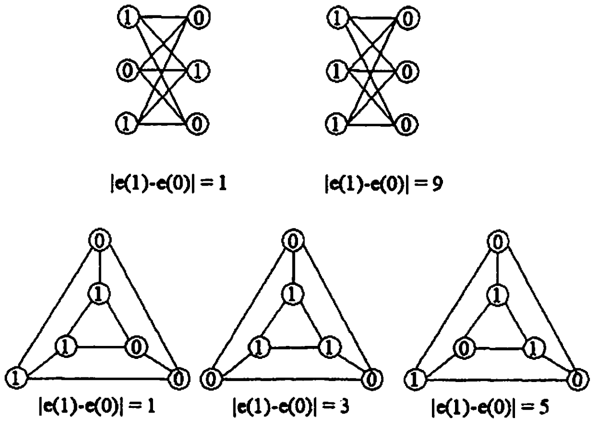


Figure 1.

<u>Example 2.</u> Among the six cubic graphs of order 8, two have friendly index set {0, 4, 12} (Figure 2) and four have friendly index set {0, 4, 8} (Figure 3).

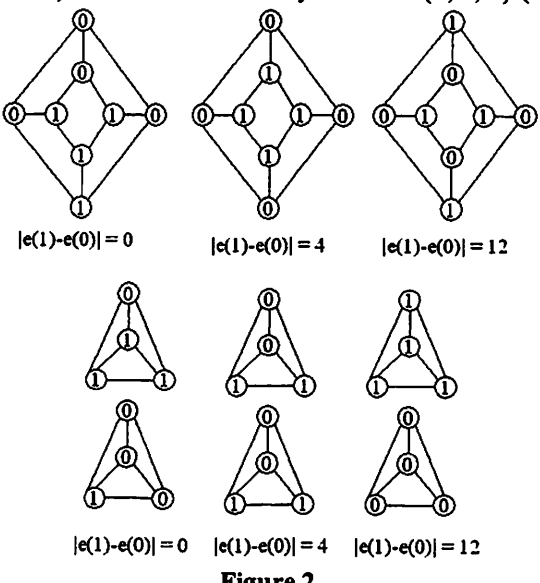
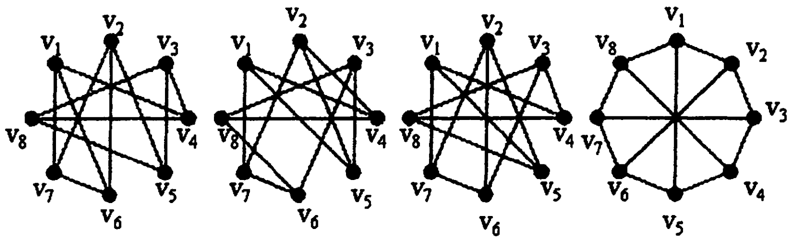


Figure 2.



G_1 G_2 G_3 G_4

Figure 3.

In the following we show the friendly index sets of G_1, G_2, G_3, and G_4.

		v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	$\|e(1)-e(0)\|$
	f_1	1	0	1	0	1	0	0	1	0
G_1	f_2	1	0	1	0	1	0	1	0	4
	f_3	1	1	1	0	0	0	0	1	8
	f_1	1	0	1	0	1	0	1	0	0
G_2	f_2	1	0	1	0	1	0	0	1	4
	f_3	1	0	0	0	1	0	1	1	8
	f_1	1	0	1	0	1	0	0	1	0
G_3	f_2	1	0	1	0	1	0	1	0	4
	f_3	1	1	0	0	0	1	0	1	8
	f_1	1	0	1	0	0	0	1	1	0
G_4	f_2	1	0	1	0	1	0	1	0	4
	f_3	1	0	1	0	0	1	0	1	8

The authors and Kwong [15] showed that for 2-regular graphs with two components, the numbers in the friendly index sets form arithmetic progressions. In this paper the friendly index sets for two classes of cubic graphs, prisms and Möbius ladders, will be completely determined. These two classes of cubic graphs are chosen to give some structure that we can handle, and to show that their friendly index sets are very different from each other. This somewhat confirms our previous statement on the difficulty to determine friendly index sets in general. The problem on the friendly index sets of general cubic graphs still seems to be beyond our reach at this moment.

2. Prisms

In this section, we will determine the friendly index set of the prism $C_n \times P_2$. Note that this graph has order 2n and size 3n. For notation, let $PM_n = C_n \times P_2$ denote the prism with $V = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$, with the vertices $u_1, u_2, \dots, u_n$ forming a cycle, the vertices $v_1, v_2, \dots, v_n$ forming a cycle, and the n edges $\{u_1, v_1\}, \{u_2, v_2\}, \dots \{u_n, v_n\}$ connecting the two cycles. To visualize the graph, assume that the u vertices form an outer cycle, while the v vertices form an inner cycle, with edges joining corresponding vertices of the two cycles.

From [17], we have the following

Lemma 2.1. Any vertex labeling (not necessarily friendly) of a cycle must have e(1) equal to an even number.

Lemma 2.2. The number of edges labeled 0 among the n edges $\{u_1, v_1\}, \{u_2, v_2\}, \dots \{u_n, v_n\}$ must be even.

Proof. An edge labeled 0 must be incident on two vertices with the same label, i.e., both vertices labeled 0 or 1. If there is an odd number of such edges, the vertices that they are incident on must have unequal numbers of 0 labels and 1 labels. Thus among the remaining vertices, there are unequal numbers of 0 labels and 1 labels. Then the remaining edge labels cannot all be 1, because an edge labeled 1 must be incident on two vertices with different labels.

Lemma 2.3. For $n \geq 2$, $FI(PM_{2n}) \subset \{6n - 4k \mid k \text{ is a non-negative integer and } 6n - 4k \geq 0\}$.

Proof. By Lemma 2.2, the number of edges labeled 0 among the 2n edges $\{u_1, v_1\}, \{u_2, v_2\}, \dots \{u_{2n}, v_{2n}\}$ must be even. Since there are altogether 2n such edges, the number of edges labeled 1 among them must also be even. Thus for the entire graph, e(1) must be even, i.e., all the possible values for e(1) are 0, 2, 4, ..., 6n – 4, 6n – 2, 6n. Then all the possible values for e(0) are 6n, 6n – 2, 6n – 4, ..., 4, 2, 0, respectively. We see that g = e(1) – e(0) = –6n, –6n – 4, –6n – 8, ..., 6n – 8, 6n – 4, 6n respectively. Taking absolute value gives the desired result.

Lemma 2.4. For $n \geq 2$, $6n - 4 \notin FI(PM_{2n})$.

Proof. The only ways to get 6n – 4 are to have e(0) = 6n – 2 and e(1) = 2, or e(0) = 2 and e(1) = 6n – 2. For the former to happen, one of the cycles must have all edges labeled 0, meaning that all vertex labels in this cycle are the same. Thus all vertex labels in the other cycle must also be the same, but different from those in the first cycle. It is then obvious that e(1) ≠ 2. For the latter to happen, one of the cycles must have all edges labeled 1, meaning that the vertex labels in this cycle alternate. It is then obvious that we cannot have exactly two edges labeled 0.

Theorem 2.1. For $n \geq 2$, $FI(PM_{2n}) = \{6n\} \cup \{6n - 8 - 4k \mid k$ is a non-negative integer and $6n - 8 - 4k \geq 0\}$. Thus $FI(PM_{2n}) = \{0, 4, 8, \ldots, 6n - 8, 6n\}$ if n is even, and $FI(PM_{2n}) = \{2, 6, 10, \ldots, 6n - 8, 6n\}$ if n is odd.

Proof. The lemmas above show that these are the only possible values in $FI(PM_{2n})$. It suffices to show that they are all attainable.

Let all the u vertices be labeled 0, and all the v vertices be labeled 1. Then $e(1) = 2n$ and $e(0) = 4n$, making $g = e(1) - e(0) = -2n$. Successively interchange the vertex labels at u_j and v_j where $j = 1, 3, 5, \ldots, 2n - 1$. Each such interchange increases e(1) by 4 and decreases e(0) by 4. These give the possible g values of $-2n + 8i$, where $i = 0, 1, 2, \ldots, n$. The absolute values are $6n$, $6n - 8$, $6n - 16$, ...

Now let all the u vertices be labeled 0, with the exception of u_3, which is labeled 1, and let all the v vertices be labeled 1, with the exception of v_1, which is labeled 0. Then $e(1) = 2n + 2$ and $e(0) = 4n - 2$, making $g = e(1) - e(0) = -2n + 4$. Successively interchange the vertex labels at u_j and v_j where $j = 5, 7, \ldots, 2n - 1$. Each such interchange increases e(1) by 4 and decreases e(0) by 4. These give the possible g values of $-2n + 4 + 8i$, where $i = 0, 1, 2, \ldots, n - 2$. The absolute values are $6n - 12$, $6n - 20$, ...

Example 3. We see from Example 2 that $FI(PM_4) = \{0, 4, 12\}$. Figure 4 shows that $FI(PM_6) = \{2, 6, 10, 18\}$.

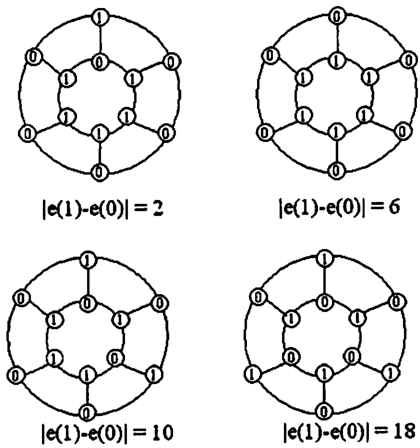


Figure 4.

Lemma 2.5. For $n \geq 1$, $6n + 3 \notin FI(PM_{2n+1})$.

Proof. In an odd cycle, there must be two neighboring vertices with the same label. Thus $e(0) \neq 0$. On the other hand, $e(1) \neq 0$, because otherwise all the vertex labels in the outer cycle must be the same, making all the vertex labels in the inner cycle the same but different from those of the outer cycle, which is not possible for $e(1) = 0$.

Lemma 2.6. For $n \geq 1$, $6n + 1 \notin FI(PM_{2n+1})$.
Proof. The only ways to get $6n + 1$ are to have $e(0) = 6n + 2$ and $e(1) = 1$, or $e(0) = 1$ and $e(1) = 6n + 2$. For $e(1) = 1$, one of the cycles must have all edges labeled 0, i.e., all vertices labeled the same way. This would require all the vertex labels in the other cycle the same but different from those of the first cycle, which is not possible for $e(1) = 1$. On the other hand, by Lemma 2.1, the number of edges labeled 0 in the outer cycle must be odd, and the number of edges labeled 0 in the inner cycle must also be odd. Using these and Lemma 2.2, $e(0)$ must be even, making $e(0) = 1$ impossible.

Lemma 2.7. For $n \geq 2$, $6n - 3 \notin FI(PM_{2n+1})$.
Proof. The only ways to get $6n - 3$ are to have $e(0) = 6n$ and $e(1) = 3$, or $e(0) = 3$ and $e(1) = 6n$. For $e(1) = 3$, using Lemma 2.1, we see that one of the cycles must have all edges labeled 0, i.e., all vertices labeled the same way. This would require all the vertex labels in the other cycle the same but different from those of the first cycle, giving $e(1) = 2n + 1$. Since $n \neq 1$, $e(1) = 3$ is impossible. On the other hand, by Lemma 2.1, the number of edges labeled 0 in the outer cycle must be odd, and the number of edges labeled 0 in the inner cycle must also be odd. Using these and Lemma 2.2, $e(0)$ must be even, making $e(0) = 3$ impossible.

Theorem 2.2. $FI(PM_3) = \{1, 3, 5\}$. For $n \geq 2$, $FI(PM_{2n+1}) = \{6n - 1\} \cup \{6n - 5 - 2k \mid k \text{ is a non-negative integer and } 6n - 5 - 2k \geq 0\}$.
Proof. By Lemmas 2.5 and 2.6, $FI(PM_3) \subset \{1, 3, 5\}$. Figure 1 shows that all these values are possible.

For $n \geq 2$, the lemmas above show that the given values are the only possible values in $FI(PM_{2n+1})$. It suffices to show that they are all attainable.

Let $u_1, u_3, \ldots, u_{2n-1}$ be labeled 0, and $u_2, u_4, \ldots, u_{2n}, u_{2n+1}$ be labeled 1. Let $v_1, v_3, \ldots, v_{2n-1}$ be labeled 1, and $v_2, v_4, \ldots, v_{2n}, v_{2n+1}$ be labeled 0. Then $e(1) = 6n + 1$ and $e(0) = 2$, making $g = e(1) - e(0) = 6n - 1$. Successively interchange the vertex labels at u_j and v_j where $j = 1, 3, 5, \ldots, 2n - 1$. Each such interchange decreases $e(1)$ by 4 and increases $e(0)$ by 4. These give the possible g values of $6n - 1 - 8i$, where $i = 0, 1, 2, \ldots, n$, i.e., $6n - 1, 6n - 9, 6n - 17, \ldots, -2n + 7, -2n - 1$.

Let u_1 and v_1 be labeled 0, u_2 and v_2, be labeled 1. Let $u_3, u_5, \ldots, u_{2n+1}$ be labeled 0, and $u_4, u_6, \ldots, u_{2n}$ be labeled 1. Let $v_3, v_5, \ldots, v_{2n+1}$ be labeled 1, and $v_4, v_6, \ldots, v_{2n}$ be labeled 0. Then $e(1) = 6n - 1$ and $e(1) = 4$, making $g = e(1) - e(0) = 6n - 5$. Successively interchange the vertex labels at u_j and v_j where $j = 4, 6, \ldots, 2n$. Each such interchange decreases $e(1)$ by 4 and increases $e(0)$ by 4. These give the possible g values of $6n - 5 - 8i$, where $i = 0, 1, 2, \ldots, n - 1$, i.e., $6n - 5, 6n - 13, 6n - 21, \ldots, -2n + 11, -2n + 3$.

Let $u_1, u_2, \ldots, u_{n+1}$ be labeled 0, and $u_{n+2}, u_{n+3}, \ldots, u_{2n+1}$ be labeled 1. Let $v_1, v_2, \ldots, v_n$ be labeled 0, and $v_{n+1}, v_{n+2}, \ldots, v_{2n+1}$ be labeled 1. Then $e(1) = 5$ and $e(0) = 6n - 2$, making $g = e(1) - e(0) = -6n + 7$. Successively interchange the vertex labels at v_j and v_{2n+1-j} where $j = n, n - 1, \ldots, 2$. Each such interchange increases $e(1)$ by 2 and decreases $e(0)$ by 2. These give the possible g values of

$-6n + 7 + 4i$, where $i = 0, 1, 2, \dots, n-1$, i.e., $-6n + 7, -6n + 11, -6n + 15, \dots, -2n + 3$.

Now we need only show that these three sequences of numbers give all the values in = $\{6n - 1\} \cup \{6n - 5 - 2k \mid k$ is a non-negative integer and $6n - 5 - 2k \geq 0\}$. Combining the first two sequences, we have $6n - 1, 6n - 5, 6n - 9, 6n - 13, 6n - 17, 6n - 21, \dots, -2n + 11, -2n + 7, -2n + 3, -2n - 1$, i.e., $6n - 1 - 4i$, where $i = 0, 1, 2, \dots, 2n$. The absolute values are $6n - 1, 6n - 5, 6n - 9, 6n - 13, 6n - 17, 6n - 21, \dots, 2n + 7, 2n + 3, 2n + 1, 2n - 1, 2n - 3, 2n - 5, \dots, 7, 5, 3, 1$, i.e., $6n - 1 - 4i$, $i = 0, 1, 2, \dots, n - 1$, and $2n + 1 - 2i$, $i = 0, 1, 2, \dots, n$. The absolute values of the third sequence are $6n - 7, 6n - 11, 6n - 15, \dots, 2n - 3$, i.e., $6n - 7 - 4i$, where $i = 0, 1, 2, \dots, n - 1$. All these absolute values together constitute $\{6n - 1\} \cup \{6n - 5 - 2k \mid k$ is a non-negative integer and $6n - 5 - 2k \geq 0\}$, with $2n + 1$ and $2n - 3$ repeating.

<u>**Example 4.**</u> Figure 5 shows that $FI(PM_5) = \{1, 3, 5, 7, 11\}$.

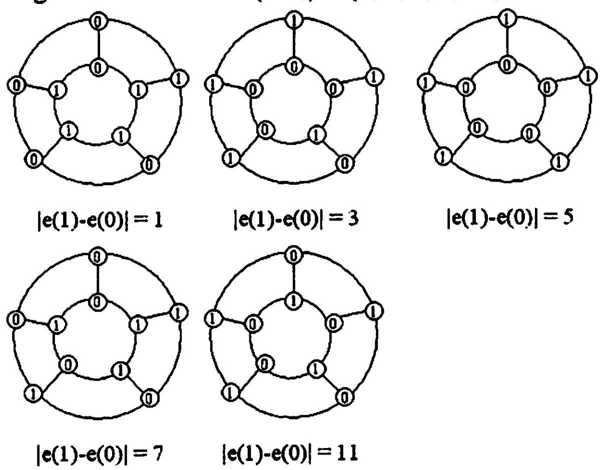


Figure 5.

3. Möbius ladders

Let n be a positive integer. The Möbius ladder (also known as the Möbius wheel) is the cycle C_{2n}, with n additional edges joining diagonally opposite vertices. We will denote this graph by M_{2n}, and its vertices by $v_1, v_2, \dots, v_{2n}$. Then the edges are $(v_1, v_2), (v_2, v_3), \dots, (v_{2n}, v_1)$ of the cycle, and the n diagonals $(v_1, v_{n+1}), (v_2, v_{n+2}), \dots, (v_n, v_{2n})$. Figure 6 shows the Möbius ladder M_{2n} for $n = 3, 4$, drawn in both the circulant form and the ladder form.

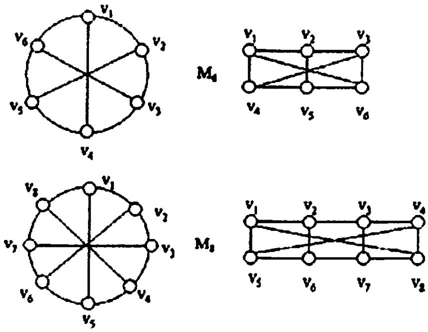


Figure 6.

<u>**Lemma 3.1.**</u> e(0) is even.
Proof. In the cycle, the number of edges labeled 1 must be even, and so the number of edges labeled 0 must also be even. A diagonal labeled 0 must link together vertices with the same label. An odd number of diagonals labeled 0 would link together vertices with unequal 0 labels and 1 labels. Then it is impossible for the remaining diagonals to be all labeled 1.

<u>**Lemma 3.2.**</u> $e(1) \neq 0$.
Proof. If all edges have labels 0, then all vertex labels are the same. This is not a friendly vertex labeling.

<u>**Lemma 3.3.**</u> $e(1) \neq 2$.
Proof. Assume $e(1) = 2$. Since the vertex labels cannot all be the same, there must be at least one edge in the cycle labeled 1. Thus both 1 edge labels must be in the cycle. The vertex labels must be a sequence of 0's followed by a sequence of 1's. These 0's and 1's are opposite to each other, making all diagonal labels equal to 1.

<u>**Lemma 3.4.**</u> In M_{4n}, $e(0) \neq 0$.
Proof. Otherwise, all edges are labeled 1. Then the vertex labels in the cycle must be labeled alternately as 0 and 1. If so, diagonally opposite vertices have the same label, and so all the diagonals have edge labels 0.

<u>**Corollary 3.5.**</u> $FI(M_{4n}) \subset \{6n - 4 - 4k \mid k \text{ is a non-negative integer and } 6n - 4 - 4k \geq 0\}$.
Proof. In M_{4n}, there are 6n edges. The possible values of e(0) are 2, 4, ... , 6n – 4, and the corresponding values of e(1) are 6n – 2, 6n – 4, ... , 4. Thus the possible values of |e(1) – e(0)| are 6n – 4, 6n – 8, ... , ending with 0 if n is even, ending with 2 if n is odd.

To show that all these values are attainable, we introduce the following constructions.

Construction 1 (for M_{4n}):

Label the vertices $v_1, v_2, \dots, v_{4n}$ alternately by 0 and 1. Interchange the vertex labels at v_1 and v_2. In this graph, $e(1) = 4n$ and $e(0) = 2n$, giving $g = e(1) - e(0) = 2n$. Successively interchange the vertex labels at v_{2j-1} and v_{2j} where $j = 2, 3, \dots, n$. Each such interchange increases the number of 1-diagonals by 2 and decreases the number of 0-diagonals by 2. These give the possible g values of $2n + 4i$, where $i = 0, 1, 2, \dots, n - 1$, i.e., $2n, 2n + 4, \dots, 6n - 4$.

Construction 2 (for M_{4n}, with n even):

Label the vertices $v_1, v_2, \dots, v_{4n}$ sequentially by 0 0 1 1 0 0 1 1, etc. Interchange the vertex labels at v_1 and v_{4n}. In this graph, $e(1) = 2n + 4$ and $e(0) = 4n - 4$, giving $g = e(1) - e(0) = -2n + 8$. Successively interchange the vertex labels at v_j and v_{4n-j+1} where $j = 3, 5, \dots, 2n - 1$. Each such interchange increases the number of 1-diagonals by 2 and decreases the number of 0-diagonals by 2. These give the possible g values of $-2n + 8 + 4i$, where $i = 0, 1, 2, \dots, n - 1$, i.e., $-2n + 8, -2n + 12, \dots, 2n + 4$.

We note that $-2n + 8 \leq 0$ if and only if $n \geq 4$.

Construction 3 (for M_{4n}, with n odd):

Label the vertices v_{2n-1} and v_{2n} by 1, the vertices v_{4n-1} and v_{4n} by 0. Label the vertices $v_1, v_2, \dots, v_{2n-2}$ sequentially by 0 0 1 1 0 0 1 1, etc. Label the vertices $v_{2n+1}, v_{2n+2}, \dots, v_{4n-2}$ sequentially by 0 0 1 1 0 0 1 1, etc. Interchange the vertex labels at v_1 and v_{4n-2}. In this graph, $e(1) = 2n + 4$ and $e(0) = 4n - 4$, giving $g = e(1) - e(0) = -2n + 8$. Successively interchange the vertex labels at v_j and v_{4n-1-j} where $j = 3, 5, \dots, 2n - 3$. Each such interchange increases the number of 1-diagonals by 2 and decreases the number of 0-diagonals by 2. These give the possible g values of $-2n + 8 + 4i$, where $i = 0, 1, 2, \dots, n - 2$, i.e., $-2n + 8, -2n + 12, \dots, 2n$.

We note that $-2n + 8 \leq 2$ if and only if $n \geq 3$.

__Theorem 3.1.__ $FI(M_{4n}) = \{6n - 4 - 4k \mid k \text{ is a non-negative integer and } 6n - 4 - 4k \geq 0\}$.

Proof. For $n = 1$, M_4 has 4 vertices and 6 edges. The vertex labels 0 0 1 1 give $e(1) - e(0) = 2$. For n odd and ≥ 3, Constructions 1 and 3 show that all the possible values are attainable.

For $n = 2$, M_8 has 8 vertices and 12 edges. The vertex labels 1 0 1 0 0 1 0 1 give $e(1) - e(0) = 8$. The vertex labels 1 0 1 0 1 0 0 1 give $e(1) - e(0) = 4$. The vertex labels 1 1 1 1 0 0 0 0 give $e(1) - e(0) = 0$. For n even and ≥ 4, Constructions 1 and 2 show that all the possible values are attainable.

Example 5. FI(M_{12}) = {2, 6, 10, 14}.

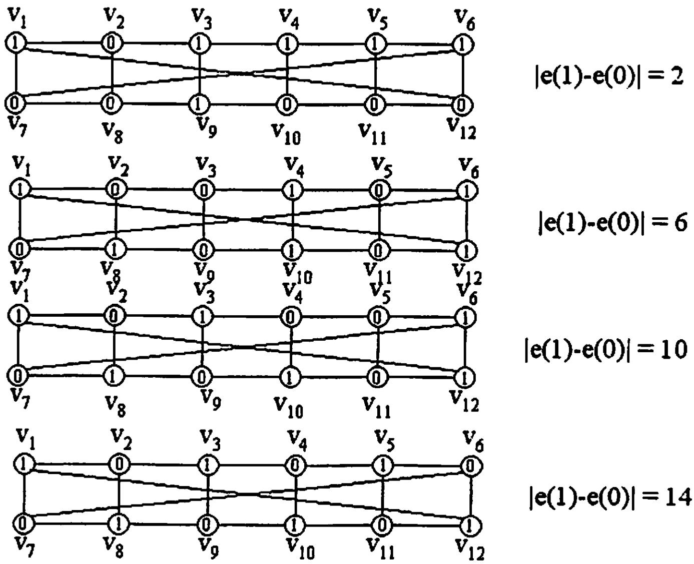
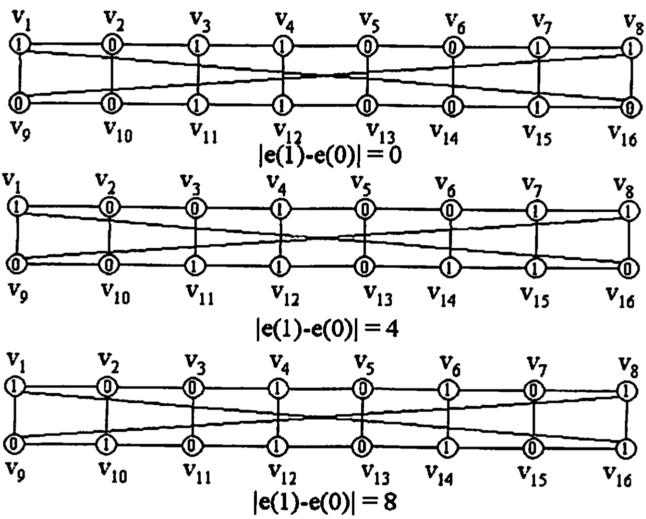
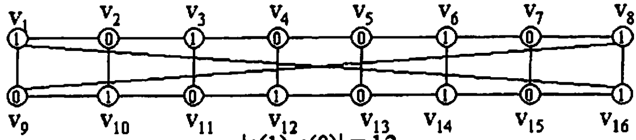


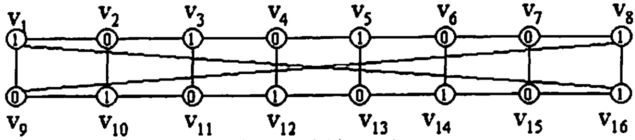
Figure 7.

Example 6. FI(M_{16}) = {0, 4, 8, 12, 16, 20}.

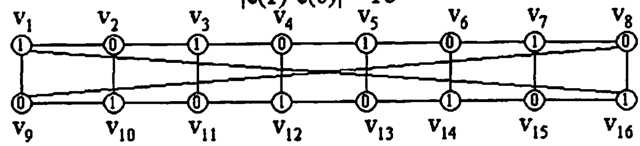




$|e(1)\text{-}e(0)| = 12$



$|e(1)\text{-}e(0)| = 16$



$|e(1)\text{-}e(0)| = 20$

Figure 8.

__Lemma 3.6.__ In M_{4n+2}, $e(0) \neq 2$.
Proof. If $e(0) = 2$, the two edges labeled 0 must be both diagonals or both in the cycle. If they are both diagonals, the edges in the cycle all have labels 1, and so the vertex labels must be alternately 0's and 1's. Then all diagonals should have 1 labels, which is a contradiction. Now assume the 0 edge labels are in the cycle. The incident vertices must be two 0's and two 1's, otherwise there must be other vertices with the same label next to each other. For these two 0's and two 1's, the vertices between them must be alternately labeled 1's and 0's. Since there are $4n + 2$ vertices, the two 0's and the two 1's cannot be exactly opposite to each other. Thus there must be two vertices with the same label diagonally opposite to each other.

__Lemma 3.7.__ In M_{4n+2}, $e(1) \neq 1$.
Proof. If $e(1) = 1$, the only edge labeled 1 must be a diagonal. Since all the edge labels in the cycle are 0, the vertex labels must all be the same. This is not a friendly vertex labeling.

__Lemma 3.8.__ In M_{4n+2}, $e(1) \neq 3$.
Proof. Assume $e(1) = 3$. Then either $e(1) = 0$ in the cycle or $e(1) = 2$ in the cycle. If $e(1) = 0$ in the cycle, all the vertex labels in the cycle must be the same. This is not a friendly vertex labeling. If $e(1) = 2$ in the cycle, then there is a unique diagonal labeled 1, i.e., a unique diagonal connecting a 0-vertex and a 1-vertex. Let this unique diagonal be (v_1, v_{2n+2}). The labels of $v_2, v_3, \ldots, v_{2n+1}$ cannot have the same label, because otherwise the labels of $v_{2n+3}, v_{2n+4}, \ldots, v_{4n+2}$ must have the same complementary label, giving all diagonals labeled 1. Thus among the vertices $v_2, v_3, \ldots, v_{2n+1}$, two adjacent vertices must have complementary labels. Then their diagonally opposite vertices must have the

same complementary labels. By considering how these 0-vertices and 1-vertices are situated relative to (v_1, v_{2n+2}), it is obvious that there must be other 1-edges.

Corollary 3.9. $FI(M_{4n+2}) \subset \{6n + 3\} \cup \{6n - 5 - 2k \mid k$ is a non-negative integer and $6n - 5 - 2k > 0\}$.

Proof. In M_{4n+2}, there are $6n + 3$ edges. The possible values of e(0) are 0, 4, 6, 8, ... , $6n - 4$, $6n - 2$, and the corresponding values of e(1) are $6n + 3$, $6n - 1$, $6n - 3$, $6n - 5$, ... , 7, 5. Thus the possible values of $|e(1) - e(0)|$ are $6n + 3$, $6n - 5$, $6n - 7$, $6n - 9$, $6n - 11$, ... , 3, 1.

To show that all these values are attainable, we introduce the following constructions.

Construction 4 (for M_{4n+2}):

Label the vertex v_{2n+1} by 1, and the vertex v_{4n+2} by 0. Label the vertices v_1, v_2, ... , v_{2n} alternately by 0 and 1. Label the vertices v_{2n+2}, v_{2n+3}, ... , v_{4n+1} alternately by 0 and 1. In this graph, $e(1) = 4n + 1$ and $e(0) = 2n + 2$, giving $g = e(1) - e(0) = 2n - 1$. Successively interchange the vertex labels at v_{2j-1} and v_{2j} where $j = 1, 2, 3, \ldots, n - 1$. Each such interchange increases the number of 1-diagonals by 2 and decreases the number of 0-diagonals by 2. These give the possible g values of $2n - 1 + 4i$, where $i = 0, 1, 2, \ldots, n - 1$, i.e., $2n - 1$, $2n + 3$, ... , $6n - 5$.

Finally interchange the vertex labels of v_{2n-1} and v_{2n}. Besides the same changes in 1-diagonals and 0-diagonals, this also increases the number of 1-edges by 2 and decreases the number of 0-edges by 2 in the cycle, making $g = 6n + 3$.

Construction 5 (for M_{4n+2}):

Label the vertex v_{2n+1} by 1, and the vertex v_{4n+2} by 0. Label the vertices v_1, v_2, ... , v_{2n} by 0, and the vertices v_{2n+2}, v_{2n+3}, ... , v_{4n+1} by 1. Interchange the vertex labels at v_1 and v_{4n+1}. In this graph, $e(1) = 2n + 3$ and $e(0) = 4n$, giving $g = e(1) - e(0) = -2n + 3$. Successively interchange the vertex labels at v_j and v_{4n+2-j} where $j = 2, 3, \ldots, n$. Each such interchange decreases the number of 1-diagonals by 2 and increases the number of 0-diagonals by 2. These give the possible g values of $-2n + 3 - 4i$, where $i = 0, 1, 2, \ldots, n - 1$, i.e., $-2n + 3$, $-2n - 1$, ... , $-6n + 7$.

Construction 6 (for M_{4n+2}, with n even):

Label the vertex v_{2n+1} by 1, and the vertex v_{4n+2} by 0. Label the vertices v_1, v_2, ... , v_{2n} sequentially by 0 0 1 1 0 0 1 1, etc. Label the vertices v_{2n+2}, v_{2n+3}, ... , v_{4n+1} sequentially by 0 0 1 1 0 0 1 1, etc. Interchange the vertex labels at v_1 and v_{4n+1}. In this graph, $e(1) = 2n + 5$ and $e(0) = 4n - 2$, giving $g = e(1) - e(0) = -2n + 7$. Successively interchange the vertex labels at v_j and v_{4n+2-j} where $j = 3, 5, \ldots, 2n - 1$. Each such interchange increases the number of 1-diagonals by 2 and decreases the number of 0-diagonals by 2. These give the possible g values of $-2n + 7 + 4i$, where $i = 0, 1, 2, \ldots, n - 1$, i.e., $-2n + 7$, $-2n + 11$, ... , $2n + 3$.

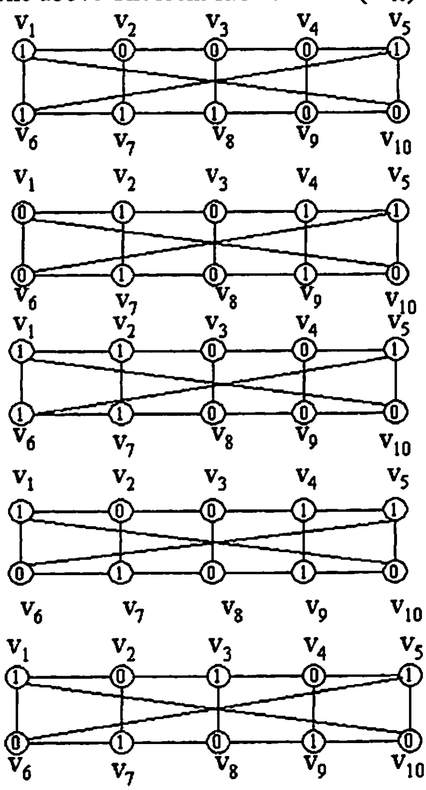
Construction 7 (for M_{4n+2}, with n odd):

Label the vertices v_{2n-1}, v_{2n} and v_{2n+1} by 1, 1 and 0, the vertices v_{4n}, v_{4n+1} and v_{4n+2} by 1, 0 and 0. Label the vertices $v_1, v_2, \ldots, v_{2n-2}$ sequentially by 0 0 1 1 0 0 1 1, etc. Label the vertices $v_{2n+2}, v_{2n+3}, \ldots, v_{4n-1}$ sequentially by 0 0 1 1 0 0 1 1, etc. Interchange the vertex labels at v_1 and v_{4n-1}. In this graph, $e(1) = 2n + 5$ and $e(0) = 4n - 2$, giving $g = e(1) - e(0) = -2n + 7$. Successively interchange the vertex labels at v_j and v_{4n-j} where $j = 3, 5, \ldots, 2n - 3$. Each such interchange increases the number of 1-diagonals by 2 and decreases the number of 0-diagonals by 2. These give the possible g values of $-2n + 7 + 4i$, where $i = 0, 1, 2, \ldots, n - 2$, i.e., $-2n + 7, -2n + 11, \ldots, 2n - 1$.

Theorem 3.2. $FI(M_{4n+2}) = \{6n + 3\} \cup \{6n - 5 - 2k \mid k$ is a non-negative integer and $6n - 5 - 2k > 0\}$.

Proof. For n odd, Constructions 4, 5 and 7 show that all the possible values are attainable. For n even, Constructions 4, 5 and 6 show that all the possible values are attainable.

Example 7. $K_{3,3}$ is isomorphic to M_6. We see in Example 1 that $FI(M_6) = \{1, 9\}$. The above Theorem shows that $FI(M_{10}) = \{1, 3, 5, 7, 15\}$.



$|e(1)-e(0)| = 1$

$|e(1)-e(0)| = 3$

$|e(1)-e(0)| = 5$

$|e(1)-e(0)| = 7$

$|e(1)-e(0)| = 15$

Figure 9.

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