On the (l, ω)-domination numbers of the circulant network *

Xin Xie[a†] Jun-Ming Xu[b]

[a]School of Mathematics and Statistics, Huangshan University
Huangshan, 245041, China

[b]Department of Mathematics, University of Science and Technology of China
Hefei, 230026, China

Abstract For an n-connected graph G, the n-wide diameter $d_n(G)$, is the minimum integer m such that for any two vertices x and y there are at least n internally disjoint paths of length at most m from x to y. For a given integer l, a subset S of $V(G)$ is called a (l, n)- dominating set of G if for any vertex $x \in V(G) - S$ there are at least n internally disjoint (di)paths of length at most l from S to x. The minimum cardinality among all (l, n)-dominating sets of G is called the (l, n)-domination number. In this paper, we obtain that the (l, ω)-domination numbers of the circulant digraph $G(d^n; \{1, d, \cdots, d^{n-1}\})$ is equal to 2 for $1 \leq \omega \leq n$ and $d_\omega(G) - (g(d, n) + \delta) \leq l \leq d_\omega(G) - 1$, where $g(d, n) = \min\{e\lceil n/2\rceil - e - 2, (\lfloor n/2\rfloor + 1)(e' - 1) - 2\}$, $\delta = 0$ for $1 \leq \omega \leq n - 1$ and $\delta = 1$ for $\omega = n$.

Keywords: Circulant network, Wide diameter, Reliability, Domination number

MR Subject Classification: 05C40 68M10 68M15 68R10

1 Introduction

This paper uses graphs to represent networks. The notions of diameter and connectivity of graphs have been treated extensively in the graph theory literature. Reliability and efficiency are important criteria in the design of interconnection networks. The distance $d_G(x, y)$ from a vertex x to another vertex y in a network G is the minimum number of edges of a path from

*The work was supported by NNSF of China (Nos.61272008; 11071233), Foundation of Anhui Province Educational Committee and Foundation of Huangshan University (No.2011xkj012).

† Corresponding author: xiexin@hsu.edu.cn

x to y. The diameter $d(G)$ is the maximum distance from one vertex to another. The connectivity $k(G)$ is the minimum number of vertices whose removal results in a disconnected or trivial network.

In order to characterize the reliability of transmission delay in a real-time parallel processing system, Hsu and Lyuu [9], Flandrin and Li [6] independently introduced n-wide diameter, which unifies diameter and connectivity. For an n-connected graph G, the distance with width n from x to y, denoted by $d_n(G;x,y)$, is the minimum number m for which there are n internally disjoint (x,y)-paths in G of length at most m. The n-wide diameter of G, i.e., the n-diameter, denoted by $d_n(G)$, is the maximum of $d_n(G;x,y)$ over all pairs (x,y) of vertices of G.

Li and Xu [10] defined a new parameter (l,n)-domination number in an n-connected undirected graph, in some sense, which can more accurately characterize the reliability of networks than the wide-diameter does. This motives us to generalize the definition to that of the digraph. Let G be an n-connected digraph, S a nonempty and proper subset of $V(G)$, x a vertex in $G-S$. For a given positive integer l, x is (l,n)-dominated by S if there are at least n internally disjoint (S,x)-dipaths of length at most l. S is said to be a (l,n)-dominating set of G if S can (l,n)-dominate any vertex in $G-S$. The minimum cardinality among all (l,n)-dominating sets of G is called the (l,n)-domination number, denoted by $\gamma_{l,n}(G)$.

The circulant networks, which were originally proposed by Elspas and Turner [5], have proved very useful in the design of fault-tolerant multiprocessor systems [1, 4]. For a positive integer N, let Z_N be the additive group of residue classes modulo N. The circulant digraph $G(N;A)$ associated with N and a subset $A \subseteq Z_N - \{0\}$ is a digraph with a vertex set Z_N and an edge set $\{xy : x,y \in Z_N \text{ and } y-x \in A\}$. It is clear that $G(N;A)$ is vertex-transitive. Every vertex in $G(N;A)$ has an indegree and an outdegree that are equal to $|A|$.

Let $N=d^n$ and $A=\{1,d,\cdots,d^{n-1}\}$, then the diameter of $G(d^n;A)$ is $n(d-1)$. Hamidoune [8] showed that the connectivity of $G(d^n;A)$ is n. Hsu and Lyuu [9], Duh and Chen [3] proved that $d_n(G(d^n;A)) = n(d-1)+1$, respectively. Liaw et al. [11] further showed that $d_\omega(G(d^n;A)) = n(d-1)$ for $1 \le \omega \le n-1$ and $d_n(G(d^n;A)) = n(d-1)+1$.

In general, to determine the (l,n)-domination number of a graph is NP-Complete since its special case of $(1,1)$, the domination number of the graph, is NP-Complete[7]. It is clear that $\gamma_{l,n}(G) = 1$ for $l \ge d_n(G)$ and $\gamma_{l,n}(G) \ge 2$ for $l < d_n(G)$. So it is of interest to show some general properties [10, 14] and values of the (l,n)-domination numbers for $l < d_n(G)$ (see, for example [12, 13, 15]).

This paper studies the (l,ω)-domination numbers of the circulant digraph $G(d^n;\{1,d,\cdots,d^{n-1}\})$ if $d \ge 4$ and $n \ge 5$, obtains $\gamma_{l,\omega}(G) = 2$ for $1 \le \omega \le n$ and $d_\omega(G) - (g(d,n)+\delta) \le l \le d_\omega(G) - 1$, where

$g(d,n) = \min\{e\lceil n/2\rceil - e - 2, (\lfloor n/2\rfloor + 1)(e' - 1) - 2\}$, $\delta = 0$ for $1 \le \omega \le n-1$ and $\delta = 1$ for $\omega = n$, $\lfloor \frac{d}{2} \rfloor = e$ and $\lceil \frac{d}{2} \rceil = e'$.

Terminologies and notations not defined here are referred to [2].

2 Preliminaries

In this paper, we take $A = \{1, d, \cdots, d^{n-1}\}$, denote $\lfloor \frac{d}{2} \rfloor$ and $\lceil \frac{d}{2} \rceil$ by e and e', respectively. Each vertex x of the circulant digraph $G(d^n; A)$ with $0 \le x < N = d^n$ can be encoded as $x = x_{n-1}d^{n-1} + x_{n-2}d^{n-2} + \cdots + x_1 d + x_0 \mod N$, where $0 \le x_{n-1}, x_{n-2}, \cdots, x_0 \le d-1$. Thus, vertex x can also be written as $(x_{n-1}, x_{n-2}, \cdots, x_0)$, and x is connected to $(x_{n-1}+1, x_{n-2}, \cdots, x_0)$, $(x_{n-1}, x_{n-2}+1, \cdots, x_0)$, $\cdots$, $(x_{n-1}, x_{n-2}, \cdots, x_0+1)$, where additions are performed modulo d^n. For $(x_{n-1}, x_{n-2}, \cdots, x_0)$, we call x_i the ith component of x.

If $x = (\cdots, x_i, x_{i-1}, \cdots, x_{j+1}, x_j, \cdots)$ with $x_{i-1}, \cdots, x_{j+1} \le e$ and $x_i, x_j > e$, call such a sequence a maximal sequence of components with values not exceeding e at $i-1$ with length $i-j-1$ and x_i its leading component.

Suppose $0 \le j_r < j_{r-1} < \cdots < j_1 \le n-1$, $0 < x_{j_r}, x_{j_{r-1}}, \cdots, x_{j_1} \le d-1$. For any vertices u, $v = u + (0, \cdots, 0, x_{j_1}, \cdots, x_{j_{r-1}}, 0, \cdots, 0, x_{j_r}, 0, \cdots, 0)$ in $G(d^n; A)$, we denote the u-v dipath

$$\begin{aligned} u \; & \rightarrow u + d^{j_r} \rightarrow u + 2d^{j_r} \rightarrow \cdots \rightarrow u + x_{j_r}d^{j_r} \\ & \rightarrow u + x_{j_r}d^{j_r} + d^{j_{r-1}} \rightarrow u + x_{j_r}d^{j_r} + 2d^{j_{r-1}} \rightarrow \cdots \rightarrow u + x_{j_r}d^{j_r} + \\ & x_{j_{r-1}}d^{j_{r-1}} \\ & \rightarrow \cdots \\ & \rightarrow u + x_{j_r}d^{j_r} + x_{j_{r-1}}d^{j_{r-1}} + \cdots + d^{j_1} \rightarrow u + x_{j_r}d^{j_r} + x_{j_{r-1}}d^{j_{r-1}} + \\ & \cdots + 2d^{j_1} \rightarrow \cdots \rightarrow v \end{aligned}$$

by $<< x_{j_r}d^{j_r}, x_{j_{r-1}}d^{j_{r-1}}, \cdots, x_{j_1}d^{j_1} >>$.

Using the method in [14], we can easily have the following proposition.

Proposition 2.1 For any digraph G of connectivity n, then
(1) $\gamma_{l+1,n}(G) \le \gamma_{l,n}(G)$, where l is a positive integer.
(2) $\gamma_{l,\omega-1}(G) \le \gamma_{l,\omega}(G)$ for $1 \le \omega \le n$.

3 Main results

Lemma 3.1 Let $o = (0, 0, \cdots, 0) \in V(G(d^n; A))$. For any vertex $x = (x_{n-1}, x_{n-2}, \cdots, x_0) \in V(G(d^n; A))$ with $x_i > 0$ for $i = 0, 1, \cdots, n-1$, then there exist n internally disjoint dipaths from o to x each with length $\sum_{i=0}^{n-1} x_i$.

Proof Construct the n dipaths from o to x as follows:

$P_s :<< x_s d^s, x_{s+1} d^{s+1}, \cdots, x_{n-1} d^{n-1}, x_0 d^0, \cdots, x_{s-1} d^{s-1} >>$ for $s = 0, 1, \cdots, n-1$.

So the length of each dipath is $\sum_{i=0}^{n-1} x_i$. ∎

Lemma 3.2 Let $u = (e, e, \cdots, e) \in V(G(d^n; A))$. For any vertex $x = (x_{n-1}, x_{n-2}, \cdots, x_0)$
$\in V(G(d^n; A))$ with $x_{n-1} > e$, if x has $l (\geq 1)$ components with values e and the others with values more than e, then there exist n internally disjoint dipaths from u to x each with length at most $\sum_{i=0}^{n-1} (x_i - e) + l(d-1)$.

Proof Assume x has maximal sequences of es at $i_1, i_2, \cdots, i_k$ with lengths $j_1, j_2, \cdots, j_k > 0$, respectively, where $k \geq 1$, $i_1 > i_2 > \cdots > i_k$. Without loss of generality, we let $\sum_{t=1}^{k} j_t = l$ and

$$x = (x_{n-1}, \cdots, x_{i_1+1}, \overbrace{e, \cdots, e}^{j_1}, x_{i_1-j_1}, \cdots, x_{i_2+1}, \overbrace{e, \cdots, e}^{j_2}, x_{i_2-j_2}, \cdots,$$
$$x_{i_k+1}, \overbrace{e, \cdots, e}^{j_k}, x_{i_k-j_k}, \cdots, x_0),$$

where $x_i > e$ for $i \neq i_t - j_t + 1, i_t - j_t + 2, \cdots, i_t$ with $t = 1, 2, \cdots, k$.

First construct the $n - l$ dipaths from u to x as follows:

$P_s :<< (x_s - e)d^s, (x_{s+1} - e)d^{s+1}, \cdots, (x_{n-1} - e)d^{n-1}, (x_0 - e)d^0, (x_1 - e)d^1, \cdots, (x_{s-1} - e)d^{s-1} >>$ for $s \neq i_t - j_t + 1, i_t - j_t + 2, \cdots, i_t$ with $t = 1, 2, \cdots, k$.

Then the length of each dipath is $\sum_{i=0}^{n-1} (x_i - e)$.

Next if $i_t - j_t + 1 \leq s \leq i_t$ for any t with $t = 1, 2, \cdots, k$, we write x as $x_{n-1} d^{n-1} + \cdots + x_{i_t+2} d^{i_t+2} + (x_{i_t+1} - 1) d^{i_t+1} + (e + d - 1) d^{i_t} + \cdots + (e + d - 1) d^{s+1} + (e + d) d^s + x_{t_t - j_t} d^{i_t - j_t} + \cdots + x_0$. , where additions are performed modulo d^n. Construct the remaining l dipaths from u to x as follows:

$P_s :<< (d-1)d^s, (d-1)d^{s+1}, \cdots, (d-1)d^{i_t}, (x_{i_t+1} - 1 - e)d^{i_t+1}, (x_{i_t+2} - e)d^{i_t+2}, \cdots, (x_{n-1} - e)d^{n-1}, (x_0 - e)d^0, (x_1 - e)d^1, \cdots, (x_{i_k - j_k} - e)d^{i_k - j_k}, \cdots, (x_{i_t - j_t} - e)d^{i_t - j_t}, d^s >>$ for $s = i_t - j_t + 1, i_t - j_t + 2, \cdots, i_t$ with $t = 1, 2, \cdots, k$.

Then the length of each dipath is $\sum_{i=0}^{n-1} (x_i - e) + \max_{1 \leq t \leq k} \{j_t\} \cdot (d-1) \leq \sum_{i=0}^{n-1} (x_i - e) + l(d-1)$ and the equality holds for $k = 1$.

It is not hard to see that all n dipaths are internally disjoint.

So the proof is complete. ∎

Lemma 3.3 Let $u = (e, e, \cdots, e) \in V(G(d^n; A))$. For any vertex $x = (x_{n-1}, x_{n-2}, \cdots, x_0) \in V(G(d^n; A))$ with $x_{n-1} > e$, if x has $l(\geq 1)$ components with values less than e and the others with values more than e, then there exist n internally disjoint dipaths from u to x each with length at most $\sum\limits_{i=0}^{n-1}(x_i - e) + (l+1)(d-1) + 1$.

Proof Assume x has maximal sequences of components with values less than e at $i_1, i_2, \cdots, i_k$ of lengths $j_1, j_2, \cdots, j_k > 0$, respectively, where $k \geq 1$, $i_1 > i_2 > \cdots > i_k$. Let $\sum\limits_{t=1}^{k} j_t = l$ and

$$x = (x_{n-1}, \cdots, x_{i_1+1}, \overbrace{x_{i_1}, \cdots, x_{i_1-j_1+1}}^{j_1}, x_{i_1-j_1}, \cdots, x_{i_2+1},$$
$$\overbrace{x_{i_2}, \cdots, x_{i_2-j_2+1}}^{j_2}, x_{i_2-j_2}, \cdots, x_{i_k+1}, \overbrace{x_{i_k}, \cdots, x_{i_k-j_k+1}}^{j_k}, x_{i_k-j_k}, \cdots, x_0),$$

where the sth component $x_s < e$ for $s = i_t - j_t + 1, i_t - j_t + 2, \cdots, i_t$ with $t = 1, 2, \cdots, k$ and the other components with values more than e.

Then we write x as

$$(x_{n-1}, \cdots, x_{i_1+2}, x_{i_1+1}-1, \overbrace{x_{i_1}+d-1, \cdots, x_{i_1-j_1+2}+d-1, x_{i_1-j_1+1}+d}^{j_1},$$
$$x_{i_1-j_1}, \cdots, x_{i_2+2}, x_{i_2+1}-1, \overbrace{x_{i_2}+d-1, \cdots, x_{i_2-j_2+2}+d-1, x_{i_2-j_2+1}+d}^{j_2},$$
$$x_{i_2-j_2}, \cdots, x_{j_k+2}, x_{j_k+1}-1, \overbrace{x_{i_k}+d-1, \cdots, x_{i_k-j_k+2}+d-1, x_{i_k-j_k+1}+d}^{j_k},$$
$$x_{i_k-j_k}, \cdots, x_0).$$

Case 1. $x_{i_1+1}, x_{i_2+1}, \cdots, x_{i_k+1} > e+1$. Using the method in Lemma 3.1, construct the n dipaths from u to x each with length $\sum\limits_{i=0}^{n-1}(x_i - e) + l(d-1)$.

Case 2. $x_{i_t+1} = e+1$ for some t with $1 \leq t \leq k$.

Subcase 2a. $t \neq 1$ or $i_1 \neq n-2$.

Assume there exist $s(s \geq 1)$ components x_{i_t+1} such that $x_{i_t+1} = e+1$. Then by Lemma 3.2, there exist n internally disjoint dipaths from u to x, where the $n-s$ dipaths each with length $\sum\limits_{i=0}^{n-1}(x_i - e) + l(d-1)$, the s dipaths each with length at most $\sum\limits_{i=0}^{n-1}(x_i - e) + (l+1)(d-1)$.

Subcase 2b. Otherwise.

We illustrate this case as $t = 1$, $i_1 = n-2$ and $t = 2$. By Lemma 3.2, first we easily construct the $n-1$ dipaths from u to x, where the $n-2$ dipaths $P_s(0 \leq s \leq n-2, s \neq i_2+1)$ each with length $\sum\limits_{i=0}^{n-1}(x_i - e) + l(d-1)$ and the one dipath P_{i_2+1} with length at most $\sum\limits_{i=0}^{n-1}(x_i - e) + (l+1)(d-1)$.

Next we write x as $x + (d, 0, \cdots, 0)$. Construct the remaining dipath P_{n-1} from u to x as $<< (d-1)d^{n-1}, P_0, 1d^{n-1} >>$, the length of the dipath is $\sum_{i=0}^{n-1}(x_i - e) + (l+1)(d-1) + 1$.

It is not hard to see that all dipaths are internally disjoint.

So the proof is complete. ∎

By Lemmas 3.2 and 3.3, we can easily have the following lemma.

Lemma 3.4 Let $u = (e, e, \cdots, e) \in V(G(d^n; A))$. For any vertex $x = (x_{n-1}, x_{n-2}, \cdots, x_0) \in V(G(d^n; A))$ with $x_{n-1} > e$, if x has $l(\geq 1)$ components with values no more than e and the others with values more than e, then there exist n internally disjoint dipaths from u to x each with length at most $\sum_{i=0}^{n-1}(x_i - e) + (l+1)(d-1) + 1$. ∎

Lemma 3.5 Let $S = \{o, u\}$ be a vertex subset of $G(d^n; A)$ with $o = (0, 0, \cdots, 0)$ and $u = (e, e, \cdots, e)$, where $d \geq 4$ and $n \geq 5$. For any vertex $x = (x_{n-1}, x_{n-2}, \cdots, x_0) \in V(G(d^n; A)) - S$ with $x_{n-1} > e$, then S can (l, n)-dominate x for $l = n(d-1) - f(d, n)$, where $f(d, n) = \min\{e\lceil n/2 \rceil - 2, (\lfloor n/2 \rfloor + 1)(e' - 1) - 1\}$.

Proof By definition of the (l, n)-dominating set, it suffices to construct n internally disjoint dipaths of length at most $n(d-1) - f(d, n)$ from S to x with $x_{n-1} > e$. We consider the following cases:

Case 1. $x_i > e$ for $i = 0, 1, \cdots, n-2$.

By vertex-transitivity and Lemma 3.1, there exist n internally disjoint dipaths from u to x each with length $\sum_{i=0}^{n-1}(x_i - e) \leq n(d-1) - ne$.

Case 2. $0 < x_i \leq e$ for $i = 0, 1, \cdots, n-2$.

Construct the n dipaths from o to x the same as in Lemma 3.1 each with length $\sum_{i=0}^{n-1} x_i \leq n(d-1) - (n-1)(e' - 1)$.

Case 3. $x_i = 0$ for $i = 0, 1, \cdots, n-2$, that is $x = (x_{n-1}, 0, \cdots, 0)$.

Subcase 3a. $x_{n-1} > e + 1$. We write x as $(x_{n-1} - 1, d-1, \cdots, d-1, d)$. Construct the n dipaths from u to x each with length $(x_{n-1} - e) + (n-1)(e' - 1) \leq n(d-1) - ne$.

Subcase 3b. $x_{n-1} = e + 1$. We write x as $(e+d, d-1, \cdots, d-1, d)$. It is similar to Subcase 2b in Lemma 3.3, construct the n dipaths from u to x each with length at most $x_{n-1} + n(d-1) - ne + 1 = n(d-1) - e(n-1) + 2$.

Case 4. x has $l(1 \leq l \leq n-2)$ components with values e and the others with values more than e.

If $l \leq \lfloor n/2 \rfloor$. Construct the n dipaths from u to x each with length at most

$$\sum_{i=0}^{n-1}(x_i - e) + l(d-1) \leq n(d-1) - e(n-l) \leq n(d-1) - e\lceil n/2 \rceil.$$

If $l \geq \lfloor n/2 \rfloor + 1$. Construct the n dipaths from o to x each with length

$$\sum_{i=0}^{n-1} x_i \leq le + (n-l)(d-1) \leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e^{'} - 1).$$

Case 5. x has $l (1 \leq l \leq n-2)$ components with values not only less than e but more than 0 and the others with values more than e.

Subcase 5a. All the leading components of x are more than $e+1$.

If $l \leq \lfloor n/2 \rfloor$. By Case 1 of Lemma 3.3, construct the n dipaths from u to x each with length at most

$$\sum_{i=0}^{n-1}(x_i - e) + l(d-1) \leq n(d-1) + l(e-1) - ne \leq n(d-1) - e\lceil n/2 \rceil - \lfloor n/2 \rfloor.$$

If $l \geq \lfloor n/2 \rfloor + 1$. Construct the n dipaths from o to x each with length

$$\sum_{i=0}^{n-1} x_i \leq n(d-1) - le^{'} \leq n(d-1) - e^{'}(\lfloor n/2 \rfloor + 1).$$

Subcase 5b. Some of the leading components of x are equal to $e+1$.

If $l \leq \lfloor n/2 \rfloor - 1$. By Case 2 of Lemma 3.3, construct the n dipaths from u to x each with length at most

$$\begin{aligned}&\sum_{i=0}^{n-1}(x_i - e) + (l+1)(d-1) + 1 \\ &\leq n(d-1) + l(e-1) - e(n-1) + 2 \\ &\leq n(d-1) - (\lfloor n/2 \rfloor + e\lceil n/2 \rceil) + 3.\end{aligned}$$

If $l \geq \lfloor n/2 \rfloor$. Construct the n dipaths from o to x each with length

$$\sum_{i=0}^{n-1} x_i \leq n(d-1) - e^{'}(l+1) + 2 \leq n(d-1) - e^{'}(\lfloor n/2 \rfloor + 1) + 2.$$

Case 6. x has $l (1 \leq l \leq n-2)$ components with values 0 and the others with values more than e .

Subcase 6a. All the leading components of x are more than $e+1$. Construct the n dipaths from u to x each with length at most $\sum_{i=0}^{n-1}(x_i - e) + l(d-1) \leq n(d-1) - ne$.

Subcase 6b. Some of the leading components of x are equal to $e+1$. Construct the n dipaths from u to x each with length at most $\sum_{i=0}^{n-1}(x_i - e) + (l+1)(d-1)+1 \leq n(d-1) - e(n-1) + 2$.

Case 7. x has $l_1 (1 \leq l_1 \leq l-1)$ components with values e, $l - l_1$ components with values not only less than e but more than 0 and the others with values more than e, where $2 \leq l \leq n-2$.

Subcase 7a. All the leading components of x are more than $e+1$.

If $l \geq \lfloor n/2 \rfloor + 1$. Construct the n dipaths from o to x each with length

$$\begin{aligned}\sum_{i=0}^{n-1} x_i &\leq n(d-1) - le' + l_1 \\ &\leq n(d-1) - l(e'-1) - 1 \\ &\leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e'-1) - 1.\end{aligned}$$

If $l \leq \lfloor n/2 \rfloor$. Construct the n dipaths from u to x each with length at most

$$\begin{aligned}&\sum_{i=0}^{n-1}(x_i - e) + l(d-1) \\ &\leq n(d-1) + l(e-1) + l_1 - ne \\ &\leq n(d-1) + e(l-n) - 1 \\ &\leq n(d-1) - e\lceil n/2 \rceil - 1.\end{aligned}$$

Subcase 7b. Some of the leading components of x are equal to $e+1$.

If $l \geq \lfloor n/2 \rfloor$. Construct the n dipaths from o to x each with length

$$\begin{aligned}\sum_{i=0}^{n-1} x_i &\leq n(d-1) - (l+1)e' + l_1 + 2 \\ &\leq n(d-1) - (l+1)(e'-1) \\ &\leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e'-1).\end{aligned}$$

If $l \leq \lfloor n/2 \rfloor - 1$. Construct the n dipaths from u to x each with length at most

$$\begin{aligned}&\sum_{i=0}^{n-1}(x_i - e) + (l+1)(d-1) + 1 \\ &\leq n(d-1) + e(l+1) - ne - l + l_1 + 2 \\ &\leq n(d-1) + e(l-n+1) + 1 \\ &\leq n(d-1) - e\lceil n/2 \rceil + 1.\end{aligned}$$

Case 8. x has $l_1 (1 \leq l_1 \leq l-1)$ components with values 0, $l - l_1$ components with values not only less than e but more than 0 and the others with values more than e, where $2 \leq l \leq n-2$.

Subcase 8a. All the leading components of x are more than $e+1$.

If $l_1 \geq l - \lfloor n/2 \rfloor$. Construct the n dipaths from u to x each with length at most

$$\sum_{i=0}^{n-1}(x_i-e)+l(d-1) \leq n(d-1)+(l-l_1)(e-1)-ne \leq n(d-1)-e\lceil n/2\rceil-\lfloor n/2 \rfloor.$$

If $l_1 \leq l - \lfloor n/2 \rfloor - 1$. Construct the n dipaths from o to x each with length at most

$$\sum_{i=0}^{n-1} x_i + l_1(d-1) \leq n(d-1) - e'(l-l_1) \leq n(d-1) - e'(\lfloor n/2 \rfloor + 1).$$

Subcase 8b. Some of the leading components of x are equal to $e+1$.

If $l_1 \geq l - \lfloor n/2 \rfloor + 1$. Construct the n dipaths from u to x each with length at most

$$\begin{aligned}
&\sum_{i=0}^{n-1}(x_i - e) + (l+1)(d-1) + 1 \\
&\leq n(d-1) + (l-l_1)(e-1) - e(n-1) + 2 \\
&\leq n(d-1) - e\lceil n/2 \rceil - \lfloor n/2 \rfloor + 3.
\end{aligned}$$

If $l_1 \leq l - \lfloor n/2 \rfloor$. Construct the n dipaths from o to x each with length at most

$$\sum_{i=0}^{n-1} x_i + l_1(d-1) \leq n(d-1) - e'(l-l_1+1) + 2 \leq n(d-1) - e'(\lfloor n/2 \rfloor + 1) + 2.$$

Case 9. x has $l_1 (1 \leq l_1 \leq l-1)$ components with values 0, $l - l_1$ components with values e and the others with values more than e, where $2 \leq l \leq n-2$.

Subcase 9a. All the leading components of x are more than $e+1$.

If $l_1 \geq \lfloor n/2 \rfloor - 1$. Construct the n dipaths from u to x each with length at most

$$\sum_{i=0}^{n-1}(x_i - e) + l(d-1) \leq n(d-1) - e(n-l+l_1) \leq n(d-1) - e\lceil n/2 \rceil.$$

If $l_1 \leq l - \lfloor n/2 \rfloor - 1$. Construct the n dipaths from o to x each with length at most

$$\sum_{i=0}^{n-1} x_i + l_1(d-1) \leq n(d-1) - (l-l_1)(e'-1) \leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e'-1).$$

Subcase 9b. Some of the leading components of x are equal to $e+1$.

If $l_1 \geq l - \lfloor n/2 \rfloor + 1$. Construct the n dipaths from u to x each with length at most

$$\begin{aligned}
&\sum_{i=0}^{n-1}(x_i - e) + (l+1)(d-1) + 1 \\
&\leq n(d-1) - e(n - l + l_1 - 1) + 2 \\
&\leq n(d-1) - e\lceil n/2 \rceil + 2.
\end{aligned}$$

If $l_1 \leq l - \lfloor n/2 \rfloor$. Construct the n dipaths from o to x each with length at most

$$\begin{aligned}
&\sum_{i=0}^{n-1} x_i + l_1(d-1) \\
&\leq n(d-1) - (e' - 1)(l - l_1 + 1) + 1 \\
&\leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1) + 1.
\end{aligned}$$

Case 10. x has $l_1(1 \leq l_1 \leq l-2)$ components with values 0, $l_2(1 \leq l_2 \leq l - l_1 - 1)$ components with values not only less than e but more than 0, $l - l_1 - l_2$ components with values e and the others with values more than e, where $3 \leq l \leq n-2$.

Subcase 10a. All the leading components of x are more than $e+1$.

If $l_1 \geq l - \lfloor n/2 \rfloor$. Construct the n dipaths from u to x each with length at most

$$\sum_{i=0}^{n-1}(x_i - e) + l(d-1) \leq n(d-1) - l_2 + e(l - l_1 - n) \leq n(d-1) - e\lceil n/2 \rceil - 1.$$

If $l_1 \leq l - \lfloor n/2 \rfloor - 1$. Construct the n dipaths from o to x each with length at most

$$\begin{aligned}
&\sum_{i=0}^{n-1} x_i + l_1(d-1) \\
&\leq n(d-1) - (l - l_1)(e' - 1) - l_2 \\
&\leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1) - 1.
\end{aligned}$$

Subcase 10b. Some of the leading components of x are equal to $e+1$.

If $l_1 \geq l - \lfloor n/2 \rfloor + 1$. Construct the n dipaths from u to x each with length at most

$$\begin{aligned}
&\sum_{i=0}^{n-1}(x_i - e) + (l+1)(d-1) + 1 \\
&\leq n(d-1) - l_2 + e(l - l_1 - n + 1) + 2 \\
&\leq n(d-1) - e\lceil n/2 \rceil + 1.
\end{aligned}$$

If $l_1 \leq l - \lfloor n/2 \rfloor$. Construct the n dipaths from o to x each with length at most

$$\begin{aligned}&\sum_{i=0}^{n-1} x_i + l_1(d-1)\\&\leq n(d-1) - (l - l_1 + 1)(e' - 1) - l_2 + 1\\&\leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1).\end{aligned}$$

Case 11. x has $l(1 \leq l \leq n-2)$ components with values not only less than e but more than 0 and the others with values e.

Construct the n dipaths from o to x each with length

$$\sum_{i=0}^{n-1} x_i \leq ne - l + e' - 1 \leq n(d-1) - (n-1)(e' - 1) - 1.$$

Case 12. x has $l(1 \leq l \leq n-2)$ components with values 0 and the others with values e.

Subcase 12a. $x_{n-1} > e + 1$.

If $l \leq \lceil n/2 \rceil - 2$. Construct the n dipaths from o to x each with length at most

$$\sum_{i=0}^{n-1} x_i + l(d-1) \leq ne + (l+1)(e' - 1) \leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1).$$

If $l \geq \lceil n/2 \rceil - 1$. Construct the n dipaths from u to x each with length at most

$$\sum_{i=0}^{n-1} (x_i - e) + (n-1)(d-1) \leq n(d-1) - (l+1)e \leq n(d-1) - e\lceil n/2 \rceil.$$

Subcase 12b. $x_{n-1} = e + 1$.

If $l \leq \lceil n/2 \rceil - 1$. Construct the n dipaths from o to x each with length at most

$$\sum_{i=0}^{n-1} x_i + l(d-1) \leq ne + l(e' - 1) + 1 \leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1) + 1.$$

If $l \geq \lceil n/2 \rceil$. Construct the n dipaths from u to x each with length at most

$$\sum_{i=0}^{n-1} (x_i - e) + n(d-1) + 1 \leq n(d-1) - le + 2 \leq n(d-1) - e\lceil n/2 \rceil + 2.$$

Case 13. x has $l_1(1 \leq l_1 \leq l-1)$ components with values 0, $l - l_1$ components with values not only less than e but more than 0 and the others with values e, where $2 \leq l \leq n-2$.

Subcase 13a. $x_{n-1} > e+1$.

If $l_1 \leq \lceil n/2 \rceil - 2$. Construct the n dipaths from o to x each with length at most

$$\begin{aligned}
&\sum_{i=0}^{n-1} x_i + l_1(d-1) \\
&\leq ne - l + l_1 e' + e' - 1 \\
&\leq ne + l_1(e' - 1) + e' - 2 \\
&\leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1) - 1.
\end{aligned}$$

If $l_1 \geq \lceil n/2 \rceil - 1$. Construct the n dipaths from u to x each with length at most

$$\begin{aligned}
&\sum_{i=0}^{n-1} (x_i - e) + (n-1)(d-1) \\
&\leq n(d-1) - l - l_1(e-1) - e \\
&\leq n(d-1) - e(l_1 + 1) - 1 \\
&\leq n(d-1) - e\lceil n/2 \rceil - 1.
\end{aligned}$$

Subcase 13b. $x_{n-1} = e+1$.

If $l_1 \leq \lceil n/2 \rceil - 1$. Construct the n dipaths from o to x each with length at most

$$\begin{aligned}
&\sum_{i=0}^{n-1} x_i + l_1(d-1) \\
&\leq ne - l + l_1 e' + 1 \\
&\leq ne + l_1(e' - 1) \\
&\leq n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1).
\end{aligned}$$

If $l_1 \geq \lceil n/2 \rceil$. Construct the n dipaths from u to x each with length at most

$$\begin{aligned}
&\sum_{i=0}^{n-1} (x_i - e) + n(d-1) + 1 \\
&\leq n(d-1) - l - l_1(e-1) + 2 \\
&\leq n(d-1) - l_1 e + 1 \\
&\leq n(d-1) - e\lceil n/2 \rceil + 1.
\end{aligned}$$

Case 14. x has $l (1 \leq l \leq n-2)$ components with values 0 and the others with values not only less than e but more than 0.

Subcase 14a. $x_{n-1} > e+1$.

If $l \leq \lceil n/2 \rceil - 2$. Construct the n dipaths from o to x each with length

$$\sum_{i=0}^{n-1} x_i + l(d-1) \leq n(e-1) + (l+1)e' \leq n(d-1) - e'(\lfloor n/2 \rfloor + 1).$$

If $l \geq \lceil n/2 \rceil - 1$. Construct the n dipaths from u to x each with length at most

$$\sum_{i=0}^{n-1}(x_i - e) + (n-1)(d-1)$$
$$\leq n(d-1) - (l+1)(e-1) - n$$
$$\leq n(d-1) - e\lceil n/2\rceil - \lfloor n/2\rfloor.$$

Subcase 14b. $x_{n-1} = e+1$.

If $l \leq \lceil n/2\rceil - 1$. Construct the n dipaths from o to x each with length at most

$$\sum_{i=0}^{n-1} x_i + l(d-1) \leq n(e-1) + le^{'} + 2 \leq n(d-1) - e^{'}(\lfloor n/2\rfloor + 1) + 2.$$

If $l \geq \lceil n/2\rceil$. Construct the n dipaths from u to x each with length at most

$$\sum_{i=0}^{n-1}(x_i - e) + n(d-1) + 1$$
$$\leq n(d-1) - l(e-1) - n + 3$$
$$\leq n(d-1) - e\lceil n/2\rceil - \lfloor n/2\rfloor + 3.$$

Summarizing above cases, there exist n internally disjoint dipaths from S to x for $x_{n-1} > e$ each with length at most $\max\{n(d-1) - e\lceil n/2\rceil + 2, n(d-1) - (\lfloor n/2\rfloor + 1)(e^{'} - 1) + 1\}$.

So the proof is complete. ∎

For any vertex $x = (x_{n-1}, x_{n-2}, \cdots, x_0) \in V(G(d^n; A)) - S$ with $0 \leq x_{n-1} \leq e$, using the method in Lemma 3.5, we can also construct n internally disjoint dipaths from S to x such that length of each dipath is as small as possible. The details are omitted here and left to the reader. Next we consider the following lemma.

Lemma 3.6 Let $S = \{o, u\}$ be a vertex subset of $G(d^n; A)$ with $o = (0, 0, \cdots, 0)$ and $u = (e, e, \cdots, e)$, where $d \geq 4$ and $n \geq 5$. For any vertex $x = (x_{n-1}, x_{n-2}, \cdots, x_0) \in V(G(d^n; A)) - S$ with $0 \leq x_{n-1} \leq e$, then S can (l, n)-dominate x for $l = n(d-1) - g(d, n)$, where $g(d, n) = \min\{e\lceil n/2\rceil - e - 2, (\lfloor n/2\rfloor + 1)(e^{'} - 1) - 2\}$.

Proof It suffices to construct n internally disjoint dipaths of length at most $n(d-1) - g(d, n)$ from S to x with $0 \leq x_{n-1} \leq e$. Let $y = (d-1, x_{n-2}, \cdots, x_0)$ be a vertex in $G(d^n; A)$. According to the proof of Lemma 3.5, there exist n internally disjoint dipaths $P_i (0 \leq i \leq n-1)$ from o (or u) to y. We consider the following cases:

Case 1. The n dipaths start at vertex u.

Subcase 1a. $x_i > e$ for $i = 0, 1, \cdots, n-2$. Then we write x as $(x_{n-1} + d, x_{n-2}, \cdots, x_0)$. It is easy to construct the n dipaths from u to x each with length $\sum_{i=0}^{n-1}(x_i - e) + d \leq n(d-1) - e(n-1) + 1$.

Subcase 1b. Otherwise.

We illustrate this case as $y = (d-1, x_{n-2}, \cdots, x_{i_1+1}, \overbrace{x_{i_1}, \cdots, x_{i_1-j_1+1}}^{j_1}, x_{i_1-j_1}, \cdots, x_{i_2+1}, \overbrace{e, \cdots, e}^{j_2}, x_{i_2-j_2}, \cdots, x_0)$, where $j_1, j_2 > 1$, $x_i < e$ for $i = i_1 - j_1 + 1, \cdots, i_1$ and $x_i > e$ for $i \neq i_t - j_t + 1, \cdots, i_t$ with $t = 1, 2$. Then we write y as

$$(d-1, \cdots, x_{i_1+2}, x_{i_1+1}-1, \overbrace{x_{i_1}+d-1, \cdots, x_{i_1-j_1+2}+d-1, x_{i_1-j_1+1}+d}^{j_1},$$
$$x_{i_1-j_1}, \cdots, x_{i_2+2}, x_{i_2+1}-1, \overbrace{e+d-1, \cdots, e+d-1, e+d}^{j_2}, x_{i_2-j_2}, \cdots, x_0).$$

If $x_{i_1+1} > e+1$ and $x_{i_2+1} > e+1$. According to Case 4, Subcases 9a and 12a in Lemma 3.5, we know each length of the n dipaths is at most $n(d-1) - e\lceil n/2 \rceil$. By case 1 in Lemma 3.3, so we have $e' - 1 + \sum_{i=0}^{n-2}(x_i - e) + (j_1 + j_2)(d-1) \leq n(d-1) - e\lceil n/2 \rceil$. Without loss of generality, denote the dipath P_i by $<< P_{ii}d^i, P_{ii+1}d^{i+1}, \cdots, P_{in-1}d^{n-1}, P_{i0}d^0, \cdots, P_{ii-1}d^{i-1} >>$ for $i = 0, 1, \cdots, n-1$, where P_{ij} means links through the jth component along the dipath P_i for $j = 0, 1, \cdots, n-1$. Next we write x as $y + (x_{n-1} + 1, 0, \cdots, 0)$. Then construct the n dipaths from u to x as $<< P_{ii}d^i, P_{ii+1}d^{i+1}, \cdots, (P_{in-1} + x_{n-1} + 1)d^{n-1}, P_{i0}d^0, \cdots, P_{ii-1}d^{i-1} >>$ for $i = 0, 1, \cdots, n-1$. So the length of each dipath is $x_{n-1} + e' + \sum_{i=0}^{n-2}(x_i - e) + (j_1 + j_2)(d-1) \leq n(d-1) - e\lceil n/2 \rceil + e + 1$.

If $x_{i_1+1} = e+1$ or $x_{i_2+1} = e+1$. According to Subcases 2a in Lemma 3.3 and Subcase 9b in Lemma 3.5,, we also have $e' - 1 + \sum_{i=0}^{n-2}(x_i - e) + (j_1 + j_2 + 1)(d-1) \leq n(d-1) - e\lceil n/2 \rceil + 1$, and can construct the n dipaths from u to x each with length at most $x_{n-1} + e' + \sum_{i=0}^{n-2}(x_i - e) + (j_1 + j_2 + 1)(d-1) \leq n(d-1) - e\lceil n/2 \rceil + e + 2$.

Case 2. The n dipaths start at vertex o.

By Lemma 3.5, each length of the n dipaths is at most $n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1) + 1$. Assume the dipath P_i as $<< P_{ii}d^i, P_{ii+1}d^{i+1}, \cdots, P_{in-1}d^{n-1}, P_{i0}d^0, \cdots, P_{ii-1}d^{i-1} >>$, where $i = 0, 1, \cdots, n-1$.

If $x_{n-1} = 0$, then write x as $(d, x_{n-2}, \cdots, x_0)$, it is easy construct the n dipaths from o to x each with length at most $n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1) + 2$.

If $x_{n-1} \geq 1$, construct the n dipaths from o to x as $<< P_{ii}d^i, P_{ii+1}d^{i+1}, \cdots, (P_{in-1} - (d-1-x_{n-1}))d^{n-1}, P_{i0}d^0, \cdots, P_{ii-1}d^{i-1} >>$ for $i = 0, 1, \cdots, n-1$. Obviously, the length of each dipath is less than $n(d-1) - (\lfloor n/2 \rfloor + 1)(e' - 1) + 1$.

So the proof is complete. ∎

Finally, we can see that Proposition 2.1, Lemmas 3.5 and 3.6 yield the following theorem:

Theorem 3.7 Let $G = G(d^n; A)$ with $d \geq 4$ and $n \geq 5$. Then $\gamma_{l,\omega}(G) = 2$ for $1 \leq \omega \leq n$ and $d_\omega(G) - (g(d,n) + \delta) \leq l \leq d_\omega(G) - 1$, where $g(d,n) = \min\{e\lceil n/2 \rceil - e - 2, (\lfloor n/2 \rfloor + 1)(e' - 1) - 2\}$, $\delta = 0$ for $1 \leq \omega \leq n-1$ and $\delta = 1$ for $\omega = n$.

4 Conclusion and problems

For the circulant network $G(d^n; A)$ with $d \geq 4$ and $n \geq 5$, we prove that $\gamma_{l,\omega}(G) = 2$ for $1 \leq \omega \leq n$ and $d_\omega(G) - (g(d,n) + \delta) \leq l \leq d_\omega(G) - 1$, where $g(d,n) = \min\{e\lceil n/2 \rceil - e - 2, (\lfloor n/2 \rfloor + 1)(e' - 1) - 2\}$, $\delta = 0$ for $1 \leq \omega \leq n-1$ and $\delta = 1$ for $\omega = n$. Important and practical problems are how to choose a vertex subset S of $G(d^n; A)$ and determine the values of $\gamma_{l,\omega}(G(d^n; A))$ if $1 \leq \omega \leq n$ and $l < d_\omega(G) - (g(d,n) + \delta)$ for some special values of d and n.

References

[1] J. Bruck, R. Cypher and C.T. Ho, Fault-tolerant meshes and hypercubes with minimal numbers of spares, *IEEE Trans. Comput.*, **C-42(9)**(1993), 1089-1104.

[2] G. Chartrand and L. Lesniak, Graphs and Digraphs. Chapman and Hall, Boca Raton, FL, 2005.

[3] D.R. Duh and G.H. Chen, On the rabin number problem,*Networks*, **30(3)**(1997), 219-230.

[4] S. Dutt and J.P. Hayes, Designing fault-tolerant systems using automorphisms, *J. Parallel Distrib. Comput.*, **12**(1991), 249-268.

[5] B. Elspas and J. Turner, Graphs with circulant adjacency matrices, *J. Combin. Theory*, **9**(1970), 297-307.

[6] E. Flandrin and H. Li, Mengerian properties, hamiltonicity, and claw-free graphs, *Networks*, **24**(1994), 177-183.

[7] M.R. Garey and D.S. Johnson, Computers and intactability, A guide to the theory of NP-Completencess, San Francisco: W. H. Freeman, 1979.

[8] Y.O. Hamidoune, On the connectivity of Cayley digraphs, *European J. Combin.*, **5**(1984), 309-312.

[9] D.F. Hsu and Y.D. Lyuu, A graph-theoretical study of transmission delay and fault tolerance, *International Journal of Mini and Microcomputers*, **16**(1994), 35-42.

[10] H. Li and J.M. Xu, (d, m)-dominating number of m-connected graphs, Rapport de Recherche, LRI, URA 410 du CNRS Universite de paris-sud No. 1130, 1997,

[11] S.C. Liaw, G.J. Chang, F. Cao and D. F. Hsu, Fault-tolerant routing in circulant networks and cycle prefix networks, *Annals of Combinatorics*, **2**(1998), 165-172.

[12] C.H. Lu, J.M. Xu and K.M. Zhang, On $(d, 2)$-dominating numbers of binary undirected de bruijn graphs, *Discrete Applied Mathematics*, **105**(2000), 137-145.

[13] Y. Lu, X.M. Hou and J.M. Xu, On the $(2, 2)$-domination number of trees, *Discussions Mathematicae Graph Theory*, **30**(2010), 185-199.

[14] J.M. Xu, Topological structure and analysis of interconnection networks, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.

[15] J.M. Xu, C.H. Lu and K.M. Zhang, A new property of binary undirected de bruijn graphs, *Chinese Annals of Mathematics*, **21B**(2000), 39-42.