Binding Number and Tenacity

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Abstract

Let $T(G)$ and bind(G) be the tenacity and the binding number, respectively, of a graph G. The inequality $T(G) \geq \text{bind}(G) - 1$ was derived by D. Moazzami in [11]. In this paper, we provide a stronger lower bound on $T(G)$ that is best possible when bind(G) $\geq$ 1.

1 Introduction

We consider only nonempty, finite, simple, undirected graphs. Given two graphs G and H we use $G + H$ to denote their *join* and $G \cup H$ to denote their *disjoint union*. We define the *tenacity* of G, denoted $T(G)$, as in [9], and the *binding number* of G, denoted bind(G), as in [12]:

$$T(G) = \min\left\{ \frac{|S| + m(G-S)}{\omega(G-S)} \,\middle|\, S \subset V(G) \text{ and } \omega(G-S) \geq 2 \right\}$$

and

$$\text{bind}(G) = \min\left\{ \frac{|N(S)|}{|S|} \,\middle|\, \emptyset \neq S \subseteq V(G),\ N(S) \neq V(G) \right\},$$

where $m(G-S)$ is the order of a largest component of $G - S$, $\omega(G-S)$ is the number of components of $G - S$, and $N(S)$ is the set of neighbors of S. Let K_n be the complete graph on n vertices. Then we define $T(K_n) := n$. Notice that $T(G) \geq \frac{1}{n} > 0$ for all graphs G on n vertices (the lower bound given by $T(nK_1)$). A graph G is *b-binding* if bind(G) $\geq b$ and *t-tenacious* if $T(G) \geq t$.

*Part of this work was completed while the author was a Ph.D candidate in the Department of Mathematical Sciences at Stevens Institute of Technology.

The binding number and the tenacity are both members of a class of vulnerability parameters of a graph that are often used to study network stability. Other parameters that fall under this heading are vertex-connectivity, edge-connectivity, and toughness. Discussions of relationships between multiple vulnerability parameters can be found in [1] and [11].

In terms of computational complexity, determining the tenacity of a graph is NP-hard (D. Moazzami, personal communication, May 29, 2010). However, Cunningham [10] has shown that bind(G) is tractable. Therefore, one can benefit from knowing how $T(G)$ compares to bind(G). Such a relationship was provided by Moazzami [11] in the form of Theorem 1.1 below.

Theorem 1.1. *Let G be a graph. Then* $T(G) \geq \text{bind}(G) - 1$.

Our main goal is to replace the lower bound of Theorem 1.1 with one that is best possible when bind(G) ≥ 1. We will then show how certain information about the degrees of vertices of G can help to strengthen this relationship. For this second task, we require the notion of a *best monotone P theorem* for a graph property P. The next few paragraphs contain a summary of ideas originally formalized in [4] and [3].

The *degree sequence* of a graph G is a list of the degrees of all the vertices of G, with repetition if multiple vertices have the same degree. In this paper the degree sequences are in nondecreasing order. If π is a degree sequence of length n, then we typically denote it as $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$. At times we may utilize exponents to indicate the number of times a degree appears, e.g., $\pi = (2, 2, 2, 2, 4) = 2^4 4^1$. Given two sequences $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ and $\pi' = (d'_1 \leq d'_2 \leq \cdots \leq d'_n)$, we say that π' *majorizes* π, denoted $\pi' \geq \pi$, if $d'_i \geq d_i$ for all i, e.g., $2^3 3^2 \geq 2^5$. A sequence $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ is a *graphical sequence* if there exists a graph G with π as its degree sequence, such a graph G is called a *realization* of the sequence π. Now, a graphical sequence π can have more than one distinct realization. However, given a property P, it may be every realization of π has the property P, in which case we say that π is *forcibly P*. For example, the graphical sequence $\pi = 2^5$ is forcibly hamiltonian.

Assume that we are given a graphical sequence π and a property P. It is sometimes the case that we have conditions for determining when π is forcibly P. A theorem that declares a sequence π to be forcibly P, rendering no result if π fails to meet the conditions of the theorem, is called a *forcibly P theorem* (or simply *P theorem*). For instance, Chvátal provides such a sufficient condition for hamiltonicity in [8].

Theorem 1.2. *Let $\pi = (d_1 \le \cdots \le d_n)$ be a graphical sequence, with $n \ge 3$. If $d_i \le i < \dfrac{n}{2}$ implies $d_{n-i} \ge n-i$, then π is forcibly hamiltonian.*

Thus, Theorem 1.2 is a forcibly hamiltonian theorem. In addition, this theorem possesses other interesting properties, which we now discuss.

If T is a P theorem, then T is *monotone* if whenever T declares π forcibly P it also declares π' forcibly P for all $\pi' \ge \pi$. Clearly, Theorem 1.2 is monotone. Another property that Theorem 1.2 has is that if π fails the given condition for some $i < \frac{n}{2}$, then the sequence $\pi' = i^i(n-i-1)^{n-2i}(n-1)^i$ majorizes π and has a realization $G' = K_i + (K_{n-2i} \cup \overline{K_i})$ that is not hamiltonian. This leads us to our next definition. A P theorem T_0 is *weakly optimal* if whenever a sequence π fails the conditions of T_0, there exists a sequence $\pi' \ge \pi$ such that π' has a realization without P. So, Theorem 1.2 is weakly optimal. Finally, a P theorem T_0 is *best monotone* if T_0 is monotone and weakly optimal. Best monotone P theorems have the following appealing property.

Theorem 1.3. *Let T_0 be a best monotone P theorem. Then given any other monotone P theorem T, if T declares a graphical sequence to be forcibly P, then T_0 will also declare it to be forcibly P.*

Proof of Theorem 1.3: Let T_0 be a best monotone P theorem and let π be a graphical sequence. Assume that T is another monotone P theorem and that T declares π to be forcibly P. If T_0 does not declare π to be forcibly P, then there exists $\pi' \ge \pi$ with a realization G' not having the property P. However, since T is monotone, π' is forcibly P, a contradiction. ∎

We see that Theorem 1.2 is best monotone with respect to the property of hamiltonicity. Of course, best monotone theorems exist for other graph properties as well. For instance, in [6] Boesch showed that the following theorem of Bondy for vertex-connectivity [7] (stated here in the form given in [6]) is best monotone.

Theorem 1.4. *Let $\pi = (d_1 \le \cdots \le d_n)$ be a graphical sequence with $n \ge 2$, and let $1 \le k \le n-1$. If $d_i \le i+k-2$ implies $d_{n-k+1} \ge n-i$, for $1 \le i \le \dfrac{1}{2}(n-k+1)$, then π is forcibly k-connected.*

Recently, various authors have taken up the task of finding best monotone theorems for other graph properties, such as edge-connectivity [3], toughness [2], and b-binding [5]. Continuing along these lines, we derive a best monotone theorem for the property of being t-tenacious in the last section of this paper.

We now use the idea of a best monotone theorem to define a special class of graphical sequences. Let P be a graph property and let $\mathrm{BM}(P)$ denote the set of graphical sequences that satisfy a best monotone P theorem. We say that a graphical sequence π is *best monotone* P if $\pi \in \mathrm{BM}(P)$. For example, $\pi = 2^2 3^2 4^1 \in \mathrm{BM}(\text{hamiltonian})$, since π satisfies Theorem 1.2. Given two properties P_1 and P_2 such that P_1 implies P_2, it is clear that if π is forcibly P_1, then π is forcibly P_2. However, we can say more.

Theorem 1.5. *Let P_1, P_2 be graph properties such that P_1 implies P_2 and let π be a graphical sequence. Then $\pi \in \mathrm{BM}(P_1)$ implies $\pi \in \mathrm{BM}(P_2)$.*

Proof of Theorem 1.5: Suppose to the contrary that $\pi \in \mathrm{BM}(P_1)$, but $\pi \notin \mathrm{BM}(P_2)$. Then there exists a graphical sequence $\pi' \geq \pi$ having a realization G' without property P_2. Since P_1 implies P_2, G' also does not have property P_1. However, $\pi \in \mathrm{BM}(P_1)$ and $\pi' \geq \pi$ together imply that $\pi' \in \mathrm{BM}(P_1)$, and thus every realization of π' has property P_1, a contradiction. ∎

2 Best Possible Upper Bound on the Binding Number

We start with the following theorem.

Theorem 2.1. *Let G be a graph on $n \geq 2$ vertices. Then*

$$(*) \qquad \mathrm{bind}(G) < \max\left\{\frac{T(G)}{2} + 1,\, T(G)\right\},$$

and the upper bound is best possible.

Before proving Theorem 2.1, we show that $(*)$ is best possible. Assume that $T(G) = \frac{c}{d} > 0$, where $\frac{c}{d}$ is in lowest terms. If $T(G) = \frac{c}{d} < 2$, then the upper bound in $(*)$ is $T(G)/2 + 1$. Consider the graphs $G := K_{(cm-2)} + dmK_2$, for $m \geq 3$. Let $v \in V(dmK_2)$. Taking $S := V(K_{(cm-2)})$ and $S' := V(dmK_2) - \{v\}$, we have that

$$T(G) = \frac{|S| + m(G-S)}{\omega(G-S)} = \frac{(cm-2)+2}{dm} = \frac{c}{d}$$

and

$$\text{bind}(G) = \frac{|N(S')|}{|S'|} = \frac{(cm-2)+2dm-1}{2dm-1} = \frac{cm-2}{2dm-1}+1 < \frac{c}{2d}+1.$$

Thus

$$\text{bind}(G) = \frac{cm-2}{2dm-1}+1 \uparrow \frac{c}{2d}+1 = \frac{T(G)}{2}+1.$$

Next, if $T(G) = \frac{c}{d} \geq 2$, then the upper bound in (*) is $T(G)$. Consider the graphs $G := K_{(cm-1)} + dmK_1$, for $m \geq 2$. Taking $S := V(K_{(cm-1)})$ and $S' := V(dmK_1)$, we have that

$$T(G) = \frac{|S| + m(G-S)}{\omega(G-S)} = \frac{(cm-1)+1}{dm} = \frac{c}{d},$$

and

$$\text{bind}(G) = \frac{|N(S')|}{|S'|} = \frac{cm-1}{dm} = \frac{c}{d} - \frac{1}{dm} \uparrow \frac{c}{d} = T(G).$$

In each case the limit is from below, and so the upper bound cannot be improved.

Proof of Theorem 2.1: If $G = K_n$, then $\text{bind}(G) = n-1 < n = T(G)$ and we are done. So assume that G is noncomplete. Then, there exists $X \subset V(G)$ such that $\frac{|X|+m(G-X)}{\omega(G-X)} = T(G)$ and $\omega := \omega(G-X) \geq 2$. Let $A_1, ..., A_\omega$ be the components of $G - X$, with $|A_1| \geq ... \geq |A_\omega|$. Define $x := |X|$ and $a_i := |A_i|$ for each i. Then $T(G) = (x+a_1)/\omega$.

If $a_1 = 1$, let $j = 0$. Otherwise, suppose $a_1, \ldots, a_j \geq 2$, but $a_{j+1} = \cdots a_\omega = 1$, for $1 \leq j \leq \omega$. We consider three cases.

Case 1. $j = 0$.

Let $S := V(\omega K_1)$. Then

$$\text{bind}(G) \leq \frac{|N(S)|}{|S|} \leq \frac{x}{\omega} < \frac{x+1}{\omega} = T(G).$$

Thus, $\text{bind}(G) < T(G) \leq \max\left\{\frac{T(G)}{2}+1, T(G)\right\}$.

Case 2. $0 < j < \omega$.

Let $S := \bigcup_{i=2}^{\omega} V(K_{a_i}) \cup \{v\}$, where $v \in V(K_{a_1})$, so that $|S| = a_2 + \cdots + a_j + (\omega - j) + 1 \geq \omega + j - 1$. Then

$$\begin{aligned} \text{bind}(G) &\leq \frac{|N(S)|}{|S|} \leq \frac{x + a_1 + a_2 + \cdots + a_j - 1}{a_2 + \cdots + a_j + (\omega - j) + 1} \\ &= 1 + \frac{x + a_1 - 2 - (\omega - j)}{a_2 + \cdots + a_j + (\omega - j) + 1}. \end{aligned}$$

If $x + a_1 - 2 - (\omega - j) < 0$, then $\text{bind}(G) < 1 \leq \max\left\{\frac{T(G)}{2} + 1, T(G)\right\}$ and we are done. So assume that $x + a_1 - 2 - (\omega - j) \geq 0$. Recalling that $T(G) = (x + a_1)/\omega$, we have

$$\begin{aligned} \text{bind}(G) &\leq 1 + \frac{x + a_1 - 2 - (\omega - j)}{a_2 + \cdots + a_j + (\omega - j) + 1} \\ &\leq \frac{T(G)\omega + 2j - 3}{\omega + j - 1} \\ &< \frac{T(G)\omega + 2j - 2}{\omega + j - 1}. \end{aligned}$$

Let $F(j) := \frac{T(G)\omega + 2j - 2}{\omega + j - 1}$. Then $F(j)$ achieves its maximum when $j = \omega - 1$ if $T(G) \leq 2$ and when $j = 1$ if $T(G) \geq 2$. Thus

$$\text{bind}(G) < F(j) \leq \begin{cases} \frac{(T(G)+2)\omega - 4}{2\omega - 2} & \text{if } T(G) \leq 2 \\ T(G) & \text{if } T(G) \geq 2. \end{cases}$$

Since $\omega \geq 2$, $\frac{(T(G)+2)\omega-4}{2\omega-2} \leq \frac{T(G)}{2} + 1$ when $T(G) \leq 2$. Therefore,

$$\text{bind}(G) < \left\{ \begin{array}{ll} \frac{T(G)}{2} + 1 & \text{if } T(G) \leq 2 \\ T(G) & \text{if } T(G) \geq 2 \end{array} \right\} = \max\left\{\frac{T(G)}{2} + 1, T(G)\right\}.$$

Case 3. $j = \omega$.

Let $S := \bigcup_{i=2}^{\omega} V(K_{a_i}) \cup \{v\}$, where $v \in V(K_{a_1})$. Define $\theta := a_2 + \cdots + a_\omega + 1$, so that $|S| = \theta \geq 2\omega - 1$. Then

$$\text{bind}(G) \leq \frac{|N(S)|}{|S|} \leq \frac{x + a_1 + \theta - 2}{\theta} = 1 + \frac{x + a_1 - 2}{\theta} = 1 + \frac{T(G)\omega - 2}{\theta}.$$

We know that $T(G)\omega - 2 \geq 0$, since $T(G) = \frac{x+a_1}{\omega} \geq \frac{2}{\omega}$. So

$$\text{bind}(G) \leq 1 + \frac{T(G)\omega - 2}{2\omega - 1} \leq \begin{cases} \frac{T(G)}{2} + 1 & \text{if } T(G) \leq 4 \\ \frac{2T(G)+1}{3} & \text{if } T(G) \geq 4. \end{cases}$$

It follows that

$$\text{bind}(G) < \left\{ \begin{array}{ll} \frac{T(G)}{2} + 1 & \text{if } T(G) \leq 2 \\ T(G) & \text{if } T(G) \geq 2 \end{array} \right\} = \max\left\{\frac{T(G)}{2} + 1, T(G)\right\}.$$

■

By rephrasing the inequality (∗), we obtain a lower bound on $T(G)$ with respect to bind(G) that is best possible when bind$(G) \geq 1$.

Theorem 2.2. *Let G be a graph on $n \geq 2$ vertices. Then*

$$T(G) > \min\{2(\text{bind}(G) - 1), \text{bind}(G)\},$$

and the lower bound is best possible when bind$(G) \geq 1$.

It is clear that this lower bound on $T(G)$ implies Theorem 1.1.

We now show that Theorem 2.2 is best possible. If $1 \leq \text{bind}(G) < 2$, then the lower bound on $T(G)$ given by Theorem 2.2 is $2(\text{bind}(G) - 1)$. Consider the graph $G := K_{(c-d)m} + \left(\frac{dm+1}{2}\right) K_2$, with $d \leq c < 2d$, $m \geq 1$, and dm odd. Let v be a vertex in $\left(\frac{dm+1}{2}\right) K_2$. Taking $S := V\left(K_{(c-d)m}\right)$ and $S' := V\left(\left(\frac{dm+1}{2}\right) K_2\right) - \{v\}$, we have

$$T(G) = \frac{|S| + m(G-S)}{\omega(G-S)} = \frac{(c-d)m+2}{\frac{dm+1}{2}} = 2\left(\frac{(c-d)m+2}{dm+1}\right).$$

and

$$\text{bind}(G) = \frac{|N(S')|}{|S'|} = \frac{(c-d)m+dm}{dm} = \frac{c}{d} < 2,$$

Thus

$$T(G) = 2\left(\frac{(c-d)m+2}{dm+1}\right) \downarrow 2\left(\frac{c-d}{d}\right) = 2(\text{bind}(G) - 1).$$

Next if bind$(G) \geq 2$, the lower bound on $T(G)$ given by Theorem 2.2 is bind(G). Consider the graphs $G := K_{cm} + dmK_1$, with $c \geq 2d$ and $m \geq 1$. Taking $S := V(dmK_1)$ and $S' := V(K_{cm})$, we have

$$\text{bind}(G) = \frac{|N(S)|}{|S|} = \frac{cm}{dm} = \frac{c}{d} \geq 2,$$

and

$$T(G) = \frac{|S'| + m(G - S')}{\omega(G - S')} = \frac{cm + 1}{dm} \downarrow \frac{c}{d} = \text{bind}(G).$$

If bind$(G) < 1$, all we can say is $T(G) > 0$. Indeed, given $b < 1$ and a connected graph H with $|H| \geq 2$ and bind$(H) = b$, let $G := H \cup (m-1)K_2$. Then bind(G) = bind(H) = b and $T(G) \leq |H|/m$, which can be made arbitrarily small.

3 A Best Monotone Degree Improvement Over Theorem 2.2

As stated, Theorem 2.2 is best possible when bind$(G) \geq 1$. However, in this section we will introduce a class of graphs that are known to satisfy a stronger result. Recall that for a given graph property P and a graphical sequence π, we say that π is best monotone P if π satisfies a best monotone P theorem, and we denote this by $\pi \in \text{BM}(P)$. Before progressing, we present best monotone theorems for the properties of being b-binding and t-tenacious. The following two theorems appear in [5].

Theorem 3.1. *Let $0 < b \leq 1$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq 2$. If*

(i) $d_i \leq \lceil bi \rceil - 1 \implies d_{n-\lceil bi \rceil+1} \geq n - i$, *for* $1 \leq i \leq \left\lfloor \frac{n}{b+1} \right\rfloor$, *and*

(ii) $d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor$,

then π is forcibly b-binding.

Theorem 3.2. *Let $b \geq 1$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq \lceil b + 1 \rceil$. If*

(i) $d_i \leq n - \left\lfloor \frac{n-i}{b} \right\rfloor - 1 \implies d_{\lfloor \frac{n-i}{b} \rfloor + 1} \geq n - i$, *for* $1 \leq i \leq \left\lfloor \frac{n}{b+1} \right\rfloor$, *and*

(ii) $d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor$,

then π is forcibly b-binding.

We can also prove the following.

Theorem 3.3. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence with $n \geq 2$ and let t be a real number with $\dfrac{2}{n-1} \leq t \leq n$. If*

$$(\dagger) \quad d_{\lfloor \frac{n+1}{t+1} \rfloor + 1} \geq n - \left\lfloor \frac{n+1}{t+1} \right\rfloor,$$

then π is forcibly t-tenacious.

Proof of Theorem 3.3: Let $\pi = (d_1 \leq d_2 \leq ... \leq d_n)$, with $n \geq 2$, satisfy ($\dagger$) for a fixed t with $\frac{2}{n-1} \leq t \leq n$. Assume that π has a realization G that is not t-tenacious. Then there exists $S \subset V(G)$ with $\omega(G-S) \geq 2$ such that $T(G) = \frac{|S|+m(G-S)}{\omega(G-S)} < t$. Define $s := |S|$, $m := m(G-S) \geq 1$, and $\omega := \omega(G-S)$ so that $T(G) = \frac{s+m}{\omega} < t$.

Since every vertex not in S has degree at most $s+m-1$, we have $d_{n-s} \leq s+m-1$. Define $j := s+m-1 \geq s$.

Claim. $j \leq n - \left\lfloor \dfrac{n+1}{t+1} \right\rfloor - 1.$

Proof of Claim: Note that $\omega > \frac{s+m}{t}$ and $n \geq s+m+(\omega-1)$. Therefore,

$$n - j = n - s - m + 1 \geq \omega > \frac{s+m}{t} = \frac{j+1}{t}.$$

Thus

$$\begin{aligned} j &< \frac{tn}{t+1} - \frac{1}{t+1} = n - \frac{n}{t+1} - \frac{1}{t+1} \\ &= n - \left(\frac{n+1}{t+1}\right) \leq n - \left\lfloor \frac{n+1}{t+1} \right\rfloor, \end{aligned}$$

proving the Claim. □

It follows that

$$d_{\lfloor \frac{n+1}{t+1} \rfloor + 1} \leq d_{n-j} \leq d_{n-s} \leq j \leq n - \left\lfloor \frac{n+1}{t+1} \right\rfloor - 1,$$

a contradiction. ■

All three theorems above are best monotone for their respective properties. Clearly, each are monotone. The weak optimality of Theorems 3.1 and 3.2 is demonstrated in [5]. To see that Theorem 3.3 is weakly optimal, notice that if π fails to satisfy ($\dagger$), then

$$\pi' = \left(n - \left\lfloor \frac{n+1}{t+1} \right\rfloor - 1\right)^{(\lfloor \frac{n+1}{t+1} \rfloor + 1)} (n-1)^{(n - \lfloor \frac{n+1}{t+1} \rfloor - 1)}$$

majorizes π and has a realization $G' = K_{n-\lfloor \frac{n+1}{t+1} \rfloor -1} + \overline{K_{\lfloor \frac{n+1}{t+1} \rfloor +1}}$ with

$$\begin{aligned} T(G') &= \frac{\left(n - \left\lfloor \frac{n+1}{t+1} \right\rfloor - 1\right) + 1}{\left\lfloor \frac{n+1}{t+1} \right\rfloor + 1} = \frac{n - \left(\left\lfloor \frac{n+1}{t+1} \right\rfloor + 1\right) + 1}{\left\lfloor \frac{n+1}{t+1} \right\rfloor + 1} \\ &< \frac{n - \frac{n+1}{t+1} + 1}{\frac{n+1}{t+1}} = t. \end{aligned}$$

From Theorem 2.2 we know that if π is a graphical sequence of length n, then

$$(\pi \text{ forcibly } b\text{-binding}) \Longrightarrow (\pi \text{ forcibly } t\text{-tenacious})$$
$$\text{for } t = \max\left\{\frac{1}{n}, \min\{2(b-1), b\}\right\}. \quad (1)$$

However if $\pi \in \text{BM}(b\text{-binding})$, we get a stronger result.

Theorem 3.4. *Let $n \geq 2$ and b be fixed with $\frac{1}{n-1} \leq b \leq n-1$. If $\pi = (d_1 \leq \cdots \leq d_n)$ is a graphical sequence, then*

$$\pi \in \text{BM}(b - \textit{binding}) \Rightarrow \pi \in \text{BM}\left(\frac{(n+1)b+1}{n} - \textit{tenacious}\right).$$

Since $\frac{(n+1)b+1}{n} > b \geq \min\{2(b-1), b\}$, this is an improvement over (1).

Proof of Theorem 3.4: Let $b \geq \frac{1}{n-1}$, and assume that $\pi \in$ BM(b − binding). Then

$$d_{\lfloor \frac{n}{b+1} \rfloor +1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor,$$

by condition (ii) of Theorem 3.1 or 3.2.

Let $t = \frac{(n+1)b+1}{n}$. Then $\frac{2}{n-1} \leq t \leq n$ and

$$\left\lfloor \frac{n+1}{t+1} \right\rfloor = \left\lfloor \frac{n+1}{\frac{(n+1)b+1}{n} + 1} \right\rfloor = \left\lfloor \frac{n(n+1)}{(b+1)(n+1)} \right\rfloor = \left\lfloor \frac{n}{b+1} \right\rfloor.$$

Thus

$$d_{\lfloor \frac{n+1}{t+1} \rfloor +1} = d_{\lfloor \frac{n}{b+1} \rfloor +1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor = n - \left\lfloor \frac{n+1}{t+1} \right\rfloor,$$

and therefore π satisfies (†) for $t = \dfrac{(n+1)b+1}{n}$. ■

If a graph G is b-binding, then $T(G) > \min\{2(b-1), b\}$ by Theorem 2.2. Now, let P be a graph property. Define

$$\mathrm{BMG}(P) := \{G | \, G \text{ is a realization of some } \pi \in \mathrm{BM}(P)\}.$$

Then by Theorem 3.4, for any $G \in \mathrm{BMG}(b\text{-binding})$ it follows that $T(G) \geq \frac{(n+1)b+1}{n} > \min\{2(b-1), b\}$. What is interesting here is how Theorem 2.2 is best possible, for $b \geq 1$, but any graph in BMG(b-binding) satisfies a stronger inequality when $b \geq \frac{1}{n-1}$. So, knowing that G is in BMG(b-binding) provides a tighter bound on $T(G)$ than the one we get when G is simply known to be b-binding.

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